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# Computation of the Greatest Regular Equivalence

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**Abstract.** The notion of social roles is a centerpiece of most sociological theoretical considerations. Regular equivalences were introduced by White and Reitz in [29] as the least restrictive among the most commonly used definitions of equivalence in social network analysis. In this paper we consider a generalization of this notion to a bipartite case. We define a pair of regular equivalences on a two-mode social network and we provide an algorithm for computing the greatest pair of regular equivalences.

## 1. Introduction

One of the main problems of the social network analysis is to find similarities between entities which indicate that they have the same role or position in a network. These similarities were formalized first by Lorrain and White [27], Breiger et al. [8] and Burt [9] by the concept of a structural equivalence. Two entities are considered to be structurally equivalent if they have identical links to the rest of the network. Structural equivalences are extensively studied in [1, 2, 4, 16–19, 21, 22]. In order to generalize the concept of a structural equivalence, White and Reitz [29] introduced the notion of a regular equivalence. Two entities are said to be regularly equivalent if they are equally related to equivalent others [5, 20]. Afterwards, a regular equivalences have been studied in numerous papers (cf. [23, 24]).

The regular equivalence approach is important because it provides a method for identifying "roles" from the patterns of ties present in a network. Rather than relying on attributes of actors to define social roles and to understand how social roles give rise to patterns of interaction, regular equivalence analysis seeks to identify social roles by identifying regularities in the patterns of network ties – whether or not the occupants of the roles have names for their positions. The regular equivalences enable the clustering of the set of actors only with respect to their relationship to each other. The aim of this paper is to introduce the generalization of the notion of regular equivalence which provides the clustering based on the actors relationship to some other group of actors (e.g. the group of students can be clustered by their interest in attending the certain group of exams).

We consider a two-mode network – an ordered triple $(A, B, R)$, where $A$ and $B$ are non-empty sets and $R$ is a relation between $A$ and $B$, and we define the pair of regular equivalences $(E, F)$, as the pair of equivalences $(E, F)$, on $A$ and $B$ respectively, which satisfies $E \circ R = R \circ F$. Similar relational equalities have been extensively studied by Ćirić, Ignjatović et others in [10–15, 24–26], where the greatest solutions to these

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equalities were computed. Based on the general ideas of this study and of the well known Paige-Tarjan partition refinement procedure [28], we provide an efficient algorithm for computing the greatest pair of regular equivalences.

The paper is organized as follows. In Section 2 we recall some basic properties of relations in general, and of equivalence relations. In particular, we define the right and left residuals. In Section 3 we define pairs of regular equivalences on a two-mode network, and we examine their main properties. Section 4 contains our main results on the computation of the greatest pair of regular equivalences on a network. Specifically, we provide an algorithm for computing the greatest pair of regular equivalences on a network and we give an illustrative computational example.

## 2. Preliminaries

Let $A$ be a non-empty set. Any subset $R \subseteq A \times A$ is called a *relation on* $A$, and equality, inclusion, union and intersection of relations on $A$ are defined as for subsets of $A \times A$. The set of all relations on $A$ will be denoted by $\mathcal{R}(A)$. The *inverse* of a relation $R \in \mathcal{R}(A)$ is a relation $R^{-1} \in \mathcal{R}(A)$ defined by $(b, a) \in R^{-1}$ if and only if $(a, b) \in R$, for all $a, b \in A$. For a relation $\varphi \in \mathcal{R}(A)$ we define a subset $\operatorname{Dom}\varphi$ of $A$ and $\operatorname{Im}\varphi$ of $A$ by $\operatorname{Dom}\varphi = \{a \in A \mid (\exists b \in A)\,(a, b) \in \varphi\}$ and $\operatorname{Im}\varphi = \{b \in A \mid (\exists a \in A)\,(a, b) \in \varphi\}$. We call $\operatorname{Dom}\varphi$ the *domain* of $\varphi$ and $\operatorname{Im}\varphi$ the *image* of $\varphi$.

For non-empty set $A$, and relations $R, S \in \mathcal{R}(A)$, the *composition* of $R$ and $S$ is a relation $R \circ S \in \mathcal{R}(A)$ defined by

$$(a, c) \in (R \circ S) \iff (\exists b \in A)\Big((a, b) \in R \wedge (b, c) \in S\Big), \tag{1}$$

for all $a, c \in A$. For non-empty set $A$, a relation $R \in \mathcal{R}(A)$, and subsets $Y, Z \subseteq A$, we define subsets $Y \circ R \subseteq A$ and $R \circ Z \subseteq A$ by

$$b \in Y \circ R \iff (\exists a \in A)\Big(a \in Y \wedge (a, b) \in R\Big), \quad a \in R \circ Z \iff (\exists b \in A)\Big((a, b) \in R \wedge b \in Z\Big), \tag{2}$$

for all $a, b \in A$. To simplify our notation, for a non-empty set $A$ and subsets $Y, Z \subseteq A$ we will write

$$Y \circ Z = \begin{cases} 1 & \text{if } Y \cap Z \neq \emptyset, \\ 0 & \text{if } Y \cap Z = \emptyset, \end{cases} \tag{3}$$

i.e., $Y \circ Z$ is the truth value of the statement "$Y \cap Z \neq \emptyset$".

