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# A Companion of Ostrowski Type Integral Inequality Using a 5-Step Kernel with Some Applications

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**Abstract.** The aim of this paper is to establish a new version of Ostrowski's type integral inequality. The results are obtained by using a new type of kernel with five sections. Applications to a composite quadrature rule and to Cumulative Distributive Functions are considered.

## 1. Introduction

In 1938, Ostrowski [10] established the following interesting integral inequality.

**Theorem 1.1.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$, whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on $(a, b)$, i.e*

$$\left\|f'\right\|_\infty = \sup_{t\in[a,b]} \left|f'(t)\right| < \infty$$

*then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_\infty . \tag{1}$$

This inequality has powerful applications in numerical integration, probability and optimization theory, statistics, and integral operator theory.

The integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals is known in literature as Grüss inequality [9], which is given below.

**Theorem 1.2.** *Let $f,\ g : [a, b] \to \mathbb{R}$ be integrable functions such that $\varphi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$, for some constants $\varphi,\ \Phi,\ \gamma,\ \Gamma$ and $x \in [a, b]$. Then*

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$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx.\frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma).$$

In [5], Guessab and Schmeisser proved the following Ostrowski's inequality:

Let $f\colon [a,b] \to \mathbb{R}$ satisfy the Lipschitz condition i.e, $\left|f(t)-f(s)\right| \leq M\left|t-s\right|$. Then for all $x \in \left[a, \frac{a+b}{2}\right]$, we have

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{8} + 2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)M. \tag{2}$$

In (2), the point $x = \frac{3a+b}{4}$ yields the following trapezoid type inequality.

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{8}M.$$

Some generalization of ostrowski type inequalities are also done in [12]-[17]. In [3], Dragomir proved the inequalities for mappings of bounded variation. In [2], Barnett et. al proved some Ostrowski and generalized trapezoid inequality. Dragomir [4] and Liu [6] established some companions of ostrowski type integral inequalities. Alomari [1] proved the following inequality:

Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$. If $f' \in L^1[a,b]$ and $\gamma \leq f'(t) \leq \Gamma$, for all $t \in [a,b]$, then the inequality

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{8}(b-a)(\Gamma-\gamma).$$

Recently Liu [7], used a 3-step kernel to prove some ostrowski type inequalities. He has demonstrated improvement in approximation errors. More recently Qayyum et. al [18]-[21] proved some ostrowski type inequalities for $L_1$ norm, $L_\infty$ norm and $L_p$ norm.

In all the references mentioned, authors proved their results by using kernels with two or three sections. In this paper we introduce a new kernel which has five sections that further generalize various results. By using this special type of kernel, one can obtain different type of useful and interesting results. We will derive our inequalities using Grüss inequality, Cauchy inequality and Diaz-Metcalf inequality. Finally, some obtained inequalities will then be applied for quadrature formula and for cumulative distributive function.

## 2. Main Results

Before we state and prove our main theorem, we need to prove the following lemma.

**Lemma 2.1.** *Let us define the kernel $P(x,t)$ as:*

$$P(x,t) = \begin{cases} t-a, & t \in (a, \frac{a+x}{2}], \\ t-\frac{3a+b}{4}, & t \in (\frac{a+x}{2}, x], \\ t-\frac{a+b}{2}, & t \in (x, a+b-x], \\ t-\frac{a+3b}{4}, & t \in (a+b-x, \frac{a+2b-x}{2}], \\ t-b, & t \in (\frac{a+2b-x}{2}, b], \end{cases} \tag{3}$$

*for all $x \in \left[a, \frac{a+b}{2}\right]$, the following identity holds:*

$$\frac{1}{b-a}\int_a^b P(x,t) f'(t)\,dt = \frac{1}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{4}$$

*Proof.* From (3), using integration by parts, we get required identity (4). □

We now give our main theorem.

