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New Type of Generalized Difference Sequence Space of Non-Absolute Type and Some Matrix Transformations

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Abstract. In the present paper, we introduce a new difference sequence space $r^q_B(u,p)$ by using the Riesz mean and the B- difference matrix. We show $r^q_B(u,p)$ is a complete linear metric space and is linearly isomorphic to the space $l(p)$. We have also computed its α-, β- and γ-duals. Furthermore, we have constructed the basis of $r^q_B(u,p)$ and characterize a matrix class $(r^q_B(u,p), l_\infty)$.

1. Introduction, Background and Notation

We denote the set of all sequences (real or complex) by ω. Any subspace of ω is called the sequence space. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, of real numbers and of complex numbers, respectively. Let l_∞, c and c_0 denote the space of all bounded, convergent and null sequences, respectively. Also, by cs, l_1 and $l(p)$ we denote the spaces of all convergent, absolutely and p-absolutely convergent series, respectively.

Let X be a real or complex linear space, h be a function from X to the set $\mathbb{R}$ of real numbers. Then, the pair (X,h) is called a paranormed space and h is a paranorm for X, if the following axioms are satisfied :
(pn.1) $h(\theta) = 0$,
(pn.2) $h(-x) = h(x)$,
(pn.3) $h(x + y) \leq h(x) + h(y)$, and
(pn.4) scalar multiplication is continuous, that is, $|\alpha_n - \alpha| \to 0$ and $h(x_n - x) \to 0$ imply $h(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and x's in X, where θ is a zero vector in the linear space X. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$ and $M = max\{1, H\}$. Then, the linear space $l(p)$ was defined by Maddox [10] as follows

$$l(p) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}$$

which is complete space paranormed by

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$$h_1(x) = \left[\sum_k |x_k|^{p_k}\right]^{1/M}.$$

We shall assume throughout the text that $p_k^{-1} + p_k^{'-1} = 1$ provided $1 < inf p_k \leq H < \infty$.

Let X, Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, the matrix A defines the A-transformation from X *into* Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $A \in (X : Y)$, we mean the characterizations of matrices from X to Y *i.e.*, $A : X \to Y$. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x. For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}. \tag{1}$$

Let (q_k) be a sequence of positive numbers and let us write, $Q_n = \sum_{k=0}^{n} q_k$ for $n \in \mathbb{N}$. Then the matrix $R^q = (r^q_{nk})$ of the Riesz mean (R, q_n) is given by

$$r^q_{nk} = \begin{cases} \frac{q_k}{Q_n}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

The Riesz mean (R, q_n) is regular if and only if $Q_n \to \infty$ as $n \to \infty$ [17].

Recently, Neyaz and Hamid [18] introduced the sequence space $r^q(u, p)$ as

$$r^q(u,p) = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_k q_j x_j \right|^{p_k} < \infty \right\}, \ (0 < p_k \leq H < \infty).$$

Kizmaz [8] defined the difference sequence spaces $Z(\triangle)$ as follows

$$Z(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in Z\},$$

where $Z \in \{l_\infty, c, c_0\}$ and $\triangle x_k = x_k - x_{k-1}$.

Altay and Başar [2] defined the sequence space of p-bounded variation bv_p which is defined as

$$bv_p = \left\{ x = (x_k) \in \omega : \sum_k |x_k - x_{k-1}|^p < \infty \right\}, \ 1 \leq p < \infty.$$

With the notation of (1), the space bv_p can be re-defined as

$$bv_p = (l_p)_\triangle, \ 1 \leq p < \infty$$

where, $\triangle$ denotes the matrix $\triangle = (\triangle_{nk})$ and is defined as

$$\triangle_{nk} = \begin{cases} (-1)^{n-k}, & \text{if } n-1 \leq k \leq n, \\ 0, & \text{if } k < n-1 \text{ or } k > n. \end{cases}$$

Neyaz and Hamid [19] introduced the space $r^q(\triangle_u^p)$ as :

$$r^q(\triangle_u^p) = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_k q_j \triangle x_j \right|^{p_k} < \infty \right\},$$

where $(0 < p_k \leq H < \infty)$.

