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Statistical $\alpha\beta-$Summability and Korovkin Type Approximation Theorem

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Abstract. In this study, we define $[N^{\gamma}, \alpha\beta]_q-$summability and statistical $(N^{\gamma}, \alpha\beta)$ summability. We also establish some inclusion relation and some related results for this new summability methods. Further we apply Korovkin type approximation theorem through statistical $(N^{\gamma}, \alpha\beta)$ summability and we apply the classical Bernstein operator to construct an example in support of our result. Furthermore, we present a rate of convergence which is uniform in Korovkin type theorem by statistical $(N^{\gamma}, \alpha\beta)$ summability.

1. Introduction, Notations and Known Results

The study of the Korovkin-type approximation theory is a well-established area of research, which concern with the problem of approximation a function f by means of a sequence A_n of positive linear operators. The concept of statistical convergence for sequence real numbers was defined by Fast [1] and Steinhaus [2] independently in 1951. Statistical convergence has recently became an area of active research. Currently, researchers in statistical convergence have devoted their effort to statistical approximation [4–12]. It is well-known that every convergent sequence is statistically convergent but converse is not always true. Also, statistically convergent sequence do not need to be bounded. So, this type convergence is quite effective in the approximation theory. First we recall the following definitions:

Let K be a subset of $\mathbb{N}$, the set of natural numbers and $K_n = \{k \leq n : k \in K\}$. The natural density of K is defined by $\delta(K) = \lim_n \frac{1}{n}|K_n|$ provided it exists, where $|K_n|$ denotes the cardinality of set K_n. A sequence $x = (x_k)$ is called statistically convergent ($st-$convergent) to the number ℓ, denoted by $st - \lim x = \ell$, for each $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - \ell| \geq \epsilon\}$ has natural density zero, that is

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - \ell| \geq \epsilon\}| = 0.$$

The idea $\alpha\beta-$statistical convergence was introduced by Aktuğlu in [20] as follows:

Let $\alpha(n)$ and $\beta(n)$ be two sequences positive number which satisfy the following conditions:

(i) α and β are both non-decreasing,

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(ii) $\beta(n) \geq \alpha(n)$,

(iii) $\beta(n) - \alpha(n) \longrightarrow \infty$ as $n \longrightarrow \infty$

and let Λ denote the set of pairs (α, β) satisfying (i)-(iii). For each pair $(\alpha, \beta) \in \Lambda$, $0 < \gamma \leq 1$ and $K \subset \mathbb{N}$, we define $\delta^{\alpha,\beta}(K, \gamma)$ in the following way

$$\delta^{\alpha,\beta}(K, \gamma) = \lim_{n\to\infty} \frac{\left|K \cap P_n^{\alpha,\beta}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}},$$

where $P_n^{\alpha,\beta}$ in the closed interval $[\alpha(n), \beta(n)]$. A sequence $x = (x_k)$ is said to be $\alpha\beta-$statistically convergent of order γ to ℓ or $S^{\gamma}_{\alpha\beta}-$ convergent, if

$$\delta^{\alpha,\beta}(\{k : |x_k - \ell| \geq \epsilon\}, \gamma) = \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : |x_k - \ell| \geq \epsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0$$

and denote $st^{\gamma}_{\alpha\beta} - \lim x = \ell$ or $x_k \longrightarrow \ell[S^{\gamma}_{\alpha\beta}]$, where $S^{\gamma}_{\alpha\beta}$ denotes the set of all $\alpha\beta-$statistically convergent of order γ. Recently, Karakaya and Karaisa [14] have introduced weighted $\alpha\beta-$statistical convergence of order γ, $[\overline{N}_{\alpha\beta}, s]$ and $(\overline{N}_{\alpha\beta}, s)$ summability methods. They have examined some inclusion relation and proved Korovkin type approximation theorems through weighted $\alpha\beta-$statistical convergence.

