A Fixed Point Approach to the Fuzzy Stability of an AQCQ-Functional Equation

Choonkil Park\textsuperscript{a}, Dong Yun Shin\textsuperscript{b}, Reza Saadati\textsuperscript{c}, Jung Rye Lee\textsuperscript{d}

\textsuperscript{a}Research Institute of Natural Sciences, Hanyang University, Seoul 133-791, Republic of Korea
\textsuperscript{b}Department of Mathematics, University of Seoul, Seoul 130-743, Republic of Korea
\textsuperscript{c}Department of Mathematics, Iran University of Science and Technology, Tehran, Iran
\textsuperscript{d}Department of Mathematics, Daejin University, Kyeonggi 487-711, Republic of Korea

Abstract. In [32, 33], the fuzzy stability problems for the Cauchy additive functional equation and the Jensen additive functional equation in fuzzy Banach spaces have been investigated. Using the fixed point method, we prove the Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation

\[ f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y) \]  \hspace{1cm} (1)

in fuzzy Banach spaces.

1. Introduction and Preliminaries

Katsaras [27] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [19, 29, 45]. In particular, Bag and Samanta [3], following Cheng and Mordeson [10], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [28]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 32–34] to investigate a fuzzy version of the generalized Hyers-Ulam stability for the functional equation (0.1) in the fuzzy normed vector space setting.

Definition 1.1. [3, 32–34] Let \( X \) be a real vector space. A function \( N : X \times \mathbb{R} \rightarrow [0, 1] \) is called a fuzzy norm on \( X \) if for all \( x, y \in X \) and all \( s, t \in \mathbb{R} \),

- \( (N_1) \) \( N(x, t) = 0 \) for \( t \leq 0 \);
- \( (N_2) \) \( x = 0 \) if and only if \( N(x, t) = 1 \) for all \( t > 0 \);
Every solution of the quartic functional equation is said to be a functional equation. It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (3), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping. Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

\[
\begin{align*}
(N_1) &\ N(cx, t) = N(x, \frac{t}{c}) \text{ if } c \neq 0; \\
(N_2) &\ N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}; \\
(N_3) &\ N(x, \cdot) \text{ is a non-decreasing function of } \mathbb{R} \text{ and } \lim_{t \to \infty} N(x, t) = 1; \\
(N_4) &\ \text{for } x \neq 0, N(x, \cdot) \text{ is continuous on } \mathbb{R}.
\end{align*}
\]

The pair $(X, N)$ is called a fuzzy normed vector space. The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [32, 35].

**Definition 1.2.** [3, 32–34] Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n \to \infty} x_n = x$.

**Definition 1.3.** [3, 32, 33] Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_n + p - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces $X$ and $Y$ is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to $x_0$ in $X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on $X$ (see [4]).


The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [41] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Cholewa [11] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group. Czerwik [12] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 5, 6, 18, 23, 26, 40, 43, 44]).

In [25], Jun and Kim considered the following cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x).$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (2), which is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic mapping.

In [30], Lee et al. considered the following quartic functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y).$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (3), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping.
Theorem 1.4. [7, 14] Let \((X,d)\) be a complete generalized metric space and let \(J : X \to X\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then for each given element \(x \in X\), either
\[
d(f^n x, f^{n+1} x) = \infty
\]
for all nonnegative integers \(n\) or there exists a positive integer \(n_0\) such that
1. \(d(f^n x, f^{n+1} x) < \infty\), \(\forall n \geq n_0\);
2. the sequence \([f^n x]\) converges to a fixed point \(y'\) of \(J\);
3. \(y'\) is the unique fixed point of \(J\) in the set \(Y = \{ y \in X | d(f^n x, y) < \infty \}\);
4. \(d(y, y') \leq \frac{1}{1-L} d(y, Jy)\) for all \(y \in Y\).

In 1996, G. Isac and Th.M. Rassias [24] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [8, 9, 35–38]).

The theory of probabilistic normed spaces (briefly, PN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The PN-spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics particularly in string theory and \(e^\infty\) where studied by El. Naschie [15]. The Hyers-Ulam stability of different functional equations in random normed spaces, PN-spaces and fuzzy normed spaces has been recently studied in [31]-[34].

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (1) in fuzzy Banach spaces for an odd case. In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (1) in fuzzy Banach spaces for an even case.

Throughout this paper, assume that \(X\) is a vector space and that \((Y, N)\) is a fuzzy Banach space.


One can easily show that an odd mapping \(f : X \to Y\) satisfies (1) if and only if the odd mapping mapping \(f : X \to Y\) is an additive-cubic mapping, i.e.,
\[
f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x).
\]
It was shown in [17, Lemma 2.2] that \(g(x) := f(2x) - 2f(x)\) and \(h(x) := f(2x) - 8f(x)\) are cubic and additive, respectively, and that \(f(x) = \frac{1}{6}g(x) - \frac{1}{3}h(x)\).

