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On Wijsman Ideal Convergent Set of Sequences Defined by an Orlicz Function

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Abstract. In this study, our main topics are Wijsman ideal convergence and Orlicz function. We define Wijsman ideal convergent set of sequences defined by an Orlicz function where $\mathcal{I}$ is an ideal of the subset of positive integers $\mathbb{N}$. We also obtain some inclusion theorems.

1. Preliminaries and Notation

Statistical convergence of sequences of points was introduced by Steinhaus [21] and Fast [7] and later Schoenberg reintroduced this concept and he established some basic properties of statistical convergence and also studied the concept as a summability method [20]. The last twenty years this concept has been applied in various areas.

Let K be a subset of the set of all natural numbers $\mathbb{N}$ and $K_n = |k \leq n : k \in K|$ where the vertical bars indicate the number of elements in the enclosed set. The natural density of K is defined by $\delta(K) := \lim_{n\to\infty} n^{-1}|\{k \leq n : k \in K\}$. Now we recall some definitions and results on statistical convergence.

Definition 1.1. *(Fast, [7]) A number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim x_k = L$. Statistical convergence is a natural generalization of ordinary convergence. If $\lim x_k = L$, then $st - \lim x_k = L$. The converse does not hold in general.

$\mathcal{I}-$convergence is an important notion in our area and that is based on the notion of an ideal of the subset of positive integers. Kostyrko et al. [14] introduced the notion of $\mathcal{I}-$convergence in a metric space in 2000. Esi and Hazarika ([5] , [6]), Hazarika and Savas [9], Savas ([17] , [18] , [19]), Kişi et al. ([12] , [13]) and many others dealt with $\mathcal{I}-$convergence and Orlicz function. Now we state the definitions of ideal and filter.

Definition 1.2. *A non-empty family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if $\emptyset \in \mathcal{I}$, for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.*

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An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 1.3. *A non-empty family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if $\emptyset \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.*

If $\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter in $\mathbb{N}$.

Definition 1.4. *Let $\mathcal{I}$ be a non-trivial ideal of subsets in $\mathbb{N}$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}-$convergent to L if and only if for each $\varepsilon > 0$ the set*

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$$

belongs to $\mathcal{I}$. This is denoted by $\mathcal{I} - \lim_{n\to\infty} x_n = L$.

Now we have some easy but important examples about $\mathcal{I}-$convergence.

Example 1.5. *Take for $\mathcal{I}$ class the $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal and $\mathcal{I}_f$ $-$convergence coincides with the usual convergence.*

Example 1.6. *Denote by $\mathcal{I}_d$ the class of all $A \subset \mathbb{N}$ which has natural density zero. Then $\mathcal{I}_d$ is an admissible ideal and $\mathcal{I}_d$-convergence coincides with the statistical convergence.*

Recently, Das, Savas and Ghosal [3] introduced new notions, namely $\mathcal{I}-$statistical convergence and $\mathcal{I}-$lacunary statistical convergence.
Now we will carry these definitions to set of sequences and we obtain Wijsman $\mathcal{I}-$convergence.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a\in A} \rho(x, A).$$

Definition 1.7. (*Baronti and Papini*, [2])*Let (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$ for all $k \in \mathbb{N}$ we say that the sequence $\{A_k\}$ is Wijsman convergent to A if*

$$\lim_{k\to\infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim_{k\to\infty} A_k = A$.

As an example, consider the following sequence of circles in (x, y)-plane:

$$A_k = \{(x, y) : x^2 + y^2 + 2kx = 0\}.$$

As $k \to \infty$ the sequence is Wijsman convergent to y-axis $A = \{(x, y) : x = 0\}$.

Definition 1.8. (*Baronti and Papini*, [2]) *Let (X, d) be a metric space. For any non-empty closed subset A_k of X for all $k \in \mathbb{N}$ we say that the sequence $\{A_k\}$ is bounded if*

$$\sup_k d(x, A_k) < \infty$$

for each $x \in X$. In this case we write $\{A_k\} \in L_\infty$.