In [3] the generalized difference matrix $B = (b_{nk})$ is defined as :

$$b_{nk} = \begin{cases} r, & \text{if } k = n, \\ s, & \text{if } k = n-1 \\ 0, & \text{if } 0 \leq k < n-1 \ \ or \ \ k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$, $r, s \in \mathbf{R} - \{0\}$. The matrix B can be reduced to difference matrix $\triangle$ incase $r = 1$, $s = -1$.

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors viz., [1, 4–7, 13, 16, 18, 19].

2. The Riesz Sequence Space $r_B^q(u, p)$ of Non-Absolute Type

In this section, we define the Riesz sequence space $r_B^q(u, p)$, and prove that the space $r_B^q(u, p)$ is a complete paranormed linear space and show it is linearly isomorphic to the space $l(p)$.

Define the sequence $y = (y_k)$, which will be frequently used, by the $R_u^q B$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k(q) = \frac{1}{Q_k} \left\{ \sum_{j=0}^{k-1} u_k (q_j.r + q_{j+1}.s) x_j + u_k q_k.r.x_k \right\}, \quad (k \in \mathbf{N}). \tag{2}$$

Following Başar and Altay [1], Mursaleen et al [14, 15], Neyaz and Hamid [18, 19], Başarir and Öztürk [4], we define the sequence space $r_B^q(u, p)$ as the set of all sequences such that $R_u^q B$ transform of it is in the space $l(p)$, that is,

$$r_B^q(u, p) = \{ x = (x_k) \in \omega : y_k(q) \in l(p) \}.$$

Note that if we take $r = 1$ and $s = -1$, the sequence spaces $r_B^q(u, p)$ reduces to $r^q(\Delta_u^p)$, introduced by Neyaz and Hamid [19]. Also, if $(u_k) = e = (1, 1, ...)$, the sequence spaces $r_B^q(u, p)$ reduces to $r_B^q(p)$ Başarir [3].

With the notation of (1) that

$$r_B^q(u, p) = \{l(p)\}_{R_u^q}.$$

Now, we prove the following theorem which is essential in the text.

Theorem 2.1. $r_B^q(u,p)$ is a complete linear metric space paranormed by h_B, defined as

$$h_B(x) = \left[\sum_k \left|\frac{1}{Q_k}\left(\sum_{j=0}^{k-1} u_k(q_j.r + q_{j+1}.s)x_j + q_k u_k.rx_k\right)\right|^{p_k}\right]^{\frac{1}{M}},$$

where $\sup_k p_k = H$ and $M = max\{1, H\}$.

Proof. The linearity of $r_B^q(u,p)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the inequalities which are satisfied for $z, x \in r_B^q(u,p)$ [11]

$$\left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1} u_k(q_j.r + q_{j+1}.s)(x_j + z_j) + q_k.r.u_k(x_k + z_k)\right|^{p_k}\right]^{\frac{1}{M}}$$

$$\leq \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1} u_k(q_j.r + q_{j+1}.s)x_j + q_k.r.u_k x_k\right|^{p_k}\right]^{\frac{1}{M}} + \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1} u_k(q_j.r + q_{j+1}.s)z_j + q_k.r.u_k z_k\right|^{p_k}\right]^{\frac{1}{M}} \quad (3)$$

and for any $\alpha \in \mathbf{R}$ [12]

$$|\alpha|^{p_k} \leq max(1, |\alpha|^M). \quad (4)$$

It is clear that, $h_B(\theta) = 0$ and $h_B(x) = h_B(-x)$ for all $x \in r_B^q(u,p)$. Again the inequality (3) and (4), yield the subadditivity of h_B and