One can easily show that an even mapping \(f : X \to Y\) satisfies (1) if and only if the even mapping \(f : X \to Y\) is a quadratic-quartic mapping, i.e.,
\[
f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + 2f(2y) - 8f(y).
\]
It was shown in [16, Lemma 2.1] that \(g(x) := f(2x) - 4f(x)\) and \(h(x) := f(2x) - 16f(x)\) are quartic and quadratic, respectively, and that \(f(x) = \frac{1}{8}g(x) - \frac{1}{2}h(x)\).

For a given mapping \(f : X \to Y\), we define
\[
Df(x, y) := f(x + 2y) + f(x - 2y) - 4f(x + y) - 4f(x - y) + 6f(x)
\]
\[
- f(2y) - f(-2y) + 4f(y) + 4f(-y)
\]
for all \(x, y \in X\).

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equation \(Df(x, y) = 0\) in fuzzy Banach spaces: an odd case.
Theorem 2.1. Let \( \varphi : X^2 \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with
\[
\varphi(x, y) \leq \frac{L}{8} \varphi(2x, 2y)
\]
for all \( x, y \in X \). Let \( f : X \to Y \) be an odd mapping satisfying
\[
N(Df(x, y), t) \geq \frac{t}{t + \varphi(x, y)}
\]
for all \( x, y \in X \) and all \( t > 0 \). Then
\[
C(x) := \lim_{n \to \infty} 8^n \left( f \left( \frac{x}{2^{n-1}} \right) - 2f \left( \frac{x}{2^n} \right) \right)
\]
exists for each \( x \in X \) and defines a cubic mapping \( C : X \to Y \) such that
\[
N(f(2x) - 2f(x) - C(x), t) \geq \frac{(8 - 8L)t}{(8 - 8L)t + 5L(\varphi(x, x) + \varphi(2x, x))}
\]
for all \( x \in X \) and all \( t > 0 \).

Proof. Letting \( x = y \) in (4), we get
\[
N(f(3y) - 4f(2y) + 5f(y), t) \geq \frac{t}{t + \varphi(y, y)}
\]
for all \( y \in X \) and all \( t > 0 \).

Replacing \( x \) by \( 2y \) in (4), we get
\[
N(f(4y) - 4f(3y) + 6f(2y) - 4f(y), t) \geq \frac{t}{t + \varphi(2y, y)}
\]
for all \( y \in X \) and all \( t > 0 \).

By (6) and (7),
\[
N(f(4y) - 10f(2y) + 16f(y), 4t + t) \geq \min \{ N(4f(3y) - 4f(2y) + 5f(y), 4t),
N(f(4y) - 4f(3y) + 6f(2y) - 4f(y), t) \}
\]
\[
\geq \frac{t}{t + \varphi(y, y) + \varphi(2y, y)}
\]
for all \( y \in X \) and all \( t > 0 \). Letting \( y := \frac{x}{2} \) and \( g(x) := f(2x) - 2f(x) \) for all \( x \in X \), we get
\[
N(g(x) - 8g \left( \frac{x}{2} \right), 5t) \geq \frac{t}{t + \varphi \left( \frac{x}{2} \right) + \varphi \left( \frac{2}{2} \right)}
\]
for all \( x \in X \) and all \( t > 0 \).

Consider the set
\[
S := \{ g : X \to Y \}
\]
and introduce the generalized metric on \( S \):
\[
d(g, h) = \inf_{\mu \in \mathbb{R}^*_+} N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}, \quad \forall x \in X, \forall t > 0,
\]
where, as usual, \( \inf \varphi = +\infty \). It is easy to show that \( (S, d) \) is complete. (See the proof of [31, Lemma 2.1].)

Now we consider the linear mapping \( J : S \to S \) such that
\[
Jg(x) := 8g \left( \frac{x}{2} \right)
\]
for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{1 + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$N(Jg(x) - Jh(x), L t) = N\left(8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right), L t\right) \geq \frac{\frac{Lt}{8} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (9) that

$$N\left(g(x) - 8g\left(\frac{x}{2}\right), \frac{5Lt}{8}\right) \geq \frac{t}{1 + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq \frac{5L}{8}$.

By Theorem 1.4, there exists a mapping $C : X \to Y$ satisfying the following:

1. $C$ is a fixed point of $J$, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x)$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping $C$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$ 

This implies that $C$ is a unique mapping satisfying (10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(g(x) - C(x), \mu t) \geq \frac{t}{1 + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

2. $d(f^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$N-\lim_{n \to \infty} 8^n g\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

3. $d(g, C) \leq \frac{1}{1 + L} d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{5L}{8 - 8L}.$$ 

This implies that the inequality (5) holds.
By (4),
\[
N\left(8^n D g\left(\frac{x}{2^n}, \frac{y}{2^n}\right), s^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}
\]
for all \(x, y \in X\), all \(t > 0\) and all \(n \in \mathbb{N}\). So
\[
N\left(8^n D g\left(\frac{x}{2^n}, \frac{y}{2^n}\right), t\right) \geq \frac{t}{t + \varphi(x, y)}
\]
for all \(x, y \in X\), all \(t > 0\) and all \(n \in \mathbb{N}\). Since \(\lim_{n \to \infty} \frac{t}{t + \varphi(x, y)} = 1\) for all \(x, y \in X\) and all \(t > 0\),
\[
N\left(D C(x, y), t\right) = 1
\]
for all \(x, y \in X\) and all \(t > 0\). Thus the mapping \(C : X \to Y\) is cubic, as desired.

