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# Modified Homotopy Perturbation Method for Solving Linear Second-Order Fredholm Integro–Differential Equations

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**Abstract.** In this paper, we propose a modification to homotopy perturbation method and improve to accelerate the rate of convergence in solving linear second-order Fredholm integro-differential equations. Some examples are given to show that this method is easy to apply and the results is obtained very fast.

## 1. Introduction

The integro-differential equations which is combination of differential and Fredholm-Volterra equations have attracted much attention, recently, due to its applications in many areas. It can be used to model many problems of science and theoretical physics such as engineering, biological models, electrostatics, control theory of industrial mathematics, [1, 2]. In the recent literature there is a growing interest to investigate and solve these type of equations for instance [3–6], and various other problems involving special functions of mathematical physics, see [17] as well as their extensions and generalizations to fractional operators, see [18].

The homotopy perturbation method was proposed by He [12] and received much concern. This method has been successfully applied by many authors, such as the works in[7–9]. Later, the modifications of (HPM) was introduced for solving integral and integro-differential equations, see [10] where some modifications of HPM was made by introducing accelerating parameters for solving linear Fredholm integral equations and applied in [11]. The modified homotopy perturbation method (MHPM) by [10] was used to solve linear Fredholm type integro-differential equations with separable kernel. In [10, 11] the method was applied with simple accelerating parameters for solving second-order Fredholm type integro-differential equation. This new modification was based on HPM [12, 13] and an improved version of it is given in [11].

In this work, we combined Sumudu transform with improved homotopy perturbation method (IHPM) and study the integro-differential equations. In particular, we find the exact solution of the Fredholm type

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integro-differential equation of second order with constant coefficients

$$u^{''}(x) = n\,u^{'}(x) + m\,u(x) + \int_0^x k(x,t)u(x)dt + f(x), \qquad a \leq x \leq b \tag{1}$$

subject to the following initial conditions

$$u(0) = A, \qquad u^{'}(0) = B \tag{2}$$

where $k(x,t))$ is the kernel and $m, n, A, B$ are real constant.

## 2. Homotopy Perturbation Method

The basic idea of (HPM) is introduced as follows:
Consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{3}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \tag{4}$$

where $A$ is a general differential operator, $B$ is a boundary operator, $f(r)$ is a known analytic function, $\Gamma$ is the boundary of the domain $\Omega$.

In general, the operator $A$ can be divided into two parts $L$ and $N$, where $L$ is linear, while $N$ is nonlinear. Eq. (3) therefor can be rewritten as follows

$$L(u) + N(u) - f(r) = 0. \tag{5}$$

By the homotopy technique [14, 15]. We construct a homotopy $v(r,p) : \Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \;\; r \in \Omega \tag{6}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{7}$$

where $p \in [0,1]$is an embedding parameter, $u_0$ is an initial approximation of eq. (3) which satisfies the boundary conditions.

From equations (6) and (7) we have

$$H(v,0) = L(v) - L(u_0) = 0, \tag{8}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{9}$$

The changing in the process of $p$ from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology this is known as deformation and $L(v) - L(u_0)$, and $A(v) - f(r)$ are called homotopic.

Now, assume that the solution of equations (6) and (7) can be expressed as

$$v = v_0 + pv_1 + p^2 v_2 + \dots. \tag{10}$$

The approximate solution of Eq. (3) can be obtained by setting $p = 1$.

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots. \tag{11}$$

### 3. Modified Homotopy Perturbation Method

This scheme combines Sumudu transform with an improved homotopy perturbation method (IHPM) to be able to solve this type of fractional integro-differential equations with kernel $\sum_{i=1}^{N} g_i(x)h_i(t)$.
The first step, we consider the special case $k(x,t) = g(x)h(t)$, so we define a new convex homotopy perturbation [16] as

$$\begin{aligned} H(u,p,m) \quad &= (1-p)\left(u^{''}(x) - n\,u^{'}(x) - w\,u(x) - f(x)\right) \\ &+p\left(u^{''}(x) + n\,u^{'}(x) + m\,u(x) - \int_a^b k(x,t)u(x)dt\right) \\ &+p(1-p)m\,k^{*}r = 0, \end{aligned} \tag{12}$$