Definition 1.9. (*Baronti and Papini*, [2]) *Let (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$ for all $k \in \mathbb{N}$ we say that the sequence $\{A_k\}$ is Wijsman Cesáro summable to A if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_k) = d(x, A)$$

for each $x \in X$ and we say that $\{A_k\}$ is Wijsman strongly Cesáro summable to A if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| = 0$$

for each $x \in X$.

In 2012, Nuray and Rhoades presented Wijsman statistical convergence for set of sequences. After this definition, Ulusu and Nuray presented the concept of Wijsman lacunary statistical convergence in 2012.

Definition 1.10. (*Nuray and Rhoades*, [16]) *Let (X, d) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$ for all $k \in \mathbb{N}$ we say that the sequence $\{A_k\}$ is Wijsman statistically convergent to A if for $\varepsilon > 0$ and for each $x \in X$ we have*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |d(x, A_k) - d(x, A)| \ge \varepsilon\}| = 0.$$

In this case we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$ where WS denotes the set of Wijsman statistically convergent sequences.

Definition 1.11. *(Kişi and Nuray,* [12]*) Let (X, d) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subseteq X$ for all $k \in \mathbb{N}$ we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$−convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$ the set,*

$$A(x, \varepsilon) = \{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \ge \varepsilon\}$$

belongs to $\mathcal{I}$. In this case we write $\mathcal{I}_W - \lim A_k = A$ or $A_k \to A(\mathcal{I}_W)$ where $\mathcal{I}_W$ is the set of Wijsman $\mathcal{I}$−convergent sequences.

As an example, consider the following sequence. Let $X = \mathbb{R}^2$ and $\{A_k\}$ be a sequence as follows:

$$A_k = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 2ky = 0\right\} & \text{if, } k \ne n^2 \\ \left\{(x, y) \in \mathbb{R}^2 : y = -1\right\} & \text{if, } k = n^2 \end{cases}$$

and

$$A = \left\{(x, y) \in \mathbb{R}^2 : y = 0\right\}.$$

The sequence $\{A_k\}$ is not Wijsman convergent to the set A. But if we take $\mathcal{I} = \mathcal{I}_d$ then $\{A_k\}$ is Wijsman $\mathcal{I}$−convergent to set A, where $\mathcal{I}_d$ is the ideal of sets which have zero density.

Definition 1.12. *(Kişi and Nuray,* [13]*) Let (X, d) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subseteq X$ for all $k \in \mathbb{N}$ we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$−statistically convergent to A or $S(\mathcal{I}_W)$-convergent to A if for each $\varepsilon > 0$, for each $x \in X$ and $\delta > 0$ we have,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \le n : |d(x, A_k) - d(x, A)| \ge \varepsilon\}| \ge \delta\right\} \in \mathcal{I}.$$

In this case, we write $A_k \to A\,(S\,(\mathcal{I}_W))$. The class of all Wijsman $\mathcal{I}-$statistically convergent sequences will be denoted by $S\,(\mathcal{I}_W)$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Definition 1.13. *(Ulusu and Nuray,* [22]*) Let (X,ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$ for all $k \in \mathbb{N}$ we say that the sequence $\{A_k\}$ is Wijsman lacunary statistically convergent to A if $\{d(x,A_k)\}$ is lacunary statistically convergent to $d(x,A)$; i.e., for $\varepsilon > 0$ and for each $x \in X$ we have*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| = 0.$$

In this case we write $S_\theta - \lim_W = A$ or $A_k \to A(WS_\theta)$.

Recall that an Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function M satisfies the Δ_2-condition if there exits a constant $K > 0$ such that $M(2u) \leq KM(u)$ for all $u \geq 0$. Note that if $0 < \lambda < 1$, then $M(\lambda x) \leq \lambda M(x)$ for all $x \geq 0$.
If convexity of Orlicz function M is replaced by $M(x+y) = M(x) + M(y)$ then this function is called Modulus function, which was presented and discussed by Maddox [15].

2. Main Results

Definition 2.1. *Let (X,d) be a metric space and θ be a lacunary sequence. A set of sequence $\{A_k\}$ is said to be Wijsman strongly $\mathcal{I}-$lacunary convergent to A if for each $\varepsilon > 0$ and for each $x \in X$ we have,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x,A_k) - d(x,A)| \geq \varepsilon\right\} \in \mathcal{I}.$$

In this case we write $A_k \overset{\mathcal{I}-W[N_\theta]}{\to} A$.