**Corollary 2.2.** Let \(\theta \geq 0\) and let \(p\) be a real number with \(p > 3\). Let \(X\) be a normed vector space with norm \(\|\cdot\|\). Let \(f : X \to Y\) be an odd mapping satisfying
\[
N\left(D f(x, y), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\]
for all \(x, y \in X\) and all \(t > 0\). Then
\[
C(x) := N^{-1} \lim_{n \to \infty} 8^n \left(f\left(\frac{x}{2^n-1}\right) - 2f\left(\frac{x}{2^n}\right)\right)
\]
exists for each \(x \in X\) and defines a cubic mapping \(C : X \to Y\) such that
\[
N\left(f(2x) - 2f(x) - C(x), t\right) \geq \frac{(2^p - 8)t}{(2^p - 8)t + 5(3 + 2^p)\theta\|x\|^p}
\]
for all \(x \in X\) and all \(t > 0\).

**Proof.** The proof follows from Theorem 2.1 by taking
\[
\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)
\]
for all \(x, y \in X\). Then we can choose \(L = 2^{3-p}\) and we get the desired result.

**Theorem 2.3.** Let \(\varphi : X^2 \to [0, \infty)\) be a function such that there exists an \(L < 1\) with
\[
\varphi(x, y) \leq 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)
\]
for all \(x, y \in X\). Let \(f : X \to Y\) be an odd mapping satisfying (4). Then
\[
C(x) := N^{-1} \lim_{n \to \infty} \frac{1}{8^n} \left(f\left(2^{n+1}x\right) - 2f(2^n x)\right)
\]
exists for each \(x \in X\) and defines a cubic mapping \(C : X \to Y\) such that
\[
N\left(f(2x) - 2f(x) - C(x), t\right) \geq \frac{(8 - 8L)t}{(8 - 8L)t + 5\varphi(x, x) + 5\varphi(2x, x)}
\]
for all \(x \in X\) and all \(t > 0\).
Proof. Let \((S,d)\) be the generalized metric space defined in the proof of Theorem 2.1. Consider the linear mapping \(J : S \to S\) such that

\[
Jg(x) := \frac{1}{8}g(2x)
\]

for all \(x \in X\).

Let \(g, h \in S\) be given such that \(d(g, h) = \epsilon\). Then

\[
N(g(x) - h(x), \epsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]

for all \(x \in X\) and all \(t > 0\). Hence

\[
N(jg(x) - Jh(x), Lt) = N\left(\frac{1}{8}g(2x) - \frac{1}{8}h(2x), Lt\right)
= N\left(g(2x) - h(2x), 8Lt\right)
\geq \frac{8Lt}{8Lt + \varphi(2x, 2x) + \varphi(4x, 2x)} \geq \frac{8Lt}{8Lt + 8L(\varphi(x, x) + \varphi(2x, x))}
= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]

for all \(x \in X\) and all \(t > 0\). So \(d(g, h) = \epsilon\) implies that \(d(jg, Jh) \leq L\epsilon\). This means that

\[
d(jg, Jh) \leq Ld(g, h)
\]

for all \(g, h \in S\).

It follows from (9) that

\[
N\left(g(x) - \frac{1}{8}g(2x), \frac{5}{8}\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]

for all \(x \in X\) and all \(t > 0\). So \(d(g, jg) \leq \frac{5}{8}\).

By Theorem 1.4, there exists a mapping \(C : X \to Y\) satisfying the following:

(1) \(C\) is a fixed point of \(j\), i.e.,

\[
C(2x) = 8C(x)
\]

for all \(x \in X\). Since \(g : X \to Y\) is odd, \(C : X \to Y\) is an odd mapping. The mapping \(C\) is a unique fixed point of \(j\) in the set

\[
M = \{g \in S : d(f, g) < \infty\}.
\]

This implies that \(C\) is a unique mapping satisfying (13) such that there exists a \(\mu \in (0, \infty)\) satisfying

\[
N(g(x) - C(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]

for all \(x \in X\) and all \(t > 0\);

(2) \(d(f^n g, C) \to 0\) as \(n \to \infty\). This implies the equality

\[
\lim_{n \to \infty} \frac{1}{8^n}g(2^nx) = C(x)
\]

for all \(x \in X\);

(3) \(d(g, C) \leq \frac{1}{1 - \mu}d(g, Jg)\), which implies the inequality

\[
d(g, C) \leq \frac{5}{8 - 8L}.
\]

This implies that the inequality (12) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □
Corollary 2.4. Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 3$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (11). Then
\[
C(x) := N \lim_{x \to 0} \frac{1}{Np} \left( f \left( 2^{n+1}x \right) - 2f(2^nx) \right)
\]
exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that
\[
N \left( f(2x) - 2f(x) - C(x), t \right) \geq \frac{(8 - 2p)t}{(8 - 2p)t + 5(3 + 2p)\theta\|x\|^p}
\]
for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.3 by taking
\[
\phi(x, y) := \theta(\|x\|^p + \|y\|^p)
\]
for all $x, y \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. $\square$