$k^{*}r = \int_a^b k(x,t)u(x)dt$ or

$$u^{''}(x) - n\,u^{'}(x) - w\,u(x) - f(x) - pg(x)\int_a^b h(t)u(x)dt + mpk*r - mp^2k*r = 0. \tag{13}$$

Substituting equation Eq.(10) into (12) and equating the terms with identical powers of $p$, we obtain

$$p^0 : u_0^{''}(x) - n\,u_0{}^{'}(x) - w\,u_0(x) - f(x) = 0, \qquad u_0(0) = A,\ u_0{}^{'}(0) = B, \tag{14}$$

and the solution with Sumudu transform is given by

$$u_0(x) = S^{-1}\left(\frac{F(s) + As^{-2} + Bs^{-1} - nAs^{-1}}{s^{-2} - ns^{-1} - w}\right) \tag{15}$$

$$p^1 : u_1^{''}(x) - n\,u_1{}^{'}(x) - w\,u_1(x) + mk^{*}r = 0, \qquad u_1(0) = 0,\ u_1{}^{'}(0) = 0, \tag{16}$$

or

$$u_1^{''}(x) - n\,u_1{}^{'}(x) - w\,u_1(x) = (1-m)k^{*}r, \qquad u_1(0) = 0,\ u_1{}^{'}(0) = 0, \tag{17}$$

$$k*r = \int_a^b k(x,t)u_0(x)dt, \tag{18}$$

$$u_1(x) = (1-m)\,S^{-1}\left(\frac{K^{*}(s)}{s^{-2} - ns^{-1} - w}\right) \tag{19}$$

$$\begin{aligned} &p^2 : u_2^{''}(x) - n\,u_2{}^{'}(x) - w\,u_2(x) - \int_a^b k(x,t)u_1(t)dt - mk^{*}r = 0 \\ &\quad u_2(0) = 0,\ u_2{}^{'}(0) = 0, \\ &u_2^{''}(x) - n\,u_2{}^{'}(x) - w\,u_2(x) = (1-m)\int_a^b k(x,t)\mathbb{S}^{-1}\left(\tfrac{K^{*}(s)}{s^{-2}-ns^{-1}-w}\right)(t)dt + mk^{*}r \\ &u_2^{''}(x) - n\,u_2{}^{'}(x) - w\,u_2(x) = (1-m)\,\gamma g(x) + mg(x)\int_a^b h(t)u_0(t)dt \\ &\qquad\qquad = [(1-m)\,\gamma + mk_1{}^{*}rg(x)] \end{aligned}$$

$$\gamma = \int_a^b h(t)\mathbb{S}^{-1}\left[\frac{K^*(s)}{s^{-2} - ns^{-1} - w}\right]dt, \tag{20}$$

$$k_1{}^* r = \int_a^b h(t)u_0(x)dt \tag{21}$$

hence we have

$$u_2(x) = [(1-m)\,\gamma + mk_1{}^* r]\, S^{-1}\left\{\frac{G(s)}{s^{-2} - ns^{-1} - w}\right\} \tag{22}$$

$$p^3 : u_3''(x) - n\,u_3{}'(x) - w\,u_3(x) - \int_a^b k(x,t)u_1(t)dt = 0, \quad u_3(0) = 0,\, u_3{}'(0) = 0.$$

In general

$$p^n : u_n''(x) - n\,u_n{}'(x) - w\,u_n(x) - \int_a^b k(x,t)u_{n-1}(t)dt = 0, \quad u_n(0) = 0,\;\; u_n{}'(0) = 0, n = 4, 5, \ldots.$$