$$h_B(\alpha x) \leq max(1, |\alpha|)h_B(x).$$

Let $\{x^n\}$ be any sequence of points of the space $r_B^q(u,p)$ such that $h_B(x^n - x) \to 0$ and (α_n) is a sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality,

$$h_B(x^n) \leq h_B(x) + h_B(x^n - x)$$

holds by subadditivity of h_B, $\{h_B(x^n)\}$ is bounded and we thus have

$$h_B(\alpha_n x^n - \alpha x) = \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1} u_k(q_j.r + q_{j+1}.s)(\alpha_n x_j^n - \alpha x_j) + u_k q_k.r(\alpha_n x_k^n - \alpha x_k)\right|^{p_k}\right]^{\frac{1}{M}}$$

$$\leq |\alpha_n - \alpha|^{\frac{1}{M}}\, h_B(x^n) + |\alpha|^{\frac{1}{M}}\, h_B(x^n - x)$$

which tends to zero as $n \to \infty$. That is to say, that the scalar multiplication is continuous. Hence, h_B is paranorm on the space $r_B^q(u,p)$.

It remains to prove the completeness of the space $r_B^q(u,p)$. Let $\{x^j\}$ be any Cauchy sequence in the space $r_B^q(u,p)$, where $x^i = \{x_0^i, x_1^i, ...\}$. Then, for a given $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that

$$h_B(x^i - x^j) < \epsilon \tag{5}$$

for all $i, j \geq n_0(\epsilon)$. Using definition of h_B and for each fixed $k \in \mathbb{N}$ that

$$\left|(R_u^q Bx^i)_k - (R_u^q Bx^j)_k\right| \leq \left[\sum_k \left|(R_u^q Bx^i)_k - (R_u^q Bx^j)_k\right|^{p_k}\right]^{\frac{1}{M}} < \epsilon$$

for $i, j \geq n_0(\epsilon)$, which leads us to the fact that $\{(R_u^q Bx^0)_k, (R_u^q Bx^1)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say, $(R_u^q Bx^i)_k \to ((R_u^q Bx)_k$ as $i \to \infty$. Using these infinitely many limits $(R_u^q Bx)_0, (R_u^q Bx)_1, \ldots$, we define the sequence $\{(R_u^q Bx)_0, (R_u^q Bx)_1, ...\}$. From (5) for each $m \in N$ and $i, j \geq n_0(\epsilon)$,

$$\sum_{k=0}^{m} \left|(R_u^q Bx^i)_k - (R_u^q Bx^j)_k\right|^{p_k} \leq h_B(x^i - x^j)^M < \epsilon^M. \tag{6}$$

Take any $i, j \geq n_0(\epsilon)$. First, let $j \to \infty$ in (6) and then $m \to \infty$, we obtain

$$h_B(x^i - x) \leq \epsilon.$$

Finally, taking $\epsilon = 1$ in (6) and letting $i \geq n_0(1)$. we have by Minkowski's inequality for each $m \in \mathbb{N}$ that

$$\left[\sum_{k=0}^{m} \left|(R_u^q Bx)_k\right|^{p_k}\right]^{\frac{1}{M}} \leq h_B(x^i - x) + h_B(x^i) \leq 1 + h_B(x^i)$$

which implies that $x \in r_B^q(u, p)$. Since $h_B(x - x^i) \leq \epsilon$ for all $i \geq n_0(\epsilon)$, it follows that $x^i \to x$ as $i \to \infty$, hence we have shown that $r_B^q(u, p)$ is complete, hence the proof .

If we take $r = 1$, $s = -1$ in the theorem 2.1, then we have the following result which was proved by Neyaz and Hamid [19].