In this work, we introduce $[N^{\gamma}, \alpha\beta]_q-$summability and statistical $(N^{\gamma}, \alpha\beta)$ summability methods. Further we establish some inclusion relation and some related results for this new summability methods. Furthermore, we prove Korovkin's theorem through statistical $(\alpha\beta)$ summability order γ. The main motivation of this paper is to define $[N^{\gamma}, \alpha\beta]_q-$summability and statistical $(N^{\gamma}, \alpha\beta)$ summability methods, which include statistical $(C, 1)$ summability, statistical lacunary summabilitiy and statistical $\lambda-$convergent. Korovkin's theorem have applied only for $\gamma = 1$ to statistical summabilitiy so far. But in this study we prove Korovkin's theorem for $0 < \gamma \leq 1$. So, our results obtained here for $0 < \gamma \leq 1$ are new and more comprehensive in literature.

2. Statistical Summability Results

In this section, we introduce $[N^{\gamma}, \alpha\beta]_q-$summability and statistical $(N^{\gamma}, \alpha\beta)$ summability methods. We establish some inclusion relation and some related results for this new summability methods.

Definition 2.1. *Given a sequence $x = (x_n)$ for which*

$$z_n^{\gamma}(x) = \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \sum_{k\in P_n^{\alpha,\beta}} x_k,$$

(i) A sequence $x = (x_k)$ is said to be $(\alpha\beta)-$summable of order γ to ℓ, if $z_n^{\gamma}(x) \longrightarrow \ell$ as $n \longrightarrow \infty$ and we can write as $(N^{\gamma}, \alpha\beta)$. Similarly, for $\gamma = 1$ the sequence $x = (x_k)$ is said to be $(\alpha\beta)-$summable to ℓ, if $z_n(x) \longrightarrow \ell$ as $n \longrightarrow \infty$.

(ii) A sequence $x = (x_k)$ is said to be statistical $(\alpha\beta)-$summable to ℓ or briefly statistically $(N^{\gamma}, \alpha\beta)$ summable of order γ to ℓ if for every $\epsilon > 0$ the set $K_{\epsilon}(\alpha\beta) := \{k \in \mathbb{N} : |z_k^{\gamma}(x) - \ell| \geq \epsilon\}$ has natural density zero, i.e., $\delta(K_{\epsilon}(\alpha\beta)) = 0$. In this case we write $\delta^{\gamma}(\alpha\beta) - \lim x = \ell$. That is

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : |z_k^{\gamma}(x) - \ell| \geq \epsilon\}\right| = 0.$$

This definition includes the following special cases:

(i) Let λ_n be a none-decreasing sequence of positive numbers tending to ∞ such that $\lambda_n \leq \lambda_n + 1$, $\lambda_1 = 1$. If we take $\gamma = 1$, $\alpha(n) = n - \lambda_n + 1$ and $\beta(n) = n$ then statistical $(N^\gamma, \alpha\beta)$ summability is reduced to statistical $\lambda-$convergent, and $[N^\gamma, \alpha\beta]_q$ is reduced to strongly λ_q- convergence [18].

(ii) If we take $\gamma = 1$, $\alpha(n) = 1$ and $\beta(n) = n$ for all n then statistical $(N^\gamma, \alpha\beta)$ summability is reduced to statistical $(C, 1)$ summability introduced in [16, 17] .

(iii) Recall that a lacunary sequence $\theta = \{k_r\}$ is an increasing integer sequence such that $k_0 = 0$ and $h_r := k_r - k_{r-1}$. If we take $\gamma = 1$, $\alpha(r) = k_{r-1} + 1$ and $\beta(r) = k_r$; then $P^{\alpha,\beta}(r) = [k_{r-1} + 1, k_r]$. But because of $[k_{r-1} + 1, k_r] \cap \mathbb{N} = (k_{r-1}, k_r] \cap \mathbb{N}$, we have statistical $(N^\gamma, \alpha\beta)$ summability is reduced to statistical lacunary summability introduced in [19].

(iv) If we take $\alpha(r) = k_{r-1} + 1$ and $\beta(r) = k_r$; then $[N^\gamma, \alpha\beta]_q$ is reduced to $N^\gamma_\theta(p)$ [3].