Theorem 2.5. Let $\phi : [0, \infty) \to [0, \infty)$ be a function such that there exists an $L < 1$ with
\[
\phi(x, y) \leq \frac{L}{2} \phi(2x, 2y)
\]
for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (4). Then
\[
A(x) := N \lim_{x \to 0} 2^n \left( f \left( \frac{x}{2^{n+1}} \right) - 2f \left( \frac{x}{2^n} \right) \right)
\]
exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that
\[
N \left( f(2x) - 8f(x) - A(x), t \right) \geq \frac{(2 - 2L)t}{(2 - 2L)t + 5L(\phi(x, x) + \phi(2x, x))}
\]
for all $x \in X$ and all $t > 0$.

Proof. Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1. Letting $y := \frac{x}{2}$ and $h(x) := f(2x) - 8f(x)$ for all $x \in X$ in (8), we get
\[
N \left( h(x) - 2h \left( \frac{x}{2} \right), 5t \right) \geq \frac{t}{t + \phi \left( \frac{x}{2}, \frac{x}{2} \right) + \phi \left( x, \frac{x}{2} \right)}
\]
for all $x \in X$ and all $t > 0$.

Now we consider the linear mapping $J : S \to S$ such that
\[
Jh(x) := 2h \left( \frac{x}{2} \right)
\]
for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then
\[
N(g(x) - h(x), \varepsilon t) \geq \frac{\varepsilon t}{t + \phi(x, x) + \phi(2x, x)}
\]
for all $x \in X$ and all $t > 0$. Hence
\[
N(Jg(x) - Jh(x), Let) = N \left( 2g \left( \frac{x}{2} \right) - 2h \left( \frac{x}{2} \right), Let \right) \geq \frac{\varepsilon t}{t + \phi \left( \frac{x}{2}, \frac{x}{2} \right) + \phi \left( x, \frac{x}{2} \right) + \phi(2x, x)}
\]
\[
= \frac{\varepsilon t}{t + \phi(x, x) + \phi(2x, x)}
\]
for all \( x \in X \) and all \( t > 0 \). So \( d(g, h) = \varepsilon \) implies that \( d(Jg, Jh) \leq L\varepsilon \). This means that
\[
d(Jg, Jh) \leq Ld(g, h)
\]
for all \( g, h \in S \).

It follows from (15) that
\[
N\left(h(x) - 2h(\frac{x}{2}), \frac{5L}{2}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]
for all \( x \in X \) and all \( t > 0 \). So \( d(h, Jh) \leq \frac{5L}{2} \).

By Theorem 1.4, there exists a mapping \( A : X \to Y \) satisfying the following:

(1) \( A \) is a fixed point of \( J \), i.e.,
\[
A\left(\frac{x}{2}\right) = \frac{1}{2}A(x)
\]
for all \( x \in X \). Since \( h : X \to Y \) is odd, \( A : X \to Y \) is an odd mapping. The mapping \( A \) is a unique fixed point of \( J \) in the set
\[
M = \{g \in S : d(f, g) < \infty\}.
\]
This implies that \( A \) is a unique mapping satisfying (16) such that there exists a \( \mu \in (0, \infty) \) satisfying
\[
N(h(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]
for all \( x \in X \) and all \( t > 0 \);
(2) \( d(J^n h, A) \to 0 \) as \( n \to \infty \). This implies the equality
\[
N- \lim_{n \to \infty} 2^n h\left(\frac{x}{2^n}\right) = A(x)
\]
for all \( x \in X \);
(3) \( d(h, A) \leq \frac{1}{1 - L}d(h, Jh) \), which implies the inequality
\[
d(h, A) \leq \frac{5L}{2 - 2L}.
\]
This implies that the inequality (14) holds.

The rest of the proof is similar to the proof of Theorem 2.1. \( \square \)

**Corollary 2.6.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( p > 1 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \to Y \) be an odd mapping satisfying (11). Then
\[
A(x) := N- \lim_{n \to \infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right)
\]
exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that
\[
N(f(2x) - 8f(x) - A(x), t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 5(3 + 2^p)\theta\|x\|^p}
\]
for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 2.5 by taking
\[
\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)
\]
for all \( x, y \in X \). Then we can choose \( L = 2^{1-p} \) and we get the desired result. \( \square \)
Theorem 2.7. Let \( \varphi : X^2 \rightarrow [0, \infty) \) be a function such that there exists an \( L < 1 \) with
\[
\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)
\]
for all \( x, y \in X \). Let \( f : X \rightarrow Y \) be an odd mapping satisfying (4). Then
\[
A(x) := \lim_{n \to \infty} \frac{1}{2^n} \left(f\left(2^{n+1}x\right) - 8f(2^n x)\right)
\]
exists for each \( x \in X \) and defines an additive mapping \( A : X \rightarrow Y \) such that
\[
N \left(f(2x) - 8f(x) - A(x), t\right) \geq \frac{(2 - 2L)t}{(2 - 2L)t + 5\varphi(x, x) + 5\varphi(2x, x)}
\]
for all \( x \in X \) and all \( t > 0 \).