Corollary 2.2. $r^q(\triangle_u^p)$ is a complete linear metric space paranormed by $h_\triangle$, defined as

$$h_\triangle(x) = \left[\sum_k \left|\frac{1}{Q_k}\left(\sum_{j=0}^{k-1} u_k(q_j - q_{j+1})x_j + q_k u_k x_k\right)\right|^{p_k}\right]^{\frac{1}{M}},$$

where $\sup_k p_k = H$and $M = max\{1, H\}$.

Note that one can easily see the absolute property does not hold on the spaces $r_B^q(u, p)$, that is $h_B(x) \neq h_B(|x|)$ for atleast one sequence in the space $r_B^q(u, p)$ and this says that $r_B^q(u, p)$ is a sequence space of non-absolute type.

Theorem 2.3. The sequence space $r_B^q(u, p)$ of non-absolute type is linearly isomorphic to the space $l(p)$, where $0 < p_k \leq H < \infty$.

Proof. To prove the theorem, we should show the existence of a linear bijection between the spaces $r_B^q(u,p)$ and $l(p)$, where $0 < p_k \le H < \infty$. With the notation of (2), define the transformation T from $r_B^q(u,p)$ to $l(p)$ by $x \to y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y \in l(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{n=0}^{k-1} (-1)^{k-n} \left(\frac{s^{k-n-1}}{r^{k-n} q_{n+1}} + \frac{s^{k-n}}{r^{k-n+1} q_n} \right) Q_n u_k^{-1} y_n + \frac{Q_k u_k^{-1} y_k}{r.q_k}$$

Then,

$$h_B(x) = \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k (q_j.r + q_{j+1}.s) x_j + u_k q_k.r.x_k \right|^{p_k} \right]^{\frac{1}{M}}$$

$$= \left[\sum_k \left| \sum_{j=0}^{k} \delta_{kj} y_j \right|^{p_k} \right]^{\frac{1}{M}}$$

$$= \left[\sum_k \left| y_k \right|^{p_k} \right]^{\frac{1}{M}} = h_1(y) < \infty,$$

where,

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases}$$

Thus, we have $x \in r_B^q(u,p)$. Consequently, T is surjective and is paranorm preserving. Hence, T is a linear bijection and this says us that the spaces $r_B^q(u,p)$ and $l(p)$ are linearly isomorphic, hence the proof.

3. Duals and Basis of $r_B^q(u,p)$

In this section, we compute α-, β- and γ-duals of $r_B^q(u,p)$ and construct its basis.

Theorem 3.1. (*i*) Let $1 < p_k \le H < \infty$ for every $k \in \mathbb{N}$. Define the sets $D_1(u,p)$ and $D_2(u,p)$ as follows

$D_1(u,p) =$

$$\bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sup_{K \in F} \sum_k \left| \sum_{n \in K} \left[\nabla_u(n,k) a_n Q_k + \frac{a_n}{r.q_n} u_n^{-1} Q_n \right] B^{-1} \right|^{p_k'} < \infty \right\}$$

and

$D_2(u,p) =$

$$\bigcup_{B>1}\left\{a=(a_k)\in\omega:\sum_k\left|\left[\left(\frac{a_k}{r.u_kq_k}+\nabla_u(n,k)\sum_{i=k+1}^{n}a_i\right)Q_k\right]B^{-1}\right|^{p_k'}<\infty\right\},$$

where

$$\nabla_u(n,k)=(-1)^{k-n}\left(\frac{s^{k-n-1}}{r^{k-n}q_{n+1}}+\frac{s^{k-n}}{r^{k-n+1}q_n}\right)u_k^{-1}.$$

Then,

$$\left[r_B^q(u,p)\right]^{\alpha}=D_1(u,p),\ \ \left[r_B^q(u,p)\right]^{\beta}=D_2(u,p)\cap\ cs,\ \ and\ \ \left[r_B^q(u,p)\right]^{\beta}=D_2(u,p).$$

(ii) Let $0<p_k\le 1$ for every $k\in\mathbb{N}$. Define the sets $D_3(u,p)$ and $D_4(u,p)$ as follows