Definition 2.2. *Let (X,d) be a metric space and θ be a lacunary sequence. A set of sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}-$lacunary statistically convergent to A if for each $\varepsilon > 0$, for each $x \in X$ and $\delta > 0$ we have,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $A_k \overset{\mathcal{I}-WS_\theta}{\to} A$.

Definition 2.3. *Let (X,d) be a metric space and M be an Orlicz function. For any non-empty closed subsets $A, A_k \subseteq X$ for all $k \in \mathbb{N}$ we say that the sequence $\{A_k\}$ is Wijsman strongly Cesáro summable to A with respect to an Orlicz function (Wijsman sense), if for each $x \in X$ we have,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} M\left(|d\,(x,A_k) - d\,(x,A)|\right) = 0.$$

This is denoted by $\{A_k\} \overset{W[C_1](M)}{\to} A$.

Definition 2.4. *Let (X,d) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$ and M be an Orlicz function. For any non-empty closed subsets $A, A_k \subseteq X$ for all $k \in \mathbb{N}$ we say that the sequence $\{A_k\}$ is strongly Cesáro summable to A (Wijsman sense) with respect to an Orlicz function and ideal if for each $\varepsilon > 0$ and for each $x \in X$ we have,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} M\left(\left|d\left(x, A_k\right) - d\left(x, A\right)\right|\right) \geq \varepsilon \right\} \in \mathcal{I}.$$

This is denoted by $\{A_k\} \overset{\mathcal{I}-W[C_1](M)}{\rightarrow} A$.

Definition 2.5. *Let (X,d) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$ and M be an Orlicz function. For any non-empty closed subsets $A, A_k \subseteq X$ for all $k \in \mathbb{N}$ we say that the sequence $\{A_k\}$ is $\mathcal{I}-$statistically convergent to A with respect to an Orlicz function (Wijsman sense) , if for each $\varepsilon, \delta > 0$ and for each $x \in X$ we have,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left\{ k \leq n : M\left(\left|d\left(x, A_k\right) - d\left(x, A\right)\right|\right) \geq \varepsilon \right\} \geq \delta \right\} \in \mathcal{I}.$$

This is denoted by $\{A_k\} \overset{S(\mathcal{I}_W)(M)}{\rightarrow} A$.

Definition 2.6. *Let (X,d) be a metric space and θ be a lacunary sequence. A set of sequence $\{A_k\}$ is said to be Wijsman strongly $\mathcal{I}-$lacunary convergent to A with respect to an Orlicz function if for each $\varepsilon > 0$ and for each $x \in X$ we have,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} M\left(\left|d\left(x, A_k\right) - d\left(x, A\right)\right|\right) \geq \varepsilon \right\} \in \mathcal{I}.$$

In this case we write $A_k \overset{(\mathcal{I}-W[N_\theta](M))}{\rightarrow} A$.

Theorem 2.7. *Let (X,ρ) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$ and M be an Orlicz function. $A, A_k \subseteq X$, for all $k \in \mathbb{N}$, are non empty closed subsets. Then we have*

(*i*) $\{A_k\} \overset{\mathcal{I}-W[C_1](M)}{\rightarrow} A \Longrightarrow \{A_k\} \overset{S(\mathcal{I}_W)}{\rightarrow} A$;

(*ii*) *If M satisfies Δ_2 condition and $\{A_k\} \overset{S(\mathcal{I}_W)}{\rightarrow} A$ for all $\{A_k\} \in L_\infty(M)$ then we have $\{A_k\} \overset{\mathcal{I}-W[C_1](M)}{\rightarrow} A$;*

(*iii*) *If M satisfies Δ_2 condition, then we have*

$$\mathcal{I} - W[C_1](M) \cap L_\infty(M) = S(\mathcal{I}_W) \cap L_\infty(M)$$

where $L_\infty(M) = \{A_k : M(d(x, A_k)) \in L_\infty, x \in X\}$.