Proof. Let \( (S, d) \) be the generalized metric space defined in the proof of Theorem 2.1. Consider the linear mapping \( J : S \rightarrow S \) such that
\[
Jh(x) := \frac{1}{2}h(2x)
\]
for all \( x \in X \).
Let \( g, h \in S \) be given such that \( d(g, h) = \varepsilon \). Then
\[
N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]
for all \( x \in X \) and all \( t > 0 \). Hence
\[
N(Jg(x) - Jh(x), \varepsilon t) = N\left(\frac{1}{2}g(2x) - \frac{1}{2}h(2x), \varepsilon t\right) = N\left(g(2x) - h(2x), 2\varepsilon t\right) \geq \frac{2Lt}{2Lt + \varphi(2x, 2x) + \varphi(4x, 2x)} = \frac{2Lt}{t + \varphi(x, x) + \varphi(2x, x)}
\]
for all \( x \in X \) and all \( t > 0 \). So \( d(g, h) = \varepsilon \) implies that \( d(Jg, Jh) \leq L\varepsilon \). This means that
\[
d(Jg, Jh) \leq Ld(g, h)
\]
for all \( g, h \in S \).
It follows from (15) that
\[
N\left(h(x) - \frac{1}{2}h(2x), \frac{5}{2}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]
for all \( x \in X \) and all \( t > 0 \). So \( d(h, Jh) \leq \frac{5}{2} \).
By Theorem 1.4, there exists a mapping \( A : X \rightarrow Y \) satisfying the following:
1. \( A \) is a fixed point of \( J \), i.e.,
\[
A(2x) = 2A(x)
\]
for all \( x \in X \). Since \( h : X \rightarrow Y \) is odd, \( A : X \rightarrow Y \) is an odd mapping. The mapping \( A \) is a unique fixed point of \( J \) in the set
\[
M = \{g \in S : d(f, g) < \infty\}.
\]
This implies that $A$ is a unique mapping satisfying (18) such that there exists a $\mu \in (0, \infty)$ satisfying
\[
N(h(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]
for all $x \in X$ and all $t > 0$;

(2) $d(f^n h, A) \to 0$ as $n \to \infty$. This implies the equality
\[
N^- \lim_{n \to \infty} \frac{1}{2^n} h(2^n x) = A(x)
\]
for all $x \in X$;

(3) $d(h, A) \leq \frac{1}{1 - L} d(h, Jh)$, which implies the inequality
\[
d(h, A) \leq \frac{5}{2 - 2L}.
\]
This implies that the inequality (17) holds.

The rest of the proof is similar to the proof of Theorem 2.1.

**Corollary 2.8.** Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $X$ be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an odd mapping satisfying (11). Then

\[
A(x) := N^- \lim_{n \to \infty} \frac{1}{2^n} \left( f \left( 2^n x \right) - 8 f(2^n x) \right)
\]
exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that
\[
N (f(2x) - 8 f(x) - A(x), t) \geq \frac{(2 - 2^p) t}{(2 - 2^p) t + 5(3 + 2^p) \theta \| x \| ^p}
\]
for all $x \in X$ and all $t > 0$.

**Proof.** The proof follows from Theorem 2.7 by taking
\[
\varphi(x, y) := \theta(\| x \| ^p + \| y \| ^p)
\]
for all $x, y \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result.

### 3. Hyers-Ulam Stability of the Functional Equation (1): an Even Case

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equation $D f(x, y) = 0$ in fuzzy Banach spaces: an even case.

**Theorem 3.1.** Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with
\[
\varphi(x, y) \leq \frac{L}{16} \varphi(2x, 2y)
\]
for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (4). Then
\[
Q(x) := N^- \lim_{n \to \infty} 16^n \left( f \left( \frac{x}{2^{n-1}} \right) - 4f \left( \frac{x}{2^n} \right) \right)
\]
exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that
\[
N (f(2x) - 4 f(x) - Q(x), t) \geq \frac{(16 - 16L) t}{(16 - 16L) t + 5L (\varphi(x, x) + \varphi(2x, x))}
\]
for all $x \in X$ and all $t > 0$. 
Proof. Letting $x = y$ in (4), we get

$$N\left( f(3y) - 6f(2y) + 15f(y), t \right) \geq \frac{t}{t + \varphi(y, y)} \tag{20}$$

for all $y \in X$ and all $t > 0$.

Replacing $x$ by $2y$ in (4), we get

$$N\left( f(4y) - 4f(3y) + 4f(2y) + 4f(y), t \right) \geq \frac{t}{t + \varphi(2y, y)} \tag{21}$$

for all $y \in X$ and all $t > 0$.