For non-empty sets $A$, arbitrary relations $R, S, T, S_1, S_2, S_i \in \mathcal{R}(A)$, where $i \in I$, and arbitrary subsets $Y, Z, V \subseteq A$, the following is true:

$$(R \circ S) \circ T = R \circ (S \circ T), \tag{4}$$

$$S_1 \subseteq S_2 \text{ implies } R \circ S_1 \subseteq R \circ S_2 \text{ and } S_1 \circ T \subseteq S_2 \circ T, \tag{5}$$

$$R \circ \Big(\bigcup_{i \in I} S_i\Big) = \bigcup_{i \in I}(R \circ S_i), \qquad \Big(\bigcup_{i \in I} S_i\Big) \circ T = \bigcup_{i \in I}(S_i \circ T) \tag{6}$$

$$(Y \circ R) \circ S = Y \circ (R \circ S), \qquad (Y \circ R) \circ Z = Y \circ (R \circ Z), \qquad (R \circ S) \circ V = R \circ (S \circ V), \tag{7}$$

Let $A$ and $B$ be non empty sets. Any subset of the Cartesian product $A \times B$ is called a *relation between* $A$ and $B$. The set of all relations between $A$ and $B$ will be denoted by $\mathcal{R}(A, B)$. Note that, despite the notation, the inverse relation $R^{-1}$ is not an inverse of the relation $R \in \mathcal{R}(A, B)$ in the sense of composition of relations, i.e., $R \circ R^{-1}$ and $R^{-1} \circ R$ are not the equality relations on $A$ in general. Let us also note that if $A$ is a finite set with $|A| = n$, then $R$ and $S$ can be treated as $n \times n$ Boolean matrices, and $R \circ S$ is their matrix product. Moreover, if we consider $Y$ and $Z$ as $1 \times n$ Boolean matrices, i.e., Boolean vectors of length $n$, then $Y \circ R$ can be treated as the matrix product of $Y$ and $R$, $R \circ Z$ as the matrix product of $R$ and $Z^t$ (the transpose of $Z$), and $Y \circ Z$ as the scalar product of vectors $Y$ and $Z$.

Recall that an *equivalence* on a set $A$ is any reflexive, symmetric and transitive relation on $A$. The set of all equivalence relations on the set $A$ is denoted by $\mathcal{E}(A)$. Let $E$ be an equivalence on a set $A$. By $E^a$ we denote the equivalence class of an element $a \in A$ with respect to $E$, i.e., $E^a = \{b \in A \mid (a,b) \in E\}$. The set of all equivalence classes of $E$ is denoted by $A/E$ and called the *factor set* of $A$ with respect to $E$.

We recall some basic properties of equivalences:

**Lemma 2.1.** *Let $E, F \in \mathcal{E}(A)$ be a equivalences on $A$. Then, relation $E \cap F$ is also a equivalence relation.*

**Lemma 2.2.** *Let $P, Q \in \mathcal{E}(A)$ and $P \subseteq Q$. Then $P \circ Q = Q$.*

For non empty sets $A$ and subsets $Y \subseteq A$ and $Z \subseteq A$ we define the *right residual* of $Z$ by $Y$ as a relation $Y \setminus Z \subseteq A \times A$, given by:

$$(a,b) \in Y \setminus Z \quad \Leftrightarrow \quad (a \in Y \Rightarrow b \in Z)$$

and the *left residual* of $Z$ by $Y$ as a relation $Z \,/\, Y \subseteq A \times A$ given with

$$(a,b) \in Z \,/\, Y \quad \Leftrightarrow \quad (b \in Y \Rightarrow a \in Z).$$

Moreover, we define a relation $Y \,|\, Z$ on $A$ as follows

$$(a,b) \in Y \,|\, Z \quad \Leftrightarrow \quad (a \in Y \Leftrightarrow b \in Z),$$

for every $a, b \in A$. Obviously, $Y \,|\, Z = Y \setminus Z \cap Y \,/\, Z$. It is easy to check that

$$Y \setminus Z = \bigcup \{R \in 2^{A\times A} \mid Y \circ R \subseteq Z\}, \quad Z \,/\, Y = \bigcup \{R \in 2^{A\times A} \mid R \circ Y \subseteq Z\},$$

and hence, the following is true

$$Y \circ R \subseteq Z \quad \Leftrightarrow \quad R \subseteq Y \setminus Z, \quad R \circ Y \subseteq Z \quad \Leftrightarrow \quad R \subseteq Z \,/\, Y.$$