$D_3(u,p) =$

$$\left\{a=(a_k)\in\omega:\sup_{K\in F}\sup_k\left|\sum_{n\in K}\left[\nabla_u(n,k)a_nQ_k+\frac{a_n}{r.q_n}u_n^{-1}Q_n\right]B^{-1}\right|^{p_k}<\infty\right\}$$

and

$$D_4(u,p)=\left\{a=(a_k)\in\omega:\sup_k\left|\left[\left(\frac{a_k}{r.u_kq_k}+\nabla_u(n,k)\sum_{i=k+1}^{n}a_i\right)Q_k\right]\right|^{p_k}<\infty\right\}.$$

Then,

$$\left[r_B^q(u,p)\right]^{\alpha}=D_3(u,p),\ \ \left[r_B^q(u,p)\right]^{\beta}=D_4(u,p)\cap\ cs,\ \ and\ \ \left[r_B^q(u,p)\right]^{\gamma}=D_4(u,p).$$

For the proof of the Theorem 3.1, we need following lemmas.

Lemma 3.2. [7] (i) Let $1<p_k\le H<\infty$. Then $A\in(l(p):l_1)$ if and only if there exists an integer $B>1$ such that

$$\sup_{K\in F}\sum_{k\in\mathbb{N}}\left|\sum_{n\in K}a_{nk}B^{-1}\right|^{p_k'}<\infty.$$

(ii) Let $0<p_k\le 1$. Then $A\in(l(p):l_1)$ if and only if

$$\sup_{K\in F}\sup_{k\in\mathbb{N}}\left|\sum_{n\in K}a_{nk}B^{-1}\right|^{p_k}<\infty.$$

Lemma 3.3. [9] (i) Let $1<p_k\le H<\infty$. Then, $A\in(l(p):l_\infty)$ if and only if there exists an integer $B>1$ such that

$$\sup_n \sum_k |a_{nk}B^{-1}|^{p'_k} < \infty. \tag{7}$$

(ii) Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (l(p) : l_\infty)$ if and only if

$$\sup_{n,k\in\mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{8}$$

Lemma 3.4. [9] Let $0 < p_k \leq H < \infty$ for every $k \in N$. Then $A \in (l(p) : c)$ if and only if (7) and (8) hold along with

$$\lim_n a_{nk} = \beta_k \ for \ k \in \mathbb{N}. \tag{9}$$

Proof of Theorem 3.1. We consider the case $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Let us take any $a = (a_n) \in \omega$. From (2) we can easily see that

$$a_n x_n = \sum_{k=0}^{n-1} \nabla_u(n,k) a_n Q_k y_k + \frac{a_n Q_n y_n}{r.q_n} u_k^{-1} = \sum_{k=0}^{n} c_{nk} y_k = (Cy)_n, \tag{10}$$

where $n \in \mathbb{N}$ and $C = (c_{nk})$ is defined by

$$c_{nk} = \begin{cases} \nabla_u(n,k) a_n Q_k, & \text{if} \quad 0 \leq k \leq n-1, \\ \dfrac{a_n Q_n}{r.q_n} u_k^{-1}, & \text{if} \quad k = n, \\ 0, & \text{if} \quad k > n, \end{cases}$$

where $k, n \in \mathbf{N}$. Thus, we deduce from (10) with Lemma 3.2 that $ax = (a_n x_n) \in l_1$ whenever $x = (x_n) \in r_B^q(u,p)$ if and only if $Cy \in l_1$ whenever $y \in l(p)$. This shows that $\left[r_B^q(u,p)\right]^\alpha = D_1(u,p)$.