By (20) and (21),

$$N\left( f(4x) - 20f(2x) + 64f(x), 4t + t \right) \geq \min \left\{ N\left( f(4f(3x) - 6f(2x) + 15f(x)), 4t \right), N\left( f(4x) - 4f(3x) + 4f(2x) + 4f(x), t \right) \right\} \tag{22}$$

for all $x \in X$ and all $t > 0$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$N\left( g(x) - 16g\left( \frac{x}{2} \right), 5t \right) \geq \frac{t}{t + \varphi\left( \frac{x}{2}, \frac{x}{2} \right) + \varphi\left( x, \frac{x}{2} \right)} \tag{23}$$

for all $x \in X$ and all $t > 0$.

Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left( \frac{x}{2} \right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$N(Jg(x) - Jh(x), \varepsilon t) = N\left( 16g\left( \frac{x}{2} \right) - 16h\left( \frac{x}{2} \right), \varepsilon t \right) \geq \frac{\varepsilon t}{\frac{t}{16} + \varphi\left( \frac{x}{2}, \frac{x}{2} \right) + \varphi\left( x, \frac{x}{2} \right) + \frac{t}{16} (\varphi(x, x) + \varphi(2x, x))} = \frac{\frac{\varepsilon t}{16}}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (23) that

$$N\left( g(x) - 16g\left( \frac{x}{2} \right), \frac{5L}{16} \varepsilon t \right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$
for all \( x \in X \) and all \( t > 0 \). So \( d(g, Jg) \leq \frac{5L}{16} \).

By Theorem 1.4, there exists a mapping \( Q : X \to Y \) satisfying the following:

1. \( Q \) is a fixed point of \( J \), i.e.,
   \[
   Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x)
   \]  

   for all \( x \in X \). Since \( g : X \to Y \) is even, \( Q : X \to Y \) is an even mapping. The mapping \( Q \) is a unique fixed point of \( J \) in the set
   \[
   M = \{ g \in S : d(f, g) < \infty \}.
   \]

   This implies that \( Q \) is a unique mapping satisfying (24) such that there exists a \( \mu \in (0, \infty) \) satisfying
   \[
   N(g(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
   \]

   for all \( x \in X \) and all \( t > 0 \);
   
   2. \( d(f^n g, Q) \to 0 \) as \( n \to \infty \). This implies the equality
   \[
   \lim_{n \to \infty} 16^n g\left(\frac{x}{2^n}\right) = Q(x)
   \]

   for all \( x \in X \);
   
   3. \( d(g, Q) \leq \frac{1}{1 - L}d(g, Jg) \), which implies the inequality
   \[
   d(g, Q) \leq \frac{5L}{16 - 16L}.
   \]

   This implies that the inequality (19) holds.

   The rest of the proof is similar to the proof of Theorem 2.1. \( \square \)

**Corollary 3.2.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( p > 4 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (11). Then

\[
Q(x) := \lim_{n \to \infty} 16^n \left( f\left(\frac{x}{2^n}\right) - 4f\left(\frac{x}{2^n}\right) - 4f\left(\frac{x}{2^{n+1}}\right)\right)
\]

exists for each \( x \in X \) and defines a quartic mapping \( Q : X \to Y \) such that

\[
N(f(2x) - 4f(x) - Q(x), t) \geq \frac{(2^p - 16)t}{(2^p - 16)t + 5(3 + 2^p)\theta \|x\|^p}
\]

for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 3.1 by taking

\[
\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)
\]

for all \( x, y \in X \). Then we can choose \( L = 2^{4-p} \) and we get the desired result. \( \square \)

**Theorem 3.3.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with

\[
\varphi(x, y) \leq 16L \varphi\left(\frac{x}{2}, \frac{y}{2}\right)
\]

for all \( x, y \in X \). Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (4). Then

\[
Q(x) := \lim_{n \to \infty} \frac{1}{16^n} \left( f\left(2^{n+1}x\right) - 4f(2^n x)\right)
\]
exists for each \( x \in X \) and defines a quartic mapping \( Q : X \to Y \) such that

\[
N\left( f(2x) - 4f(x) - Q(x), t \right) \geq \frac{(16 - 16L)t}{(16 - 16L)t + 5\varphi(x, x) + 5\varphi(2x, x)}
\]

for all \( x \in X \) and all \( t > 0 \).

**Proof.** Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping \( J : S \to S \) such that

\[
jg(x) := \frac{1}{16} g(2x)
\]

for all \( x \in X \).

Let \( g, h \in S \) be given such that \( d(g, h) = \varepsilon \). Then

\[
N(g(x) - h(x), t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]

for all \( x \in X \) and all \( t > 0 \). Hence

\[
N(\{g(x) - h(x), t \mid t \geq 16L \}) = N\left( \frac{1}{16} g(2x) - \frac{1}{16} h(2x), t \right)
\]

\[
= N\left( g(2x) - h(2x), 16Lt \right)
\]

\[
\geq \frac{16Lt}{16Lt + \varphi(2x, 2x) + \varphi(4x, 2x)} \geq \frac{16Lt}{16Lt + 16L(\varphi(x, x) + \varphi(2x, x))}
\]

\[
= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]

for all \( x \in X \) and all \( t > 0 \). So \( d(g, h) = \varepsilon \) implies that \( d(Jg, Jh) \leq L\varepsilon \). This means that

\[
d(Jg, Jh) \leq Ld(g, h)
\]

for all \( g, h \in S \).