**Lemma 2.3.** *Let $E, F \subseteq A \times A$ be equivalence relations on $A$, such that $E \subseteq F$. Then $E \subseteq F^a \,|\, F^a$, for all $a \in A$.*

*Proof.* Let $(b,c) \in E$. Then $(b,c) \in F$. Thus $b \in F^a$ if and only if $c \in F^a$, or equivalently $(b,c) \in F^a \,|\, F^a$. □

## 3. Regular Equivalences

A *two-mode social network* is an ordered triple $\mathcal{A} = (A, B, R)$, where $A$ and $B$ are non-empty sets and $R \in \mathcal{R}(A, B)$.

A pair of equivalence relations $(E, F)$, where $E \in \mathcal{E}(A)$ and $F \in \mathcal{E}(B)$, on the two-mode network $\mathcal{A}$ is called a *pair of regular equivalences* if and only if:

$$E \circ R = R \circ F. \tag{8}$$

It is not hard to prove the following theorem.

**Theorem 3.1.** *Let $\mathcal{A} = (A, B, R)$ be a two-mode social network, $E \in \mathcal{E}(A)$ and $F \in \mathcal{E}(B)$. Then, $(E, F)$ is a pair of regular equivalences if and only if the following holds:*

$$E \circ R \circ F = E \circ R \cap R \circ F. \tag{9}$$

The following theorem gives the basic characterization of regular equivalences on two-mode networks.

**Theorem 3.2.** *Let $\mathcal{A} = (A, B, R)$ be a two-mode social network, $E \in \mathcal{E}(A)$ and $F \in \mathcal{E}(B)$. Then, $(E, F)$ is a pair of regular equivalences if and only if the following holds:*

$$(E, F) \subseteq ((R \circ F^b)|(R \circ F^b), (E^a \circ R)|(E^a \circ R)), \quad \text{for all } a \in A,\ b \in B. \tag{10}$$

*Proof.* Let $(E, F)$ be a pair of regular equivalences. Then according to the previous theorem we have $E \circ R \circ F \subseteq R \circ F$. This is equivalent to:

$$(a, b) \in E \circ R \circ F \text{ implies } (a, b) \in R \circ F, \quad \text{for all } a \in A,\ b \in B.$$

Thus, whenever there exists $c \in A$ such that $(a, c) \in E$ and $(c, b) \in R \circ F$ then $(a, b) \in R \circ F$. Therefore $(a, c) \in E$ implies $(a, c) \in (R \circ F^b) / (R \circ F^b)$, for every $b \in B$.

According to the fact that $E$ is symmetric, we obtain:

$$E = E^{-1} \subseteq ((R \circ F^b) / (R \circ F^b))^{-1} = (R \circ F^b) \setminus (R \circ F^b), \quad \text{for all } b \in B.$$

So,

$$E \subseteq ((R \circ F^b) / (R \circ F^b)) \bigcap ((R \circ F^b) \setminus (R \circ F^b)) = (R \circ F^b) \mid (R \circ F^b), \quad \text{for all } b \in B.$$

In a similar way we prove that $F \subseteq (E^a \circ R)|(E^a \circ R)$, for all $a \in A$.

On the other hand, let (10) holds. Let $(a, b) \in E \circ R \circ F$, then there exists $c \in A$ such that $(a, c) \in E$ and $(c, b) \in R \circ F$, that is, $(a, c) \in E$ and $c \in R \circ F^b$. According to (10), $(a, c) \in E$ implies $(a, c) \in (R \circ F^b) \mid (R \circ F^b)$, for every $b \in B$. Therefore, $a \in R \circ F^b$, i.e., $(a, b) \in R \circ F$, which means $E \circ R \circ F \subseteq R \circ F$, that is, $E \circ R \circ F = R \circ F$, since $E$ is an equivalence relation.