Definition 2.2. *A sequence $x = (x_k)$ is said to $[N^\gamma, \alpha\beta]_q-$summable to ℓ, $(0 < q < \infty)$, if*

$$\lim_{n\to\infty} \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}} |x_k - \ell|^q \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

We denote it by $x_k \longrightarrow \ell[N^\gamma, \alpha\beta]_q$. Similarly, for $\gamma = 1$ the sequence $x = (x_k)$ is said to be $[N, \alpha\beta]_q-$ summable to ℓ.

Theorem 2.3. *Let $0 < \gamma \leq \theta \leq 1$. Then, we get $[N^\gamma, \alpha\beta]_q \subseteq [N^\theta, \alpha\beta]_q$ and the inclusion is strict for some γ, θ such that $\gamma < \theta$.*

Proof. Let $x = (x_k) \in [N^\gamma, \alpha\beta]_q$ and γ, θ be given such that $0 < \gamma \leq \theta \leq 1$. Then, we have

$$\frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}} |x_k - \ell|^q \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\theta} \sum_{k\in P_n^{\alpha,\beta}} |x_k - \ell|^q$$

which gives $[N^\gamma, \alpha\beta]_q \subseteq [N^\theta, \alpha\beta]_q$. Now, we show that this inclusion is strict. Let us consider the sequence $r = (r_k)$ define by,

$$r = (r_k) = \begin{cases} 1, & \beta(n) - \sqrt{\beta(n) - \alpha(n) + 1} + 1 \leq \beta(n), \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that

$$\frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}} |r_k - 0|^q \leq \frac{\sqrt{\beta(n) - \alpha(n) + 1}}{(\beta(n) - \alpha(n) + 1)^\gamma} = \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma-1/2}}.$$

Since $\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma-1/2}} \longrightarrow 0$ as $n \longrightarrow \infty$ for $1/2 < \gamma \leq 1$, so we have $r = (r_k) \in [N^\gamma, \alpha\beta]_q$. On the other hand, we get

$$\frac{\sqrt{\beta(n) - \alpha(n) + 1} - 1}{(\beta(n) - \alpha(n) + 1)^\theta} \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\theta} \sum_{k\in P_n^{\alpha,\beta}} |r_k - 0|^q$$

and $\frac{\sqrt{\beta(n)-\alpha(n)+1}-1}{(\beta(n)-\alpha(n)+1)^\theta} \longrightarrow \infty$ as $n \longrightarrow \infty$ for $0 < \theta < 1/2$ then, we have $r = (r_k) \notin [N^\theta, \alpha\beta]_q$. This completes the proof. □

Theorem 2.4. *Let $x = (x_k)$ is bounded. If $(\alpha\beta)-$statistical convergence of order γ to ℓ then it is statistical $(N^\gamma, \alpha\beta)-$summable to ℓ but not conversely.*

Proof. Because of $(\alpha\beta)-$statistical convergence of order γ to ℓ, $K_{\alpha\beta}(\epsilon)/(\beta(k)-\alpha(k)+1)^\gamma \longrightarrow 0$ as $k \longrightarrow \infty$, where $K_{\alpha\beta}(\epsilon) = \{j \in P_k^{\alpha,\beta} : |x_j - \ell| \geq \epsilon\}$. Then

$$\begin{aligned} |z_k^\gamma(x) - \ell| &= \left| \frac{1}{(\beta(k)-\alpha(k)+1)^\gamma} \sum_{j \in P_k^{\alpha,\beta}} x_j - \ell \right| = \left| \frac{1}{(\beta(k)-\alpha(k)+1)^\gamma} \sum_{j=\alpha(k)}^{\beta(k)} (x_j - \ell) \right| \\ &\leq \left| \frac{1}{(\beta(k)-\alpha(k)+1)^\gamma} \sum_{j \in K_{\alpha\beta}(\epsilon)} (x_j - \ell) \right| \leq \frac{1}{(\beta(k)-\alpha(k)+1)^\gamma} (\sup_j |x_j - \ell|) K_{\alpha\beta}(\epsilon) \longrightarrow 0 \end{aligned}$$

as $k \longrightarrow \infty$, this means that $z_k^\gamma(x) \longrightarrow \ell$ as $k \longrightarrow \infty$. This implies that x is $(N^\gamma, \alpha\beta)-$summable to ℓ. Therefore, x is statistical $(N^\gamma, \alpha\beta)-$summable to ℓ. □

For converse, let $\alpha(n) = 1$, $\gamma = 1$ and $\beta(n) = n$ and the sequence $y = (y_n)$ define as

$$y_n = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

Indeed, y is not $(\alpha\beta)-$statistical convergence. On the other hand y is statistical $(N^\gamma, \alpha\beta)-$summable to 0.