Further, consider the equation

$$\sum_{k=0}^{n} a_n x_n = \sum_{k=0}^{n} \left(\frac{a_k}{r.q_k} u_k^{-1} + \nabla_u(n,k) \sum_{i=k+1}^{n} a_i \right) Q_k y_k = (Dy)_n, \tag{11}$$

where $n \in \mathbb{N}$ and $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \left(\dfrac{a_k}{r.q_k} u_k^{-1} + \nabla_u(n,k) \sum\limits_{i=k+1}^{n} a_i \right) Q_k, & \text{if} \quad 0 \leq k \leq n, \\ 0, & \text{if} \quad k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. Thus we deduce from (11) with Lemma 3.3 that $ax = (a_n x_n) \in cs$ whenever $x = (x_n) \in r_B^q(u,p)$ if and only if $Dy \in c$ whenever $y \in l(p)$. Therefore, we derive from (11) that

$$\sum_k \left| \left[\left(\frac{a_k}{r.q_k} u_k^{-1} + \nabla_u(n,k) \sum_{i=k+1}^{n} a_i \right) u_k^{-1} Q_k \right] B^{-1} \right|^{p_k'} < \infty, \tag{12}$$

and $\lim_n d_{nk}$ exists and hence shows that $\left[r_B^q(u,p)\right]^\beta = D_2(u,p) \cap \ cs$.

As proved above, from Lemma 3.4 together with (12) that $ax = (a_k x_k) \in bs$ whenever $x = (x_n) \in r_B^q(u,p)$ if and only if $Dy \in l_\infty$ whenever $y = (y_k) \in l(p)$. Therefore, we again obtain the condition (12) which means that $\left[r_B^q(u,p)\right]^\gamma = D_2(u,p)$ and this completes the proof.

Theorem 3.5. Define the sequence $b^{(k)}(q) = \{b_n^{(k)}(q)\}$ of the elements of the space $r_B^q(u,p)$ for every fixed $k \in \mathbb{N}$ by

$$b_n^{(k)}(q) = \begin{cases} \frac{Q_k}{r.q_k} u_k^{-1} + \nabla_u(n,k) Q_k, & \text{if } 0 \le n \le k, \\ 0, & \text{if } n > k. \end{cases}$$

Then, the sequence $\{b^{(k)}(q)\}$ is a basis for the space $r_B^q(u,p)$ and for any $x \in r_B^q(u,p)$ has a unique representation of the form

$$x = \sum_k \lambda_k(q) b^{(k)}(q) \tag{13}$$

where, $\lambda_k(q) = (R_u^q Bx)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \le H < \infty$.

Proof. It is clear that $\{b^{(k)}(q)\} \subset r_B^q(u,p)$, since

$$R_u^q B b^{(k)}(q) = e^{(k)} \in l(p) \ for \ k \in N \tag{14}$$

and $0 < p_k \le H < \infty$, where $e^{(k)}$ is the sequence whose only non-zero term is 1 in k^{th} place for each $k \in N$.

Let $x \in r_B^q(u,p)$ be given. For every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) b^{(k)}(q). \tag{15}$$

Then, by applying $R_u^q B$ to (15) and using (14), we obtain

$$R_u^q B x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) R_u^q B b^{(k)}(q) = \sum_{k=0}^{m} (R_u^q B x)_k e^{(k)}$$

and

$$\left(R_u^q B\left(x - x^{[m]}\right)\right)_i = \begin{cases} 0, & \text{if } 0 \le i \le m \\ (R_u^q B x)_i, & \text{if } i > m \end{cases}$$

where $i, m \in \mathbb{N}$. Given $\varepsilon > 0$, there exists an integer m_0 such that

$$\left(\sum_{i=m}^{\infty} |(R_u^q B x)_i|^{p_k}\right)^{\frac{1}{M}} < \frac{\varepsilon}{2},$$

for all $m \ge m_0$. Hence,

$$\begin{aligned} h_B\left(x - x^{[m]}\right) &= \left(\sum_{i=m}^{\infty} |(R_u^q B x)_i|^{p_k}\right)^{\frac{1}{M}} \\ &\le \left(\sum_{i=m_0}^{\infty} |(R_u^q B x)_i|^{p_k}\right)^{\frac{1}{M}} \\ &< \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

for all $m \ge m_0$, which proves that $x \in r_B^q(u, p)$ is represented as (14).