In an analogue way we show that $E \circ R \circ F \subseteq E \circ R$, i.e., $E \circ R \circ F = E \circ R$. Thus, $(E, F)$ is a pair of regular equivalences. □

**Lemma 3.3.** *Let $\mathcal{A} = (A, B, R)$ be a two-mode social network, $(E, F)$ be a pair of regular equivalences and $(P, Q)$ be a pair of equivalences such that $(E, F) \subseteq (P, Q)$. Then the following holds:*

$$(E, F) \subseteq \Big((R \circ Q^b) \mid (R \circ Q^b)\,,\ (P^a \circ R) \mid (P^a \circ R)\Big), \quad \textit{for all } a \in A, b \in B. \tag{11}$$

*Proof.* According to the Theorem 3.1 the pair of regular equivalences $(E, F)$ satisfies

$$E \circ R \circ F \subseteq E \circ R \quad \text{and } E \circ R \circ F \subseteq R \circ F.$$

Since $E, F, P$ and $Q$ are equivalence relations such that $E \subseteq P$ and $F \subseteq Q$, we conclude $P \circ E = P$ and $F \circ Q = Q$. Therefore we obtain the following:

$$E \circ R \circ F \circ Q \subseteq R \circ F \circ Q, \quad P \circ E \circ R \circ F \subseteq P \circ E \circ R,$$

i.e.,

$$E \circ R \circ Q \subseteq R \circ Q, \quad P \circ R \circ F \subseteq P \circ R. \tag{12}$$

In the similar way as in the proof of the Theorem 3.2, we can prove that the first inequality in (12) implies $E \subseteq (R \circ Q^b) \mid (R \circ Q^b)$, for every $b \in B$, and the second implies $F \subseteq (P^a \circ R) \mid (P^a \circ R)$, for every $a \in A$. Therefore, inequalities in (12) imply $(E, F) \subseteq \Big((R \circ Q^b) \mid (R \circ Q^b), (P^a \circ R) \mid (P^a \circ R)\Big)$, for every $a \in A$ and $b \in B$. □

## 4. Computing the Greatest Pair of Regular Equivalences

The following theorem gives the procedure for computing the greatest pair of regular equivalences on the given network.

**Theorem 4.1.** *Let $\mathcal{A} = (A, B, R)$ be a two-mode network. Further let $E \in \mathcal{E}(A)$ and $F \in \mathcal{E}(B)$ be equivalences on $A$ and $B$ respectively.*

*Define the sequences $\{(E_k, F_k)\}_{k \in N}$ and $\{(X_k, P_k)\}_{k \in N}$ as follows: Initially for $k = 1$,*

$$(X_1, P_1) = (U_A, U_B), \tag{13}$$

$$(E_1, F_1) = (E, F) \cap \Big((R \circ U_B^b) \mid (R \circ U_B^b), (U_A^a \circ R) \mid (U_A^a \circ R)\Big) \tag{14}$$

*where $a \in A$ and $b \in B$ are arbitrary elements.*

*Further, for each $k \in N$ repeat the following step: Find $a \in A$ and $b \in B$ such that $(X_k^a, P_k^b) \neq (E_k^a, F_k^b)$ and set*

$$(X_{k+1}, P_{k+1}) = (X_k, P_k) \cap (E_k^a \,|\, E_k^a, F_k^b \,|\, F_k^b), \tag{15}$$

$$\begin{aligned}(E_{k+1}, F_{k+1}) = (E_k, F_k) &\cap \Big(R \circ F_k^b \,|\, R \circ F_k^b\,,\ E_k^a \circ R \,|\, E_k^a \circ R\Big) \\ &\cap \Big(R \circ (P_k^b - F_k^b) | R \circ (P_k^b - F_k^b)\,,\ (X_k^a - E_k^a) \circ R \,|\, (X_k^a - E_k^a) \circ R\Big),\end{aligned} \tag{16}$$

*until $(X_k, P_k) = (E_k, F_k)$. Then:*

*(a) The sequences $\{(E_k, F_k)\}_{k \in N}$ and $\{(X_k, P_k)\}_{k \in N}$ are descending;*

*(b) For every $k \in \mathbb{N}$, $E_k \subseteq X_k$ and $F_k \subseteq P_k$;*

*(c) For every $k \in \mathbb{N}$, for all $c \in A$ and $d \in B$ the following holds :*

$$E_k \subseteq (R \circ P_k^d) \,|\, (R \circ P_k^d), \quad F_k \subseteq (X_k^c \circ R) \,|\, (X_k^c \circ R); \tag{17}$$

*(d) There exists $n \in \mathbb{N}$ such that $(X_n, P_n) = (E_n, F_n)$ and $(E_n, F_n)$ is the greatest pair of regular equivalences contained in $(E, F)$.*

*Proof.* (a) This follows directly from the definition of the sequences;

We prove the statement (b) by induction on $k \in \mathbb{N}$.

For $k = 1$, it evidently holds.