Theorem 2.5. *Let γ, θ be real numbers such that $0 < \gamma \leq \theta \leq 1$ and $0 < q < \infty$. Then, we have $[N^\gamma, \alpha\beta]_q \subseteq S_{\alpha\beta}^\theta$.*

Proof. Assume that $x = (x_k) \in [N^\gamma, \alpha\beta]_q$ summable to ℓ and for $\epsilon > 0$, we get

$$\begin{aligned} \sum_{k \in P_n^{\alpha,\beta}} |x_k - \ell|^q &= \sum_{\substack{k \in P_n^{\alpha,\beta} \\ |x_k - \ell| \geq \epsilon}} |x_k - \ell|^q + \sum_{\substack{k \in P_n^{\alpha,\beta} \\ |x_k - \ell| < \epsilon}} |x_k - \ell|^q \\ &\geq \sum_{\substack{k \in P_n^{\alpha,\beta} \\ |x_k - \ell| \geq \epsilon}} |x_k - \ell|^q \geq \left| \left\{ k \in P_n^{\alpha,\beta} : |x_k - \ell| \geq \epsilon \right\} \right| \epsilon^q, \end{aligned} \tag{2}$$

Using (2), we obtain

$$\begin{aligned} \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} |x_k - \ell|^q &\geq \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \left| \left\{ k \in P_n^{\alpha,\beta} : |x_k - \ell| \geq \epsilon \right\} \right| \epsilon^q \\ &\geq \frac{1}{(\beta(n)-\alpha(n)+1)^\theta} \left| \left\{ k \in P_n^{\alpha,\beta} : |x_k - \ell| \geq \epsilon \right\} \right| \epsilon^q \end{aligned}$$

which means that $x = (x_k) \in S_{\alpha\beta}^\theta$. □

The following statements are obtained in Theorem 2.5.

Corollary 2.6. *Let γ be real number such that $0 < \gamma \leq 1$ and $0 < q < \infty$. Then we have $[N^\gamma, \alpha\beta]_q \subseteq S_{\alpha\beta}^\gamma$ and $[N^\gamma, \alpha\beta]_q \subseteq S_{\alpha\beta}$.*

Theorem 2.7. *If $x = (x_k)$ is bounded and $(\alpha\beta)-$statistical convergence of order γ to ℓ then $x_k \longrightarrow \ell[N^\gamma, \alpha\beta]_q$.*

Proof. Assume that $x = (x_k)$ is bounded and $(\alpha\beta)-$statistical convergence of γ to ℓ. Then for $\epsilon > 0$, we get $\delta^{\alpha,\beta}(K_{\alpha\beta}(\epsilon), \gamma) = 0$. Since $x = (x_k)$ is bounded, there exists $M > 0$ such that $|x_k - \ell| \leq M$ for $k \in \mathbb{N}$. We obtain that

$$\frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \sum_{k \in P_n^{\alpha,\beta}} |x_k - \ell|^q = R_1(n) + R_2(n)$$

where

$$\begin{aligned} R_1(n) &= \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \sum_{\substack{k \in P_n^{\alpha,\beta} \\ k \notin K_{\alpha\beta}(\epsilon)}} |x_k - \ell|^q, \\ R_2(n) &= \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \sum_{\substack{k \in P_n^{\alpha,\beta} \\ k \in K_{\alpha\beta}(\epsilon)}} |x_k - \ell|^q. \end{aligned}$$

If $k \notin K_{\alpha\beta}(\epsilon)$ then $R_1(n) < \epsilon^q$. For $k \in K_{\alpha\beta}(\epsilon)$ we get,

$$R_2(n) \leq (\sup |x_k - \ell|) \frac{|K_{\alpha\beta}(\epsilon)|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \leq \frac{M|K_{\alpha\beta}(\epsilon)|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \longrightarrow 0, \text{ as } n \longrightarrow \infty,$$

since $\delta^{\alpha\beta}(K_{\alpha\beta}(\epsilon), \gamma) = 0$. Therefore, $x_k \longrightarrow \ell[N^{\gamma}, \alpha\beta]_q$. □