Let us show the uniqueness of the representation for $x \in r_B^q(u, p)$ given by (13). Suppose, on the contrary; that there exists a representation $x = \sum_k \mu_k(q) b^k(q)$. Since the linear transformation T from $r_B^q(u, p)$ to $l(p)$ used in the Theorem 3 is continuous we have

$$\begin{aligned} (R_u^q B x)_n &= \sum_k \mu_k(q) \left(R_u^q B b^k(q)\right)_n \\ &= \sum_k \mu_k(q) e_n^{(k)} = \mu_n(q) \end{aligned}$$

for $n \in N$, which contradicts the fact that $(R_u^q B)_n = \lambda_n(q)$ for all $n \in \mathbb{N}$. Hence, the representation (13) is unique. This completes the proof.

4. Matrix Mappings on the Space $r_B^q(u, p)$

In this section, we characterize the matrix mappings from the space $r_B^q(u, p)$ to the space l_∞.

Theorem 4.1. (*i*) Let $1 < p_k \le H < \infty$ for every $k \in N$. Then $A \in \left(r_B^q(u, p)) : l_\infty\right)$ if and only if there exists an integer $B > 1$ such that

$$C(B)=\sup_n\sum_k\left|\left[\left(\frac{a_{nk}}{r.u_kq_k}+\nabla_u(n,k)\sum_{i=k+1}^{n}a_{ni}\right)Q_k\right]B^{-1}\right|^{p'_k}<\infty \tag{16}$$

and $\{a_{nk}\}_{k\in\mathbb{N}}\in cs$ for each $n\in\mathbb{N}$.

(ii) Let $0<p_k\le 1$ for every $k\in\mathbb{N}$. Then $A\in\left(r_B^q(u,p):l_\infty\right)$ if and only if

$$\sup_k\left|\left[\left(\frac{a_{nk}}{r.u_kq_k}+\nabla_u(n,k)\sum_{i=k+1}^{n}a_{ni}\right)Q_k\right]B^{-1}\right|^{p_k}<\infty \tag{17}$$

and $\{a_{nk}\}_{k\in\mathbb{N}}\in cs$ for each $n\in\mathbb{N}$.

Proof. We will prove (i) and (ii) can be proved in a similar fashion. So, let $A\in\left(r_B^q(u,p):l_\infty\right)$ and $1<p_k\le H<\infty$ for every $k\in\mathbb{N}$. Then Ax exists for $x\in r_B^q(u,p)$ and implies that $\{a_{nk}\}_{k\in N}\in\{r_B^q(u,p)\}^\beta$ for each $n\in\mathbb{N}$. Hence necessity of (16) holds.

Conversely, suppose that the necessities (16) hold and $x\in r_B^q(u,p)$, since $\{a_{nk}\}_{k\in\mathbb{N}}\in\{r_B^q(u,p)\}^\beta$ for every fixed $n\in\mathbb{N}$, so the A-transform of x exists. Consider the following equality obtained by using the relation (11) that

$$\sum_{k=0}^{m}a_{nk}x_k=\sum_{k}^{m}\left[\left(\frac{a_{nk}}{r.u_kq_k}+\nabla_u(n,k)\sum_{i=k+1}^{n}a_{ni}\right)Q_k\right]y_k. \tag{18}$$

Taking into account the assumptions we derive from (18) as $m\to\infty$ that

$$\sum_{k}a_{nk}x_k=\sum_{k}\left[\left(\frac{a_{nk}}{r.u_kq_k}+\nabla_u(n,k)\sum_{i=k+1}^{n}a_{ni}\right)Q_k\right]y_k. \tag{19}$$