Proof. (*i*) Suppose that $\{A_k\} \overset{\mathcal{I}-W[C_1](M)}{\rightarrow} A$. Let $\varepsilon > 0$ be given. Then we can write

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} M\left(\left|d\left(x, A_k\right) - d\left(x, A\right)\right|\right) &\geq \frac{1}{n} \sum_{\substack{k=1 \\ |d(x,A_k)-d(x,A)| \geq \varepsilon}}^{n} M\left(\left|d\left(x, A_k\right) - d\left(x, A\right)\right|\right) \\ &\geq \frac{M(\varepsilon)}{n} \left|\left\{ k \leq n : \left|d\left(x, A_k\right) - d\left(x, A\right)\right| \geq \varepsilon \right\}\right|. \end{aligned}$$

Consequently, for any $\delta > 0$ we have

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left|\left\{ k \leq n : \left|d\left(x, A_k\right) - d\left(x, A\right)\right| \geq \varepsilon \right\}\right| \geq \frac{\delta}{M(\varepsilon)} \right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} M\left(\left|d\left(x, A_k\right) - d\left(x, A\right)\right| \geq \delta\right) \right\} \in \mathcal{I}.$$

Hence $\{A_k\} \overset{S(\mathcal{I}_W)}{\rightarrow} A$.

(*ii*) Suppose that M is bounded and $\{A_k\} \overset{S(\mathcal{I}_W)}{\rightarrow} A$. Since M is bounded there exists a real number $K > 0$ such that $\sup_t M(t) \leq K$. Moreover, for any $\varepsilon > 0$ we can write

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} M\left(\left|d\left(x,A_k\right)-d\left(x,A\right)\right|\right) &= \frac{1}{n}\left[\sum_{\substack{k=1\\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}^{n} M\left(\left|d\left(x,A_k\right)-d\left(x,A\right)\right|\right)+\sum_{\substack{k=1\\ |d(x,A_k)-d(x,A)|<\varepsilon}}^{n} M\left(\left|d\left(x,A_k\right)-d\left(x,A\right)\right|\right)\right]\\ &\leq \frac{K}{n}\left|\left\{k\leq n:\left|d\left(x,A_k\right)-d\left(x,A\right)\right|\geq\varepsilon\right\}\right|+M\left(\varepsilon\right).\end{aligned}$$

Now for any $\delta > 0$ we get

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n} M\left(\left|d\left(x,A_k\right)-d\left(x,A\right)\right|\right)\geq\delta\right\}\subseteq\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\left|d\left(x,A_k\right)-d\left(x,A\right)\right|\geq\varepsilon\right\}\right|\geq\frac{\delta}{K}\right\}\in\mathcal{I}.$$

Hence $\{A_k\} \overset{\mathcal{I}-W[C_1](M)}{\rightarrow} A$.

(*iii*) The proof of this part follows from parts (*i*) and (*ii*). □

Theorem 2.8. *Let (X,ρ) be a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$, $\theta = (k_r)$ be a lacunary sequence and M be an Orlicz function. For any non-empty closed subsets A_k, $B_k \subseteq X$ for all $k \in \mathbb{N}$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for $x \in X$ we have,*

(i) (a) $A_k \overset{\mathcal{I}-W[N_\theta](M)}{\rightarrow} A \Rightarrow A_k \overset{\mathcal{I}-WS_\theta}{\rightarrow} A$;

(b) $\mathcal{I} - W[N_\theta](M) \subset \mathcal{I} - WS_\theta$;

(ii) If M satisfies Δ_2 condition and $\{A_k\} \overset{\mathcal{I}-WS_\theta}{\rightarrow} A$ for all $\{A_k\} \in L_\infty(M)$ then we have $\{A_k\} \overset{\mathcal{I}-W[N_\theta](M)}{\rightarrow} A$;

(iii) If M satisfies Δ_2 condition then $\mathcal{I} - WS_\theta \cap L_\infty(M) = \mathcal{I} - W[N_\theta](M) \cap L_\infty(M)$.