Now we find $m$ such that $u_2(x) = 0$, since if $u_2(x) = 0$ then $u_3(x) = u_4(x) = \cdots = 0$, and the solution will be obtained as $u(x) = u_0(x) + u_1(x)$, so for all values of $x$ we should have

$$[(1-m)\,\gamma + mk_1{}^* r] = 0.$$

This implies to

$$m = \frac{\gamma}{\gamma - k_1{}^* r} = \frac{\int_a^b h(t)\mathbb{S}^{-1}\left\{\frac{K^*(s)}{s^{-2}-ns^{-1}-w}\right\}dt}{\int_a^b h(t)\mathbb{S}^{-1}\left\{\frac{K^*(s)}{s^{-2}-ns^{-1}-w}\right\}dt - \int_a^b h(t)u_0(t)dt}. \tag{23}$$

Now, we consider the general case

$$k(x,t) = \sum_{i=1}^{N} g_i(x)h_i(t).$$

Here we choose the convex homotopy as follows:

$$\begin{aligned} H(u,p,m) \quad &= (1-p)\left(u''(x) - n\,u'(x) - w\,u(x) - f(x)\right) + p\left(u''(x) + n\,u'(x) + m\,u(x)\right. \\ &\left. - \int_a^b k(x,t)u(x)dt\right) + p(1-p)\sum_{i=1}^{N} m_i k^* r_i = 0, \end{aligned} \tag{24}$$

$k^* r_i = \int_a^b k(x,t)u(x)dt$. Further

$$u_0''(x) - n\,u_0{}'(x) - w\,u_0(x) - f(x) = 0 \quad u_0(0) = A,\, u_0{}'(0) = B, \tag{25}$$

then the solution with Sumudu transform is given as

$$u_0(x) = \mathbb{S}^{-1}\left(\frac{F(s) + As^{-2} + Bs^{-1} - nAs^{-1}}{s^{-2} - ns^{-1} - w}\right) \tag{26}$$

$$u_1''(x) - n\,u_1'(x) - w\,u_1(x) = \sum_{i=1}^{n}\left[\int_a^b k_i(x,t)u_0(t)dt - m_i k^* r_i\right] \tag{27}$$

$k * r_i = \int_a^b k_i(x,t)u_0(x)dt$, we have

$$u_1(x) = (1 - m_i)\,\mathbb{S}^{-1}\left(\frac{K_i{}^*(s)}{s^{-2} - ns^{-1} - w}\right) \tag{28}$$

$$\begin{aligned} u_2''(x) - n\,u_2'(x) - w\,u_2(x) &= \sum_{i=1}^{n}\left(\int_a^b k_i(x,t)u_1(t)dt + m_i k^* r_i\right) \\ &= \sum_{i=1}^{n}\left[\int_a^b k_i(x,t)\left(\sum_{j=1}^{n}\left(\left(1 - m_j\right)S^{-1}\left\{\frac{k_j^*(s)(t)}{s^{-2} - ns^{-1} - w}\right\}\right)dt\right.\right. \\ &\quad m_i k^* r_i] \\ u_n''(x) - n\,u_n'(x) - w\,u_n(x) &= \sum_{i=1}^{n}\left[\int_a^b k_i(x,t)\,u_{n-1}dt\right]. \end{aligned} \tag{29}$$

Now we find $m_i, i = 1, 2, \cdots, N$ such that $u_2(x) = 0$, since if $u_2(x) = 0$ then $u_3(x) = u_4(x) = \cdots = 0$, so from for values of $x$ we should have

$$\begin{aligned} u_2(x) = g_1(x)&\left[(1 - m_1)\gamma_1 + k^* r_1 m_1 \pm \sum_{i\neq 1}^{n}(1 - m_i)\gamma_i\right] \\ \pm g_2(x)&\left[(1 - m_2)\beta_2 + k^* r_2 m_2 \pm \sum_{i\neq 2}^{n}(1 - m_i)\beta_i\right] \\ \pm \ldots \pm g_n(x)&\left[(1 - m_n)\mu_n + k^* r_n m_n \pm \sum_{i=1}^{n-1}(1 - m_i)\mu_i\right]. \end{aligned} \tag{30}$$