It follows from (23) that

\[
N\left( g(x) - \frac{1}{16} g(2x), \frac{5}{16} t \right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]

for all \( x \in X \) and all \( t > 0 \). So \( d(g, Jg) \leq \frac{L}{16} \).

By Theorem 1.4, there exists a mapping \( Q : X \to Y \) satisfying the following:

(1) \( Q \) is a fixed point of \( J \), i.e.,

\[
Q(2x) = 16Q(x)
\]

for all \( x \in X \). Since \( g : X \to Y \) is even, \( Q : X \to Y \) is an even mapping. The mapping \( Q \) is a unique fixed point of \( J \) in the set

\[
M = \{ g \in S : d(f, g) < \infty \}.
\]

This implies that \( Q \) is a unique mapping satisfying (26) such that there exists a \( \mu \in (0, \infty) \) satisfying

\[
N(\varphi(x, x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]

for all \( x \in X \) and all \( t > 0 \);

(2) \( d(f^n g, Q) \to 0 \) as \( n \to \infty \). This implies the equality...
\[
N\lim_{n \to \infty} \frac{1}{16^n} g(2^n x) = Q(x)
\]
for all \( x \in X \);

(3) \( d(g, Q) \leq \frac{1}{16} d(g, J g) \), which implies the inequality \( d(g, Q) \leq \frac{5}{16 - 16L} \).

This implies that the inequality (25) holds.

The rest of the proof is similar to the proof of Theorem 2.1. \( \square \)

**Corollary 3.4.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( 0 < p < 4 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \).

Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (11). Then

\[
Q(x) := N \lim_{n \to \infty} \frac{1}{16^n} \left( f \left( 2^{n+1} x \right) - 4 f(2^n x) \right)
\]
exists for each \( x \in X \) and defines a quartic mapping \( Q : X \to Y \) such that

\[
N \left( f(2x) - 4 f(x) - Q(x), t \right) \geq \frac{(16 - 2^p)t}{(16 - 2^p) t + 5(3 + 2^p)\theta \|x\|^p}
\]
for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 3.3 by taking

\[
\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)
\]
for all \( x, y \in X \). Then we can choose \( L = 2^{p-4} \) and we get the desired result. \( \square \)

**Theorem 3.5.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with

\[
\varphi(x, y) \leq \frac{L}{4} \varphi(2x, 2y)
\]
for all \( x, y \in X \). Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (4). Then

\[
T(x) := N \lim_{n \to \infty} 4^n \left( f \left( \frac{x}{2^{n-1}} \right) - 16 f \left( \frac{x}{2^{n}} \right) \right)
\]
exists for each \( x \in X \) and defines a quadratic mapping \( T : X \to Y \) such that

\[
N \left( f(2x) - 16 f(x) - T(x), t \right) \geq \frac{(4 - 4L)t}{(4 - 4L) t + 5L(\varphi(x, x) + \varphi(2x, x))}
\]
for all \( x \in X \) and all \( t > 0 \).

**Proof.** Let \( (S, d) \) be the generalized metric space defined in the proof of Theorem 2.1.

Letting \( h(x) := f(2x) - 16 f(x) \) for all \( x \in X \) in (22), we get

\[
N \left( h(x) - 4h \left( \frac{x}{2} \right), 5t \right) \geq \frac{t}{t + \varphi \left( \frac{x}{2}, \frac{x}{2} \right) + \varphi \left( x, \frac{x}{2} \right)}
\]
for all \( x \in X \) and all \( t > 0 \).

Now we consider the linear mapping \( J : S \to S \) such that

\[
Jh(x) := 4h \left( \frac{x}{2} \right)
\]
for all $x \in X$.
Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{l + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$N(fg(x) - fh(x), L\varepsilon t) = N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t \right) \geq \frac{4L}{4 + \varphi\left(\frac{x}{2}, x\right) + \varphi\left(\frac{x}{2}, x\right)} \geq \frac{4L}{4 + \frac{4}{2}(\varphi(x, x) + \varphi(2x, x))}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(fg, fh) \leq L\varepsilon$. This means that

$$d(fg, fh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (28) that

$$N\left(h(x) - 4h\left(\frac{x}{2}\right), \frac{5L}{4}t\right) \geq \frac{t}{l + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(h, Jh) \leq \frac{5L}{4}$.

By Theorem 1.4, there exists a mapping $T : X \to Y$ satisfying the following:

1. $T$ is a fixed point of $f$, i.e.,

$$T\left(\frac{x}{2}\right) = \frac{1}{4}T(x)$$

for all $x \in X$. Since $h : X \to Y$ is even, $T : X \to Y$ is an even mapping. The mapping $T$ is a unique fixed point of $f$ in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$ 

This implies that $T$ is a unique mapping satisfying (29) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(h(x) - T(x), \mu t) \geq \frac{t}{l + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

2. $d(f^n h, T) \to 0$ as $n \to \infty$. This implies the equality

$$N- \lim_{n \to \infty} 4^n h\left(\frac{x}{2^n}\right) = T(x)$$

for all $x \in X$;

3. $d(h, T) \leq \frac{1}{4L}d(h, Jh)$, which implies the inequality

$$d(h, T) \leq \frac{5L}{4 - 4L}.$$ 

This implies that the inequality (27) holds.