Theorem 2.8. *Let $(\alpha, \beta) \in \Lambda$. If $\liminf_n \frac{\beta_n}{\alpha_n - 1} \geq 1$, then we have $S_{\alpha\beta} \subseteq S$, where S is denoted set of all statistically convergent sequence space.*

Proof. Assume that $\liminf_n \frac{\beta(n)}{\alpha(n)-1} \geq 1$. Then, there a exists $\theta > 0$ such that $\liminf_n \frac{\beta(n)}{\alpha(n)-1} \geq 1 + \theta$, then obtain that

$$\frac{\beta_n - \alpha(n) + 1}{\beta(n)} = 1 - \frac{\alpha(n) - 1}{\beta(n)} \geq 1 - \frac{1}{1 + \theta} = \frac{\theta}{1 + \theta}.$$

For a given $\varepsilon > 0$, we have

$$\{k \leq \beta(n) : |x_k - \ell| \geq \epsilon\} \supset \{\alpha(n) \leq k \leq \beta(n) : |x_k - \ell| \geq \epsilon\}.$$

Thus,

$$\begin{aligned} \frac{1}{\beta(n)} \left|\{k \leq \beta(n) : |x_k - \ell| \geq \epsilon\}\right| &\geq \frac{\beta(n) - \alpha(n) + 1}{\beta(n)} \frac{1}{\beta(n) - \alpha(n) + 1} \left|\{\alpha(n) \leq k \leq \beta(n) : |x_k - \ell| \geq \epsilon\}\right| \\ &\geq \frac{\theta}{1 + \theta} \frac{1}{\beta(n) - \alpha(n) + 1} \left|\left\{k \in P_n^{\alpha,\beta} : |x_k - \ell| \geq \epsilon\right\}\right| \end{aligned}$$

Since $st - \lim x = \ell$, we get $st_{\alpha\beta} - \lim x = \ell$. This step completes the proof. □

3. Application to Korovkin Type Approximation

In this section, we get an analogue of classical Korovkin theorem by using the concept of statistical $(N^{\gamma}, \alpha\beta)$ summability.

Let $C[a, b]$ be the linear space of all real-valued continuous functions f on $[a, b]$ and let A be a linear operator which maps $C[a, b]$ into itself. We say A is positive operator, if for every non-negative $f \in C[a, b]$, we have $A(f, x) \geq 0$ for $x \in [a, b]$. It is well-known that $C[a, b]$ is a Banach space with the norm given by

$$\| f \|_{C[a,b]} = \sup_{x \in [a,b]} |f(x)|.$$

The classical Korovkin approximation theorem states as follows (see [20, 21])

$$\lim_{n\to\infty} \| A_n(f,x) - f(x) \|_{C[a,b]} = 0 \Leftrightarrow \lim_{n\to\infty} \| A_n(e_i,x) - e_i \|_{C[a,b]} = 0,$$

where $e_i = x^i$, $i \in \{0,1,2\}$ and $f \in C[a,b]$.

Theorem 3.1. *Let (L_k) be a sequence of positive linear operator from $C[a,b]$ in to $C[a,b]$. Then for all $f \in C[a,b]$*

$$N^\gamma(st) - \lim_{k\to\infty} \| L_k(f,x) - f(x) \|_{C[a,b]} = 0 \tag{3}$$

if and only if

$$N^\gamma(st) - \lim_{k\to\infty} \| L_k(e_0,x) - e_0 \|_{C[a,b]} = 0, \tag{4}$$

$$N^\gamma(st) - \lim_{k\to\infty} \| L_k(e_1,x) - e_1 \|_{C[a,b]} = 0, \tag{5}$$

$$N^\gamma(st) - \lim_{k\to\infty} \| L_k(e_2,x) - e_2 \|_{C[a,b]} = 0. \tag{6}$$