Since we have $u_2(x) = 0$, then we get the following system of equations

$$\begin{cases} (k^* r_1 \pm \gamma_1)m_1 - \sum\limits_{i\neq 1}^{n} m_i\gamma_i = \gamma_1 \pm \sum\limits_{i\neq 1}^{n}\gamma_i \\ (k^* r_2 \pm \beta_2)m_2 - \sum\limits_{i\neq 2}^{n} m_i\beta_i = \beta_2 \pm \sum\limits_{i\neq 2}^{n}\gamma_i \\ \quad\cdot \\ \quad\cdot \\ \quad\cdot \\ (k^* r_n \pm \mu_n)m_1 - \sum\limits_{i=1}^{n-1} m_i\mu_i = \mu_n \pm \sum\limits_{i=1}^{n-1}\gamma_i \end{cases} \tag{31}$$

where

$$
\begin{aligned}
k^*r_i &= \sum_{i=1}^{n} \int_a^b h_i(t)u_0(t)dt, \\
\gamma_i &= \int_a^b h_1(t)\left[\sum_{i=1}^{n}\left(\mathbb{S}^{-1}\left\{\frac{k_i{}^*(s)}{s^{-2}-ns^{-1}-w}\right\}\right)\right]dt, \\
\beta_i &= \int_a^b h_2(t)\left[\sum_{i=1}^{n}\left(\mathbb{S}^{-1}\left\{\frac{k_i{}^*(s)}{s^{-2}-ns^{-1}-w}\right\}\right)\right]dt, \\
\mu_i &= \int_a^b h_n(t)\left[\sum_{i=1}^{n}\left(\mathbb{S}^{-1}\left\{\frac{k_i{}^*(s)}{s^{-2}-ns^{-1}-w}\right\}\right)\right]dt.
\end{aligned}
\tag{32}
$$

## 4. Numerical Examples

In this section, we will apply the modified homotopy perturbation method described in previous section for solving IDEs

**Example 4.1.** Consider Fredholm integro-differential equation of fractional order

$$u^{''}(x) = x - \sin x - \int_0^{\frac{\pi}{2}} xtu(t)dt, \tag{33}$$

subject to initial conditions

$$u(0) = 0, \qquad u^{'}(0) = 1 \tag{34}$$

the exact solution is given by $u(x) = \sin x$

$$
\begin{aligned}
&f(x) = x - \sin x, \qquad n = 0, \quad m = 0 \\
&g(x) = x, \qquad h(t) = t, \quad a = 0, \quad b = \tfrac{\pi}{2}
\end{aligned}
$$

$$u_0^{''}(x) = x - \sin x, \qquad u_0(0) = 0, \qquad u_0{}^{'}(0) = 1. \tag{35}$$

Using Sumudu transform we get

$$u_0(x) = \frac{x^3}{6} + \sin(x) \tag{36}$$

$$u_1^{''}(x) = (-1 - m)k^*r, \qquad u_1(0) = 0, \qquad u_1{}^{'}(0) = 0. \tag{37}$$

From (31)-(32) we get

$$k^*r = \left(\frac{\pi^5}{960} + 1\right)x, \qquad m = \frac{-\left(\frac{\pi^5}{960}\right)}{\left(\frac{\pi^5}{960} + 1\right)} \tag{38}$$

so we have,

$$u_1(x) = (-1 - m)\mathbb{S}^{-1}\left\{\frac{k^*r(s)}{s^{-2}}\right\}.$$

Thus we obtain,

$$u_1(x) = \frac{-x^3}{6}$$

and the solution will be as follows

$$\begin{aligned} u(x) &= u_0(x) + u_1(x) \\ &= \frac{x^3}{6} + \sin(x) - \frac{x^3}{6} \\ &= \sin(x). \end{aligned}$$