Suppose $E_m \subseteq X_m$, for $k = m$, and prove $E_{m+1} \subseteq X_{m+1}$. Since $\{E_k\}_{k \in N}$ is descending, we have that $E_{m+1} \subseteq E_m$, and by Lemma 2.3 we have $E_{m+1} \subseteq E_m^a \,|\, E_m^a$. Moreover, according to induction hypothesis we have $E_{m+1} \subseteq E_m \subseteq X_m$, and thus:

$$E_{m+1} \subseteq X_m \cap (E_m^a \,|\, E_m^a) = X_{m+1},$$

which was to be proved. Analogous, we prove $F_k \subseteq P_k$.

We prove the statement (c) also by induction on $k \in \mathbb{N}$.

In the case $k = 1$ directly from the definition of $E_1$ and the fact that $U_B$ has only one equivalence class, we obtain that (22) holds.

For $k = 2$, $X_2 = X_1 \cap (E_1^a \,|\, E_1^a)$ for some $a \in A$. Since $P_1 = U_B$ it has one equivalence class and we will denote it $P_1^b$. According to the definition of $P_2$, we conclude that $P_2$ is also an equivalence relation and it has exactly two equivalence classes $F_1^b$ and $P_1^b - F_1^b$. Now, according to the definition of $E_2$ we have:

$$E_2 = E_1 \cap (R \circ (P_1^a - F_1^b)) \,|\, (R \circ (P_1^b - F_1^b)) \cap (R \circ F_1^b) \,|\, (R \circ F_1^b),$$

which means,

$$E_2 \subseteq (R \circ P_2^d) \,|\, (R \circ P_2^d), \quad \text{for every } d \in B. \tag{18}$$

In a similar way we prove $F_2 \subseteq (X_2^c \circ R) \,|\, (X_2^c \circ R)$ for every $c \in A$. Hence for $k = 2$ inequality (22) holds.

Suppose that for $k = m$ inequality (22) is true. Then for every $d \in B$:

$$E_m \subseteq (R \circ P_m^d) \,|\, (R \circ P_m^d).$$

Consider the equivalence relation $P_{m+1} = P_m \cap (F_m^b \,|\, F_m^b)$, for some $b \in A$ such that $P_m^b \neq F_m^b$ . According to (b), $F_m^b \subset P_m^b$. This means that except of the equivalence class $P_m^b$, all equivalence classes of the equivalence relations $P_m$ and $P_{m+1}$ are the same. Moreover, $P_m$ has the equivalence class $P_m^b$, whereas $P_{m+1}$ instead of

$P^b_m$ has two equivalence classes $F^b_m$ and $P^b_m - F^b_m$. Therefore, for any equivalence class $P^d_{m+1}$, $d \in B$, such that $P^d_{m+1} \neq F^b_m$ and $P^d_{m+1} \neq (P^b_m - F^b_m)$, according to the induction hypothesis we have:

$$E_{m+1} \subseteq E_m \subseteq (R \circ P^d_m) \,|\, (R \circ P^d_m) = (R \circ P^d_{m+1}) \,|\, (R \circ P^d_{m+1}). \tag{19}$$

Therefore, for every $d \in B$, such that $P^d_{m+1} \neq F^b_m$ and $P^d_{m+1} \neq (P^b_m - F^b_m)$, the first inequality in (22) holds. For equivalence classes $F^b_m$ and $P^b_m - F^b_m$, according to the definition of $E_{m+1}$ the following holds:

$$E_{m+1} = E_m \cap ((R \circ F^b_m) \,|\, (R \circ F^b_m)) \cap ((R \circ (P^b_m - F^b_m)) \,|\, (R \circ (P^b_m - F^b_m))).$$

Hence, for $k = m + 1$ the first inequality (22) is satisfied. In a similar way we can show that the second inequality in (22) holds, which was to be proved.

(d) As the set $A$ and $B$ are finite, there is a finite number of relations on $A$ and $B$ so there exists $k \in N$ such that $(X_k, P_k) = (E_k, F_k)$.

Next, for such $k$, according to (c) and by Theorem 3.2, we conclude that $(E_k, F_k)$ is a pair of regular equivalences. In order to prove that $(E_k, F_k)$ is the greatest pair of regular equivalences we will show that for an arbitrary pair of regular equivalences $(P, Q)$ the following holds:

$$(P, Q) \subseteq (E_k, F_k) \qquad k \in \mathbb{N}.$$

This will be proved by induction, too. First we consider the case $k = 1$. Since $(P, Q)$ is a pair of regular equivalences and $U_A, U_B$ are equivalences such that $P \subseteq U_A$ and $Q \subseteq U_B$, the conditions of the Lemma 3.3 are satisfied, and hence,

$$(P, Q) \subseteq \Big((R \circ U^b_B) \,|\, (R \circ U^b_B), (U^a_A \circ R) \,|\, (U^a_A \circ R)\Big).$$

Now, since $(P, Q) \subseteq (E, F)$ also holds, we obtain $(P, Q) \subseteq (E_1, F_1)$.