Proof. Because of $e_i \in C[a,b]$ for $(i = 0,1,2)$, conditions (4)-(6) follow immediately from (3). Let the conditions (4)-(6) hold and $f \in C[a,b]$. By the continuity of f at x, it follows that for given $\varepsilon > 0$ there exists δ such that for all t

$$|f(x) - f(t)| < \varepsilon, \text{ whenever } \forall |t-x| < \delta. \tag{7}$$

Since f is bounded, we get

$$|f(x)| \le M, \ -\infty < x,t < \infty.$$

Hence

$$|f(x) - f(t)| \le 2M, \ -\infty < x,t < \infty. \tag{8}$$

By using (7) and (8), we have

$$|f(x) - f(t)| < \varepsilon + \frac{2M}{\delta^2}(t-x)^2, \ \forall |t-x| < \delta.$$

This implies that

$$-\varepsilon - \frac{2M}{\delta^2}(t-x)^2 < f(x) - f(t) < \varepsilon + \frac{2M}{\delta^2}(t-x)^2.$$

By using the positivity and linearity of $\{L_k\}$, we get

$$L_k(1,x)\left(-\varepsilon - \frac{2M}{\delta^2}(t-x)^2\right) < L_k(1,x)\left(f(x) - f(t)\right) \le L_k(1,x)\left(\varepsilon + \frac{2M}{\delta^2}(t-x)^2\right)$$

where x is fixed and so $f(x)$ is constant number. Therefore,

$$-\varepsilon L_k(1,x) - \frac{2M}{\delta^2}L_k((t-x)^2,x) < L_k(f,x) - f(x)L_k(1,x) < \varepsilon L_k(1,x) + \frac{2M}{\delta^2}L_k((t-x)^2,x). \tag{9}$$

On the other hand

$$\begin{aligned} L_k(f,x) - f(x) &= L_k(f,x) - f(x)L_k(1,x) + f(x)L_k(1,x) - f(x) \\ &= [L_k(f,x) - f(x)L_k(1,x) - f(x)L_k] + f(x)[L_k(1,x) - 1]. \end{aligned} \tag{10}$$

By inequality (9) and (10), we obtain

$$L_k(f,x) - f(x) < \varepsilon L_k(1,x) + \frac{2M}{\delta^2}L_k((t-x)^2,x) + f(x) + f(x)[L_k(1,x) - 1]. \tag{11}$$

Now, we compute second moment

$$\begin{aligned} L_k((t-x)^2,x) &= L_k(x^2-2xt+t^2,x) \\ &= x^2L_k(1,x)-2xL_k(t,x)+L_k(t^2,x) \\ &= [L_k(t^2,x)-x^2]-2x[L_k(t,x)-x]+x^2[L_k(1,x)-1]. \end{aligned}$$

Combing above equality with the relation (11), one can see that

$$\begin{aligned} L_k(f,x)-f(x) &< \varepsilon L_k(1,x)+\frac{2M}{\delta^2}\{[L_k(t^2,x)-x^2]-2x[L_k(t,x)-x]+x^2[L_k(1,x)-1]\}+f(x)(L_k(1,x)-1) \\ &= \varepsilon[L_n(1,x)-1]+\varepsilon+\frac{2M}{\delta^2}\{[L_k(t^2,x)-x^2]-2x[L_k(t,x)-x]+x^2[L_k(1,x)-1]\} \\ &\quad +f(x)(L_k(1,x)-1). \end{aligned}$$

Because of ε is arbitrary, we obtain

$$\begin{aligned} \| L_k(f,x)-f(x)\|_{C[a,b]} &\leq \left(\varepsilon+M+\frac{2Mb^2}{\delta^2}\right)\| L_k(e_0,x)-e_0\|_{C[a,b]}+\frac{4Mb}{\delta^2}\| L_k(e_1,x)-e_1\|_{C[a,b]} \\ &\quad +\frac{2M}{\delta^2}\| L_k(e_2,x)-e_2\|_{C[a,b]} \\ &\leq R\left(\| L_k(e_0,x)-e_0\|_{C[a,b]}+\| L_k(e_1,x)-e_1\|_{C[a,b]}+\| L_k(e_2,x)-e_2\|_{C[a,b]}\right) \end{aligned}$$