**Example 4.2.** Consider the linear Fredholm integro-differential equation :

$$u^{''}(x) = x - 2 + 60 \int_0^1 (x-t)u(t)dt, \tag{39}$$

subject to initial conditions

$$u(0) = 0, \qquad u^{'}(0) = 1 \tag{40}$$

the exact solution is given by $u(x) = x(x-1)^2$

$$\begin{array}{llll} f(x) = x - 2, & n = 0, & m = 0, & g_1(x) = x \\ g_2(x) = -1, & h_1(t) = 60 & h_2(t) = 60t, & a = 0, \ b = 1 \end{array}$$

$$u_0^{''}(x) = x - 2, \qquad u_0(0) = 0, \qquad u_0{}^{'}(0) = 1. \tag{41}$$

Using Sumudu transform we get

$$u_0(x) = \frac{x^3}{6} - x^2 + x \tag{42}$$

$$u_1^{''}(x) = (-1 - m_1)xk^*r_1 - (1 - m_2)k^*r_2, \qquad u_1(0) = 0, \qquad u_1{}^{'}(0) = 0. \tag{43}$$

From (32), we have

$$\begin{array}{lll} k^*r_1 = \frac{25}{2}, & k^*r_2 = 7, & \gamma_1 = \frac{125}{4} \\ \gamma_2 = 70, & \beta_1 = 25, & \beta_2 = 105. \end{array} \tag{44}$$

From (31) we have

$$\begin{aligned} &(k^*r_1 - \gamma_1)\, m_1 + \gamma_2 m_2 = \gamma_2 - \gamma_1 \\ &(k^*r_2 + \beta_2)\, m_2 - \beta_1 m_1 = \beta_2 - \beta_1 \end{aligned} \tag{45}$$

So from (45) we obtain $m_1 = \frac{3}{5}$ and $m_2 = \frac{5}{7}$.

Now, by substituting by the values of $m_1$ and $m_2$ we can write

$$u_1^{''}(x) = 5x - 2, \quad u_1(0) = 0, \; u_1^{'}(0) = 0.$$

By applying Sumudu transform to the above equation then take the inverse transform of the result we get

$$u_1(x) = \frac{5x^3}{6} - x^2$$

and the solution will be obtained as

$$u(x) = u_0(x) + u_1(x) = x(x-1)^2$$

which is the exact solution.

**Example 4.3.** Consider the linear Fredholm integro-differential equation :

$$u''(x) + 2u'(x) + 5u(x) = 3e^{-x}\sin(x) + \int_{-\pi}^{\pi} e^t u(t)dt, \tag{46}$$

subject to initial conditions

$$u(0) = 0, \qquad u'(0) = 2 \tag{47}$$

the exact solution is given by $u(x) = \frac{1}{2}e^{-x}\sin(2x) + e^{-x}\sin(x)$

$$f(x) = 3e^{-x}\sin(x), \qquad n = 2, \qquad m = 5,$$
$$g(x) = 1, \qquad h(t) = e^t$$

$$u_0''(x) + 2u_0'(x) + 5u_0(x) = 3e^{-x}\sin(x), \qquad u_0(0) = 0, \qquad u_0'(0) = 2. \tag{48}$$

Using Sumudu transform we get

$$u_0(x) = \frac{1}{2}e^{-x}\sin x + e^x \sin x \tag{49}$$

$$u_1''(x) + 2u_1'(x) + 5u_1(x) = (1-m)k^*r, \qquad u_1(0) = 0, \qquad u_1'(0) = 0 \tag{50}$$

$$k^*r = \int_{-\pi}^{\pi} 1e^t u_0(t)dt = 0 \tag{51}$$

then

$$u_1(x) = 0.$$

So the solution is

$$u(x) = u_0(x) + u_1(x) = \frac{1}{2}e^{-x}\sin x + e^x \sin x.$$

## 5. Conclusion

In this work, based on HPM and improved version of it the IDEs with initial conditions have been solved. As it was seen in previous section the exact solution of the test problems are calculated by using modified homotopy perturbation method. We noted that in all the equations we are solved the solution we got in three terms of HPM series solutions while the same solutions have been obtained in two term of MHPM series solutions. This is demonstrated that the modified procedure is quite efficient to determine the solution closed form also. Further, this method is very simple and the results are obtained very fast.

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