Proof. (*i*) *a*) Suppose that $A_k \overset{\mathcal{I}-W[N_\theta](M)}{\rightarrow} A$. Let $\varepsilon > 0$ be given. Then we can write

$$\frac{1}{h_r}\sum_{k\in I_r} M\left(\left|d(x,A_k)-d(x,A)\right|\right) = \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x,A_k)-d(x,A)|\geq\varepsilon}} M\left(\left|d(x,A_k)-d(x,A)\right|\right)+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x,A_k)-d(x,A)|<\varepsilon}} M\left(\left|d(x,A_k)-d(x,A)\right|\right)$$

and so

$$\frac{1}{h_r}\sum_{k\in I_r} M\left(\left|d(x,A_k)-d(x,A)\right|\right) \geq \frac{M\left(\varepsilon\right)}{h_r}\left|\left\{k\in I_r:\left|d(x,A_k)-d(x,A)\right|\geq\varepsilon\right\}\right|.$$

Then for any $\delta > 0$

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r:\left|d(x,A_k)-d(x,A)\right|\geq\varepsilon\right\}\right|\geq\frac{\delta}{M\left(\varepsilon\right)}\right\}\subseteq\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r} M\left(\left|d(x,A_k)-d(x,A)\right|\geq\delta\right)\right\}\in\mathcal{I}.$$

This proves the result.

b) In order to establish $\mathcal{I} - W[N_\theta](M) \subseteq \mathcal{I} - WS_\theta$ is proper, for any given θ we choose $\{A_k\}$ as follows:

$$\{A_k\} = \begin{cases} \{k\} & , \text{ if } k_{r-1} < k \leq k_{r-1} + \left[\sqrt{h_r}\right] \qquad r = 1,2,... \\ \{0\} & , \text{ otherwise} \end{cases}$$

Then for any $\varepsilon > 0$,

$$\frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = \frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, \{0\})| \geq \varepsilon\}| \leq \frac{\left[\left|\sqrt{h_r}\right|\right]}{h_r}$$

and for any $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{\left[\left|\sqrt{h_r}\right|\right]}{h_r} \geq \delta\right\}.$$

Since the set on the right-hand side is a finite set and so belongs to $\mathcal{I}$, it follows that $A_k \overset{\mathcal{I}-WS_\theta}{\rightarrow} A$.

On the other hand

$$\frac{1}{h_r}\sum_{k\in I_r} M|d(x, A_k) - d(x, \{0\})| = \frac{1}{h_r}\frac{\left[\sqrt{h_r}\right]\left(\left[\sqrt{h_r}\right]+1\right)}{2}.$$

Then

$$\begin{aligned} \left\{r \in \mathbb{N} : \frac{1}{h_r}\textstyle\sum_{k\in I_r} M|d(x, A_k) - d(x, \{0\})| \geq \frac{1}{4}\right\} &= \left\{r \in \mathbb{N} : \frac{\left[\sqrt{h_r}\right]\left(\left[\sqrt{h_r}\right]+1\right)}{h_r} \geq \frac{1}{2}\right\} \\ &= \{m, m+1, m+2, ...\} \end{aligned}$$

for some $m \in \mathbb{N}$ which belongs to $F(\mathcal{I})$, since $\mathcal{I}$ is admissible. So $A_k \overset{\mathcal{I}-W[N_\theta](M)}{\nrightarrow} \{0\}$.

(*ii*) Suppose that M is bounded and $A_k \overset{\mathcal{I}-WS_\theta}{\rightarrow} A$. Since M is bounded there exists a real number $K > 0$ such that $\sup_t M(t) \leq K$. Moreover, for any $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{h_r}\sum_{k\in I_r} M(|d(x, A_k) - d(x, A)|) &= \frac{1}{h_r}\left[\sum_{\substack{k=1 \\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}^{n} M(|d(x, A_k) - d(x, A)|) + \sum_{\substack{k\in I_r \\ |d(x,A_k)-d(x,A)|<\varepsilon}} M(|d(x, A_k) - d(x, A)|)\right] \\ &\leq \frac{K}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| + M(\varepsilon). \end{aligned}$$

Consequently, we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} M(|d(x, A_k) - d(x, A)|) \geq \varepsilon\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \frac{\varepsilon}{K}\right\} \in \mathcal{I}.$$

This proves the result.