where $R=\max\left(\varepsilon+M+\frac{2Mb^2}{\delta^2},\frac{4Mb}{\delta^2}\right)$.
Finally, replacing $L_k(t,x)$ by $T_k(t,x)=\frac{1}{(\beta(k)-\alpha(k)+1)^{\gamma}}\sum_{j\in P_k^{\alpha,\beta}}L_j(t,x)$ and for $\varepsilon^{'}>0$, we can write

$$\begin{aligned} \mathcal{M}: &= \left\{k\in\mathbb{N}:\| T_k(e_0,x)-e_0\|_{C[a,b]}+\| T_k(e_1,x)-e_1\|_{C[a,b]}+\| T_k(e_2,x)-e_2\|_{C[a,b]}\geq\frac{\varepsilon^{'}}{R}\right\}, \\ \mathcal{M}_1: &= \left\{k\in\mathbb{N}:\| T_k(e_0,x)-e_0\|_{C[a,b]}\geq\frac{\varepsilon^{'}}{3R}\right\}, \\ \mathcal{M}_2: &= \left\{k\in\mathbb{N}:\| T_k(e_1,x)-e_1\|_{C[a,b]}\geq\frac{\varepsilon^{'}}{3R}\right\}, \\ \mathcal{M}_3: &= \left\{k\in\mathbb{N}:\| T_k(e_2,x)-e_2\|_{C[a,b]}\geq\frac{\varepsilon^{'}}{3R}\right\}. \end{aligned}$$

Then, $\mathcal{M}\subset\mathcal{M}_1\cup\mathcal{M}_2\cup\mathcal{M}_3$, so we have $\delta(\mathcal{M})\leq\delta(\mathcal{M}_1)+\delta(\mathcal{M}_2)+\delta(\mathcal{M}_3)$. Thus, by conditions (4)-(6), we obtain

$$N^{\gamma}(st)-\lim_{k\to\infty}\| L_k(f,x)-f(x)\|_{C[a,b]}=0.$$

which completes the proof. □

We remark that our Theorem 3.1 is stronger than that of classical Korovkin approximation theorem as well as Theorem of Gadjiev and Orhan [15]. For this purpose, we get the following example:

Example 3.2. *Considering the sequence of Bernstein operators*

$$B_n(f,x)=\sum_{k=0}^{n}f\left(\frac{n}{k}\right)\binom{n}{k}x^k(1-x)^{n-k};\ x\in[0,1].$$

We define the sequence of linear operators as $K_n:C[0,1]\longrightarrow C[0,1]$ *with* $K_n(f,x)=(1+y_n)B_n(f,x)$, *where* $y=(y_n)$ *is defined in (1). Then,* $B_n(1,x)=1$, $B_n(t,x)=x$ *and* $B_n(t^2,x)=x^2+\frac{x-x^2}{n}$ *and sequence* (K_n) *satisfies the conditions (4)-(6). Therefore, we get*

$$N^{\gamma}(st)-\lim_{k\to\infty}\| K_n(f,x)-f(x)\|_{C[a,b]}=0.$$

On the other hand, we have $K_n(f,0) = (1+y_n)f(0)$*, since* $B_n(f,0) = f(0)$*, thus we obtain*

$$\| K_n(f,x) - f(x) \|_{\infty} \geq |K_n(f,0) - f(0)| \geq y_n |f(0)|.$$

One can see that (K_n) *is not satisfy the classical Korovkin theorem as well as Theorem of Gadjiev and Orhan [15], since* y *is statistical* $(N^{\gamma}, \alpha\beta)$*−summable to* 0 *but neither convergent nor statistical convergent.*

4. Rate of Statistical Summability $(N^{\gamma}, \alpha\beta)$

In this section, we estimate rate of statistical summability $(N^{\gamma}, \alpha\beta)$ of a sequence of positive linear operators defined $C[a,b]$ into $C[a,b]$. Now, we give following definition:

Definition 4.1. *Let* (u_n) *be a positive non-increasing sequence. We say that the sequence* $x = (x_k)$ *is a statistical summable* $(N^{\gamma}, \alpha\beta)$ *to* ℓ *with the rate* $o(u_n)$ *if for every,* $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{u_n} \left| \left\{ k \leq n : |z_k^{\gamma} - \ell| \geq \epsilon \right\} \right| = 0.$$

At this point, we can write $x_k - \ell = N^{\gamma}(st) - o(u_n)$*.*

As usual we have the following auxiliary result.