Next, suppose that $(P, Q) \subseteq (E_m, F_m)$ holds and prove $(P, Q) \subseteq (E_{m+1}, F_{m+1})$.

According to (b), $(E_m, F_m) \subseteq (X_m, P_m)$. Also, according to Lemma 2.3, $E_m \subseteq E^a_m \,|\, E^a_m$ and $F_m \subseteq F^b_m \,|\, F^b_m$, for all $a \in A$ and $b \in B$. Thus $E_m \subseteq X_{m+1}$ and $F_m \subseteq P_{m+1}$.

Therefore, by the induction hypothesis $(P, Q) \subseteq (E_m, F_m) \subseteq (X_{m+1}, P_{m+1})$ the conditions of the Lemma 3.3 are satisfied and we obtain:

$$(P, Q) \subseteq \Big((R \circ P^b_{m+1}) \,|\, (R \circ P^b_{m+1}),\ (X^a_{m+1} \circ R) \,|\, (X^a_{m+1} \circ R)\Big), \quad \text{for all } a \in A, b \in B.$$

This directly implies that $(P, Q) \subseteq (E_{m+1}, F_{m+1})$, which was to be proved. □

In the sequel, according to the previous theorem, for a given two-mode network $\mathcal{A}$, and a pair of equivalences $(E, F)$ we provide an algorithm for computing the greatest pair of regular equivalences contained in $(E, F)$.

**Algorithm 4.2 (***Construction of the greatest pair of regular equivalences***).** The input of this algorithm is a two-mode network $\mathcal{A}$, and equivalences $E \in \mathcal{E}(A)$ and $F \in \mathcal{E}(B)$, and the output is the greatest pair of regular equivalences contained in $(E, F)$.

The procedure builds the sequences of pairs of equivalences $\{(X_k, P_k)\}_{k \in N}$ and $\{(E_k, F_k)\}_{k \in N}$ in the following way:

**(A1)** In the first step we set $(X_1, P_1) = (U_A, U_B)$, where $(U_A, U_B)$ is the pair of universal relations on $A$ and $B$, and we compute $(E_1, F_1)$ using formula (14), where $a \in A$ and $b \in B$ are arbitrary elements;

**(A2)** After the $k$-th step let the pairs of equivalences $(X_k, P_k)$ and $(E_k, F_k)$ be computed;

**(A3)** In the next step we do the following: If there exists a pair $(a, b) \in A \times B$, such that the pair of classes $(X^a_k, P^b_k)$ is not equal to $(E^a_k, F^b_k)$, then using formula (15) we compute the pair $(X_{k+1}, P_{k+1})$ and using formula (16) we compute $(E_{k+1}, F_{k+1})$. Otherwise, if such a pair of elements does not exist, the procedure terminates and the last computed pair $(X_k, P_k)$ is the greatest pair of regular equivalences contained in $(E, F)$.

The initial pair of equivalences $(X_1, P_1)$, is the pair $(U_A, U_B)$, that is, $R_1$, as well as $P_1$ has only one equivalence class, and, in the worst case, the last pair of equivalences $(X_k, P_k)$, for some $k \in N$, is the pair $(Id_A, Id_B)$. Therefore, in the worst case, $X_k$ has $n$ equivalence classes and $P_k$ has $m$ equivalence classes, where $n = |A|$ and $m = |B|$. According to that and the fact that in every step of this algorithm we split at least one equivalence class into two classes, we obtain that this algorithm terminates after at most $m + n - 1$ steps.

If we adopt this procedure to the case one-mode networks we obtain a method for computing the greatest regular equivalence:

**Theorem 4.3.** *Let $(A, R)$ be a network and $U = A \times A$ the universal relation on $A$, and $E \in \mathcal{E}(A)$ an equivalence on $A$.*

*Define sequences $\{E_k\}_{k\in N}$ and $\{R_k\}_{k\in N}$ of equivalences on $A$ as follows: Initially for $k = 1$*

$$R_1 = U, \qquad E_1 = E \cap \Big((R \circ U^a)|(R \circ U^a)\Big),$$

*where $a$ is an arbitrary element of $A$.*

*Further, for each $k \in N$ repeat the following: Find $a \in A$ such that $R_k^a \neq E_k^a$ and set*

$$R_{k+1} = R_k \cap (E_k^a|E_k^a), \tag{20}$$

$$E_{k+1} = E_k \cap \Big((R \circ (R_k^a - E_k^a))|(R \circ (R_k^a - E_k^a))\Big) \cap \Big((R \circ E_k^a)|(R \circ E_k^a)\Big), \tag{21}$$