Now, by combining (19) and the following inequality which holds for any $B>0$ and any complex numbers $a,\ b$

$$|ab|\le B\left(\left|aB^{-1}\right|^{p'}+|b|^p\right)$$

with $p^{-1}+p'^{-1}=1$ [10, 16], one can easily see that

$$\sup_{n\in N}\left|\sum_k a_{nk}x_k\right|\le\sup_{n\in N}\sum_k\left|\left[\left(\frac{a_{nk}}{r.u_kq_k}+\nabla_u(n,k)\sum_{i=k+1}^{n}a_{ni}\right)\right]Q_k\right||y_k|$$
$$\le B\left[C(B)+h_1^B(y)\right]<\infty.$$

This shows that $Ax\in l_\infty$ whenever $x\in r_B^q(u,p)$.

This completes the proof.

References

[1] B. Altay and F. Başar, On the paranormed Riesz sequence space of nonabsolute type, Southeast Asian Bulletin of Mathematics 26 (2002) 701–715.
[2] B. Altay and F. Başar, On the space of sequences of p-bounded variation and related matrix mappings, Ukrainian Mathematical Journal 55(1) (2003) 136–147.
[3] M. Başarir, New type of generalised difference sequence space, Filomat 24(4) (2010) 35–52.
[4] M. Başarir and M. Öztürk, On The Riesz difference sequence space, Rendiconti del Circolo Matematico di Palermo 57 (2008) 377–389.
[5] A. H. Ganie, N. A. Sheikh, Some new paranormed sequence spaces of non-absolute type and matrix transformations, Advanced and Development in Mathematical Sciences 1-2(3)(2012) 1–14.
[6] A.H. Ganie and N.A. Sheikh, On some new sequence space of non-absolute type and matrix transformations, Journal of the Egyptian Mathematical Society 21(2013) 34–40.
[7] K.-G. Gross-Erdmann, Matrix transformations between the sequence spaces of Maddox, Journal of Mathematical Analysis and Applications 180 (1993) 223–238.
[8] H. Kizmaz, On certain sequence, Canadian Mathematical Bulletin 24(2) (1981) 169–176.
[9] C.G. Lascarides and I.J. Maddox, Matrix transformations between some classes of sequences, Mathematical Proceedings of the Cambridge Philosophical Society 68 (1970) 99–104.
[10] I.J. Maddox, Elements of Functional Analysis , 2*nd*ed., The University Press, Cambridge, 1988 .
[11] I.J. Maddox, Paranormed sequence spaces generated by infinite matrices, Mathematical Proceedings of the Cambridge Philosophical Society 64 (1968), 335–340.
[12] I.J. Maddox, Spaces of strongly summable sequences, Quarterly Journal of Mathematics-Oxford 18(2) (1967), 345–355.
[13] E. Malkowsky, M. Mursaleen and S. Suantai, The dual spaces of sets of difference sequences of order m and matrix transformations, Acta Mathematica Sinica, English Series 23(3) (2007) 521–532.
[14] M. Mursaleen and A.K. Noman, On some new difference sequence spaces of non-absolute type, Math. Comput. Mod., 52 (2010), 603-617.
[15] M. Mursaleen, Generalised difference sequence spaces, Journal of Mathematical Analysis and Applications 203 (1995) 738–745.
[16] P.-N. Ng and P.-Y. Lee, Cesàro sequences spaces of non-absolute type, Commentationes Mathematicae. Prace Matematyczne 20(2) (1978) 429-433.
[17] G. M. Petersen, Regular matrix transformations. McGraw-Hill Publishing Co. Ltd., London-New York-Toronto, 1966.
[18] N.A. Sheikh and A.H. Ganie, A new paranormed sequence space and some matrix transformations, Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis 28(1)(2012) 47–58.
[19] N.A. Sheikh and A.H. Ganie, New paranormed sequence space and some matrix transformations, WSEAS Transactions on Mathematics 8(12) (2013) 852–859.