Lemma 4.2. *Let* (a_n) *and* (b_n) *be two positive non-increasing sequences. Let* $x = (x_k)$ *and* $y = (y_k)$ *be two sequences such that* $x_k - L_1 = N^{\gamma}(st) - o(a_n)$ *and* $y_k - L_1 = N^{\gamma}(st) - o(b_n)$*. Then we have*

(i) $\alpha(x_k - L_1) = N^{\gamma}(st) - o(a_n)$ *for any scalar* α*,*

(ii) $(x_k - L_1) \pm (y_k - L_2) = N^{\gamma}(st) - o(c_n)$*,*

(iii) $(x_k - L_1)(y_k - L_2) = N^{\gamma}(st) - o(a_n b_n)$*,*

where $c_n = \max\{a_n, b_n\}$*.*

Before proceeding further, let us give basic definition and notation on the concept of the modulus of continuity. The modulus of continuity of f, $\omega(f,\delta)$ is defined by

$$\omega(f,\delta) = \sup_{\substack{|x-y|\leq\delta \\ x,y\in[a,b]}} |f(x) - f(y)|.$$

It is well-known that for a function $f \in C[a,b]$,

$$\lim_{n\to 0^+} \omega(f,\delta) = 0$$

for any $\delta > 0$

$$|f(x) - f(y)| \leq \omega(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{12}$$

Theorem 4.3. *Let* (L_k) *be sequence of positive linear operator from* $C[a,b]$ *into* $C[a,b]$*. Assume that*

(a) $\| L_k(1,x) - x \|_{C[a,b]} = N^{\gamma}(st) - o(u_n)$,

(b) $\omega(f,\psi_k) = N^{\gamma}(st) - o(v_n)$ *where* $\psi_k = \sqrt{L_k[(t-x)^2, x]}$.

Then for all $f \in C[a,b]$*, we get*

$$\| L_k(f,x) - f(x) \|_{C[a,b]} = N^{\gamma}(st) - o(z_n)$$

where $z_n = \max\{u_n, v_n\}$*.*

Proof. Let $f \in C[a,b]$ and $x \in [a,b]$. From (10) and (12), we can write

$$\begin{aligned}
|L_k(f,x) - f(x)| &\leq L_k(|f(t) - f(x)|; x) + |f(x)||L_k(1,x) - 1| \\
&\leq L_k\left(\frac{|x-y|}{\delta} + 1; x\right)\omega(f,\delta) + |f(x)||L_k(1,x) - 1| \\
&\leq L_k\left(\frac{(t-x)^2}{\delta^2} + 1; x\right)\omega(f,\delta) + |f(x)||L_k(1,x) - 1| \\
&\leq \left(L_k(1,x) + \frac{1}{\delta^2}L_k\left((t-x)^2; x\right)\right)\omega(f,\delta) + |f(x)||L_k(1,x) - 1| \\
&= L_k(1,x)\omega(f,\delta) + \frac{1}{\delta^2}L_k\left((t-x)^2; x\right)\omega(f,\delta) + |f(x)||L_k(1,x) - 1|.
\end{aligned}$$

By choosing $\sqrt{\psi_k} = \delta$, we obtain

$$\begin{aligned}
\| L_k(f,x) - f(x) \|_{C[a,b]} &\leq \| f \|_{C[a,b]} \| L_k(1,x) - x \|_{C[a,b]} + 2\omega(f,\psi_k) + \omega(f,\psi_k) \| L_k(1,x) - x \|_{C[a,b]} \\
&\leq H\{\| L_k(1,x) - x \|_{C[a,b]} + \omega(f,\psi_k) + \omega(f,\psi_k) \| L_k(1,x) - x \|_{C[a,b]}\},
\end{aligned}$$

where $H = \max\{2, \| f \|_{C[a,b]}\}$. From Definition 4.1, conditions (a) and (b), we get the desired the result. □

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