*until $R_k = E_k$. Then:*

*(a) Sequences $\{E_k\}_{k\in\mathbb{N}}$ and $\{R_k\}_{k\in N}$ are descending;*

*(b) For every $k \in \mathbb{N}$, $E_k \subseteq R_k$;*

*(c) For all $k \in \mathbb{N}$ and $c \in A$ the following holds :*

$$E_k \subseteq (R \circ R_k^c)|(R \circ R_k^c); \tag{22}$$

*(d) For every $k \in \mathbb{N}$, $R_k$ and $E_k$ are equivalence relations;*

*(e) The procedure terminates after at most $|A| - 1$ steps and the last computed equivalence $E_n$ is the greatest regular equivalence contained in $E$.*

According to the previous theorem we obtain an algorithm for computing the greatest regular equivalence.

**Algorithm 4.4.** *The inputs of the algorithm are a network $(A, R)$ and an equivalence $Ein\mathcal{E}(A)$. The algorithm computes the greatest regular equivalence on this network contained in E.*

*The procedure constructs the sequence of equivalences $\{R_k\}_{k\in\mathbb{N}}$ and $\{E_k\}_{k\in\mathbb{N}}$, in the following way:*

*(A1)* *In the first step we set*

$$R_1 = U$$
$$E_1 = E \cap \Big((R \circ U^a)|(R \circ U^a)\Big).$$

*(A2)* *After the kth step let $R_k$ and $E_k$ be equivalences that have been constructed.*

*(A3)* *In the next step we do the following: If there exists $a \in A$ such that $R_k^a \neq E_k^a$ then construct the equivalence $R_{k+1}$ by means of the formula (20) and the equivalence $E_{k+1}$ by means of the formula (21). Otherwise, the procedure of constructing the sequence $\{R_k\}_{k\in\mathbb{N}}$ and $\{E_k\}_{k\in\mathbb{N}}$ terminates and $E_{k+1}$ is the greatest right invariant equivalence on $\mathcal{A}$ contained in E.*

Let us analyze the computational time of this algorithm. Let $n$ denote the number of actors of $A$. In the step (A1) in order to compute $E_1$, we first need to compute the composition of $R$ and $U^a$, and the computational time of this step is $O(n^2)$. Further, we need to compute the relation $(R \circ U^a)|(R \circ U^a)$, which is realized in time $O(n^2)$. Finally, the relation $E_1$ is an intersection of the relation $E$ and the previously mentioned relation, and the computational time of executing the intersection is $O(n^2)$. Hence, the computational time of the step (A1) is $O(n^2)$.

In (A3) we find $a \in A$, such that $R_k^a \neq E_k^a$, if it exists, in time $O(n^2)$.

Computing of $R_{k+1}$ using formula (20) can be done in time $O(n^2)$ and the computing of $E_{k+1}$ using formula (21) can be done in $O(n^2)$.

As it is stated by the Theorem 4.3(e), the number of steps of the previous algorithm is at most $n - 1$.

Summing up, we get that the total computation time for the whole algorithm is $O(n^3)$.

It should be noted that several algorithms for computing the greatest regular equivalence have been previously provided by several authors. The well known REGE algorithm [6, 29–31] is an iterative algorithm, within each iteration a search is implemented to optimize a matching function. The computational time of this algorithm is $O(n^5)$.

Next, a direct approach has been proposed by Batagelj et al. [2, 3, 18]. This method is based on constructing the objective function in the terms of regular equivalence and then using an optimization procedure to minimize objective function, and it performs a good results when dealing with graphs of smaller size.

Another approach has been developed by Boyd and Everett in [7]. Our method for computing the greatest regular equivalence on a one-mode network is similar to this one. However, the algorithm suggested by Boyd and Everett runs in $O(n^4)$, where $n$ is the number of states of $A$. This is due to the fact that in each step of this algorithm the relation $\pi_i$ is computed in the time $O(n^3)$.

The following example illustrates the work of the algorithm.

**Example 4.5.** *Let $\mathcal{A}$ be a two-mode network with $A = \{a_1, a_2, \ldots, a_8\}$, $B = \{b_1, b_2, \ldots, b_{12}\}$, and a relation $R$ given by the following relations:*

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

*Initial relations E and F are given by:*

$$E_{start} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \qquad F_{start} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