The rest of the proof is similar to the proof of Theorem 2.1. \qed
Corollary 3.6. Let \( \theta \geq 0 \) and let \( p \) be a real number with \( p > 2 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (11). Then

\[
T(x) := N \lim_{n \to \infty} 4^n \left( f \left( \frac{x}{2^{n-1}} \right) - 16 f \left( \frac{x}{2^n} \right) \right)
\]

exists for each \( x \in X \) and defines a quadratic mapping \( T : X \to Y \) such that

\[
N (f(2x) - 16f(x) - T(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 5(3 + 2^p)\theta \|x\|^p}
\]

for all \( x \in X \) and all \( t > 0 \).

Proof. The proof follows from Theorem 3.5 by taking

\[
\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)
\]

for all \( x, y \in X \). Then we can choose \( L = 2^{2-p} \) and we get the desired result. \( \square \)

Theorem 3.7. Let \( \varphi : X^2 \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with

\[
\varphi(x, y) \leq 4L \varphi \left( \frac{x}{2}, \frac{y}{2} \right)
\]

for all \( x, y \in X \). Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (4). Then

\[
T(x) := N \lim_{n \to \infty} \frac{1}{4^n} \left( f \left( \frac{2^n+1}{2} x \right) - 16 f \left( \frac{2^n x}{2} \right) \right)
\]

exists for each \( x \in X \) and defines a quadratic mapping \( T : X \to Y \) such that

\[
N (f(2x) - 16f(x) - T(x), t) \geq \frac{(4 - 4L)t}{(4 - 4L)t + 5\varphi(x, x) + 5\varphi(2x, x)}
\]

(30)

for all \( x \in X \) and all \( t > 0 \).

Proof. Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping \( J : S \to S \) such that

\[
Jh(x) := \frac{1}{4} h (2x)
\]

for all \( x \in X \).

Let \( g, h \in S \) be given such that \( d(g, h) = \varepsilon \). Then

\[
N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]

for all \( x \in X \) and all \( t > 0 \). Hence

\[
N(f(g(x) - Jh(x),Lt)) = N \left( \frac{1}{4} g (2x) - \frac{1}{4} h (2x), 4Lt \right)
\]

\[
= N \left( g (2x) - h (2x), 4Lt \right)
\]

\[
\geq \frac{4Lt}{4Lt + \varphi (2x, 2x) + \varphi(4x, 2x)} \geq \frac{4Lt}{4Lt + 4L(\varphi(x, x) + \varphi(2x, x))}
\]

\[
= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]
for all \( x \in X \) and all \( t > 0 \). So \( d(g, h) = \varepsilon \) implies that \( d(Jg, Jh) \leq L \varepsilon \). This means that
\[
d(Jg, Jh) \leq Ld(g, h)
\]
for all \( g, h \in S \).

It follows from (28) that
\[
N\left( h(x) - \frac{1}{4} h(2x), \frac{5}{4} t \right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]
for all \( x \in X \) and all \( t > 0 \). So \( d(h, Jh) \leq \frac{5}{4} \).

By Theorem 1.4, there exists a mapping \( T : X \to Y \) satisfying the following:

1. \( T \) is a fixed point of \( J \), i.e.,
   \[
   T(2x) = 4T(x)
   \]  (31)

for all \( x \in X \). Since \( h : X \to Y \) is even, \( T : X \to Y \) is an even mapping. The mapping \( T \) is a unique fixed point of \( J \) in the set
\[
M = \{ g \in S : d(f, g) < \infty \}.
\]

This implies that \( T \) is a unique mapping satisfying (31) such that there exists a \( \mu \in (0, \infty) \) satisfying
\[
N(h(x) - T(x), \mu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\]
for all \( x \in X \) and all \( t > 0 \);

2. \( d(J^n h, T) \to 0 \) as \( n \to \infty \). This implies the equality
\[
N\lim_{n \to \infty} \frac{1}{4^n} h(2^n x) = T(x)
\]
for all \( x \in X \);

3. \( d(h, T) \leq \frac{1}{1 - T} d(h, Jh) \), which implies the inequality
\[
d(h, T) \leq \frac{5}{4 - 4L}.
\]

This implies that the inequality (30) holds.

The rest of the proof is similar to the proof of Theorem 2.1. \( \square \)

**Corollary 3.8.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( 0 < p < 2 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \).

Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (11). Then \( T(x) := N\lim_{n \to \infty} \frac{1}{2^n} \left( f\left( 2^{n+1} x \right) - 16 f(2^n x) \right) \) exists for each \( x \in X \) and defines a quadratic mapping \( T : X \to Y \) such that
\[
N \left( f(2x) - 16 f(x) - T(x), t \right) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 5(3 + 2^p)\theta\|x\|^p}
\]
for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 3.7 by taking
\[
\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)
\]
for all \( x, y \in X \). Then we can choose \( L = 2^{p-2} \) and we get the desired result. \( \square \)
References