(*iii*) The proof of this part follows from parts (*i*) and (*ii*). □

Theorem 2.9. *For any $\theta = (k_r)$ lacunary sequence and for any Orlicz function M, $\mathcal{I}-$statistical convergence implies $\mathcal{I}-$lacunary statistical convergence for sequence of sets with respect to M if and only if* $\liminf_r q_r > 1$. *If* $\liminf_r q_r = 1$ *then there exists a bounded sequence $\{A_k\}$ which is $\mathcal{I}-$statistically convergent but not $\mathcal{I}-$lacunary statistically convergent with respect to M.*

Proof. Suppose first that $\liminf_r q_r > 1$. Then there exists $\alpha > 0$ such that $q_r > 1 + \alpha$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\alpha}{1+\alpha}.$$

Since $\{A_k\} \overset{S(I_W)(M)}{\rightarrow} A$ and for sufficiently large r, we have

$$\begin{aligned} \frac{1}{k_r} \left| k \leq k_r : M\left(\left| d(x, A_k) - d(x, A) \right| \geq \varepsilon\right) \right| &\geq \frac{1}{k_r} \left| \{k \in I_r : M\left(\left| d(x, A_k) - d(x, A) \right| \geq \varepsilon\right)\} \right| \\ &\geq \frac{\alpha}{1+\alpha} \frac{1}{h_r} \left| \{k \in I_r : M\left(\left| d(x, A_k) - d(x, A) \right| \geq \varepsilon\right)\} \right|. \end{aligned}$$

Then for any $\delta > 0$, we get

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \{k \in I_r : M\left(\left| d(x, A_k) - d(x, A) \right| \geq \varepsilon\right)\} \right| \geq \delta \right\} \\ &\subseteq \left\{ r \in \mathbb{N} : \frac{1}{k_r} \left| \{k \leq k_r : M\left(\left| d(x, A_k) - d(x, A) \right| \geq \varepsilon\right)\} \right| \geq \frac{\delta\alpha}{(1+\alpha)} \right\} \in \mathcal{I}. \end{aligned}$$

This proves the sufficiency.

Conversely, suppose that $\liminf_r q_r = 1$. Hence we can select a subsequence $\left\{k_{r_j}\right\}$ of the lacunary sequence $\theta = (k_r)$ such that

$$\frac{k_{r_j}}{k_{r_j - 1}} < 1 + \frac{1}{j} \quad \text{and} \quad \frac{k_{r_j - 1}}{k_{r_{j-1}}} > j, \text{ where } r_j \geq_{r_{j-1}} + 2.$$

Now we define a sequence $\{A_k\}$ as follows:

$$A_k = \begin{cases} x^2 + (y-1)^2 = \dfrac{1}{k^4} & , \text{ if } i \in I_{r_j}, \\ \{(0,0)\} & , \text{ otherwise.} \end{cases}$$

Then

$$\frac{1}{h_{r_j}} \sum_{k \in I_{r_j}} M\left(\left| d(x, A_k) - d(x, \{(0,0)\}) \right|\right) = K, \quad \text{for } j = 1, 2, .. (K \in \mathbb{R}^+)$$

and

$$\frac{1}{h_{r_j}} \sum_{k \in I_{r_j}} M\left(\left| d(x, A_k) - d(x, \{(0,0)\}) \right|\right) = 0, \text{ for } r \neq r_j.$$

Then it is quite clear that $\{A_k\}$ does not belong to $\mathcal{I} - W[N_\theta](M)$. Since $\{A_k\}$ is bounded then we have $\{A_k\} \overset{\mathcal{I}-WS_\theta}{\nrightarrow} A$. Next, let $k_{r_{j-1}} \leq n \leq k_{r_{j+1}-1}$ Then, from Theorem 2.1 in [3], we can write

$$\begin{aligned} &\frac{\varepsilon}{n} \left| k \leq n : M\left(\left| d(x, A_k) - d(x, \{(0,0)\}) \right|\right) \right| \\ &\leq \frac{1}{n} \sum_{k=1}^{n} M\left(\left| d(x, A_k) - d(x, \{(0,0)\}) \right|\right) \\ &\leq \frac{k_{r_{j-1}} + h_{r_j}}{k_{r_j - 1}} \leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j}. \end{aligned}$$