*Using procedure from Theorem 4.1 we obtain:*

$(X_1, P_1) = (U_A, U_B)$ *and* $(E_1, F_1) = (E_{start}, F_{start})$

*In the second step we are choosing elements* $a \in A$ *and* $b \in B$ *such that* $(X_1^a, P_1^b) \neq (E_1^a, F_1^b)$. *For* $a_1 \in A$ *we have that* $X_1^a \neq E_1^a$, *and therefore we choose* $a_1 \in A$ *and any element (e.g.* $b_1 \in B$*) from the set B. Hence,*

$$X_2 = X_1 \cap E_1^{a_1} \,|\, E_1^{a_1} = \begin{pmatrix} 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1 \end{pmatrix}$$

$$P_2 = P_1 \cap F_1^{b_1} \,|\, F_1^{b_1} = \begin{pmatrix} 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&1&1&1&1\\ 0&0&0&0&0&0&0&1&1&1&1&1\\ 0&0&0&0&0&0&0&1&1&1&1&1\\ 0&0&0&0&0&0&0&1&1&1&1&1\\ 0&0&0&0&0&0&0&1&1&1&1&1 \end{pmatrix}$$

$$E_2 = E_1 \cap \Big((R \circ (P_1^{b_1} - F_1^{b_1})) \,|\, (R \circ (P_1^{b_1} - F_1^{b_1}))\Big) \cap \Big((R \circ F_1^{b_1}) \,|\, (R \circ F_1^{b_1})\Big) = \begin{pmatrix} 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1 \end{pmatrix}$$

$$F_2 = F_1 \cap \Big(((R_1^{a_1} - E_1^{a_1}) \circ R) \,|\, ((R_1^{a_1} - E_1^{a_1}) \circ R)\Big) \cap \Big((E_1^{a_1} \circ R) \,|\, (E_1^{a_1} \circ R)\Big) = \begin{pmatrix} 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1 \end{pmatrix}$$

*In the sequel, regarding the fact that* $(R_2, P_2) \neq (E_2, F_2)$ *we choose elements* $a_1 \in A$ *and* $b_8 \in B$ *because they satisfy* $(R_2^{a_1}, P_2^{b_8}) \neq (E_2^{a_1}, F_2^{b_8})$.

$$R_3 = R_2 \cap E_2^{a_1} \,|\, E_2^{a_1} = \begin{pmatrix} 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&0&0&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1 \end{pmatrix}$$

$$P_3 = P_2 \cap F_2^{b_8} \,|\, F_2^{b_8} = \begin{pmatrix} 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1 \end{pmatrix}$$

$$E_3 = E_2 \cap \Big((R \circ (P_2^{b_8} - F_2^{b_8}))|(R \circ (P_2^{b_8} - F_2^{b_8}))\Big) \cap \Big((R \circ F_2^{b_8}) \,|\, (R \circ F_2^{b_8})\Big) = \begin{pmatrix} 1&1&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1 \end{pmatrix}$$

$$F_3 = F_2 \cap \Big(((R_2^{a_1} - E_2^{a_1}) \circ R) \,|\, ((R_2^{a_1} - E_2^{a_1}) \circ R)\Big) \cap \Big((E_2^{a_1} \circ R) \,|\, (E_2^{a_1} \circ R)\Big) = \begin{pmatrix} 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1 \end{pmatrix}$$

*Similarly as in the previous step we obtain:*

$$R_4 = R_3 \cap E_3^{a_4} \,|\, E_3^{a_4} = \begin{pmatrix} 1&1&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1 \end{pmatrix}$$

$$P_4 = P_3 \cap F_3^{b_1} \,|\, F_3^{b_1} = \begin{pmatrix} 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1 \end{pmatrix}$$

$$E_4 = E_3 \cap \Big((R \circ (P_3^{b_1} - F_3^{b_1}))|(R \circ (P_3^{b_1} - F_3^{b_1}))\Big) \cap \Big((R \circ F_3^{b_1}) \,|\, (R \circ F_3^{b_1})\Big) = \begin{pmatrix} 1&1&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1 \end{pmatrix}$$

$$F_4 = F_3 \cap \Big(((R_3^{a_4} - E_3^{a_4}) \circ R) \,|\, ((R_3^{a_4} - E_3^{a_4}) \circ R)\Big) \cap \Big((E_3^{a_4} \circ R) \,|\, (E_3^{a_4} \circ R)\Big) = \begin{pmatrix} 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1 \end{pmatrix}$$

*In the last step we have obtained that* $(R_4, P_4) = (E_4, F_4)$ *and thereby the pair of equivalences* $(E_4, F_4)$ *is the greatest regular equivalences contained in* $(E_{start}, F_{start})$.

## 5. Conclusion

In this paper we have developed an efficient algorithm for computing the greatest regular equivalences on one-mode and two-mode networks. This algorithm, in the one-mode case, perform a more efficient computation time than the existing algorithms. The method computes the exact regular equivalence. However when dealing with social networks it is usually better to have a good approximation of the solution than the exact one, which we intend to deal in our further research.

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