Hence $\{A_k\}$ is Wijsman $\mathcal{I}$−statistically convergent with respect to M for any admissible ideal $\mathcal{I}$. $\square$

3. References

[1] V. Baláž, J. Červeňanski, P. Kostyrko and T. Šalát, , *I-convergence and I-continutiy of real functions*, Acta Mathematica 5 (2002) 56 - 62.
[2] M. Baronti and P. Papini, *Convergence of sequences of sets*, Methods of Functional Analysis in Approximation Theory, Basel: Birkhauser(1986)133-155.
[3] P. Das, E. Savaş and S. Kr.,Ghosal, *On generalized of certain summability methods using ideals*, Appl. Math. Letters 36(2011) 1509-1514.
[4] A. Esi and M. Et, *Some new sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math. 31(8) (2000)967-972.
[5] A. Esi and B. Hazarika), *Lacunary summable sequence spaces of fuzzy numbers defined by ideal convergence and an Orlicz function*, Afrika Matematika, (2014) DOI: 10.1007/s13370-012-0117-3.
[6] A. Esi and B. Hazarika, *Some new generalized classes of sequences of Fuzzy numbers defined by an Orlicz function*, Annals of Fuzzy Math. Informatics 4(2)(2012) 401-406.
[7] H. Fast, *Sur la convergence statistique*, Coll. Math. 2 (1951) 241-244.
[8] B. Hazarika, *Some new sequence of fuzzy numbers defined by Orlicz functions using a fuzzy metric*, Comput. Math. Appl. 61(9) (2011), 2762-2769.
[9] B. Hazarika and E. Savas, *Some $\mathcal{I}-$convergent lambda-summable difference sequence spaces of fuzzy real numbers defined by a sequence of Orlicz functions*, Math. Comp. Modell. 54 (2011), 2986-2998.
[10] B. Hazarika, *On fuzzy real valued generalized difference $\mathcal{I}-$convergent sequence spaces defined by Musielak-Orlicz function*, J. Intell. Fuzzy Systems 25(1) (2013) 9-15.
[11] B. Hazarika, K. Tamang and B. K. Singh, , Zweier *Ideal Convergent Sequence Spaces Defined by Orlicz Function*, The J. Math. and Computer Sci. 8 (2014) 307-318.
[12] O. Kişi and F. Nuray, *New Convergence Definitions for Sequence of Sets*, Abstract and Applied Analysis., Vol. 2013(2013), Article ID:852796, 6 pages.
[13] O. Kişi and F. Nuray, On $S_\lambda(\mathcal{I})$ *−asymptotically statistical equivalence of sequence of sets,* Mathematical Analysis, Vol.2013, Article ID:602963, (2013)6 pages.
[14] P. Kostyrko,T. Šalát and W. Wilezyński, *$\mathcal{I}-$Convergence*, Real Analysis Exchange 26(2000) 669-680.
[15] I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Cambridge Philos. Soc. 100(1986) 161-166.
[16] F. Nuray and B. E. Rhoades, S*tatistical convergence of sequences of sets*, Fasciculi Mathematici 49(2012) 87-99.
[17] E. Savaş, *On Some New Sequence Spaces in 2-Normed Spaces Using Ideal Convergence and an Orlicz Function*, J. Ineq. Appl.(2010), Article Number:482392 DOI:10.1155/2010/482392.
[18] E. Savaş, *A-sequence spaces in 2-normed spaces using Ideal Convergence and an Orlicz Function*, Abst. Appl. Anal., Vol. 2011(2011), Article ID:741382.
[19] E. Savaş, Δ^m*-strongly summable sequences spaces in 2-normed spaces defined by ideal convergence and an Orlicz function*, Applied Mathematics and Computation 217(1) (2010) 271–276.
[20] I.J. Schoenberg,*The integrability of certain functions and related summability methods*, Amer. Math. Monthly 66(1959) 361-375.
[21] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. 2(1951) 73-74.
[22] U. Ulusu and F. Nuray , *Lacunary statistical convergence of sequence of sets*, Progress in Applied Mathematics 4(2)(2012) 99-109.
[23] R. A. Wijsman, *Convergence of sequences of Convex sets, Cones and Functions II*, Trans. Amer. Math. Soc., Vol 123, no:1(1966)