*2.1.* ***When*** $f' \in L^1[a,b]$ :

*2.1.1.* ***Case.1(a):***

**Theorem 2.2.** *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $(a,b)$. If $f' \in L^1[a,b]$ and $\gamma \le f'(t) \le \Gamma$, for all $t \in [a,b]$,*

*then the inequality*

$$\left|\frac{1}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{16}(b-a)(\Gamma-\gamma) \tag{5}$$

*holds for all $x \in \left[a, \frac{a+b}{2}\right]$.*

*Proof.* As we know that for all $t \in [a,b]$ and $x \in \left[a, \frac{a+b}{2}\right]$, we have

$$x - \frac{3a+b}{4} \le P(x,t) \le x - a.$$

Applying Grüss-Inequality [9] to the mappings $P(x,.)$ and $f'(.)$, we obtain

$$\left|\frac{1}{b-a}\int_a^b P(x,t)f'(t)dt - \frac{1}{(b-a)}\int_a^b P(x,t)dt\frac{1}{b-a}\int_a^b f'(t)dt\right| \le \frac{1}{16}(b-a)(\Gamma-\gamma), \tag{6}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$. It is a straightforward exercise to show that

$$\frac{1}{b-a}\int_a^b P(x,t)dt = 0 \tag{7}$$

and

$$\frac{1}{b-a}\int_a^b f'(t)dt = \frac{f(b)-f(a)}{b-a}. \tag{8}$$

Hence using (6)-(8), we get our required result (5). □

Our obtained result (5), further generalizes the results given in [1]-[3], and [7]. To emphasize the importance of the above obtained result (5), we will now discuss some corollaries.

**Corollary 2.3.** *Let $f$ is defined as in Theorem 2.2, and, additionally, if $f(x) = f(a+b-x)$, then we have*

$$\left|\frac{1}{4}\left[2f(x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{16}(b-a)(\Gamma-\gamma),$$

*for all $x \in \left[a, \frac{a+b}{2}\right]$. For instance; choose $x = a$, we have*

$$\left|\frac{3f(a) + f(b)}{4} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{16}(b-a)(\Gamma-\gamma)$$

*and choose $x = \frac{a+b}{2}$, we have*

$$\left|\frac{1}{4}\left[2f\left(\frac{a+b}{2}\right) + f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{16}(b-a)(\Gamma-\gamma).$$

**Corollary 2.4.** *If we substitute $x = a$, in (5), we get*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{16}(b-a)(\Gamma-\gamma). \tag{9}$$

**Corollary 2.5.** *If we substitute $x = \frac{a+b}{2}$, in (5), we get*

$$\left|\frac{1}{4}\left[2f\left(\frac{a+b}{2}\right) + f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{16}(b-a)(\Gamma-\gamma). \tag{10}$$

**Corollary 2.6.** *If we substitute $x = \frac{3a+b}{4}$, in (5), we get*

$$\left|\frac{1}{4}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{7a+b}{8}\right) + f\left(\frac{a+7b}{8}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{16}(b-a)(\Gamma-\gamma). \tag{11}$$

**Corollary 2.7.** *If we substitute $x = \frac{a+3b}{4}$, in (5), we get*

$$\left|\frac{1}{4}\left[f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right) + f\left(\frac{5a+3b}{8}\right) + f\left(\frac{3a+5b}{8}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{16}(b-a)(\Gamma-\gamma). \tag{12}$$

By using (3), we can prove another interesting theorem.

*2.1.2. Case.1(b):*

**Theorem 2.8.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0$, the interior of the interval $I$, and let $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$ and $\gamma \leq f'(t) \leq \Gamma$ for all $x \in [a,b]$, then the following inequality holds for all $x \in \left[a, \frac{a+b}{2}\right]$.*

$$\begin{aligned}&\left|\frac{1}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{8(b-a)}\left[3a^2 + a(b-11x) + x(4x-5b)\right](\Gamma+\gamma).\end{aligned} \tag{13}$$

*Proof.* Let

$$C = \frac{\Gamma + \gamma}{2}$$

then

$$\frac{1}{b-a}\int_a^b P(x,t) f'(t)\,dt - \frac{C}{b-a}\int_a^b P(x,t)dt = \frac{1}{b-a}\int_a^b P(x,t)\left[f'(t) - C\right]dt$$
$$= \frac{1}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt$$

where

$$\int_a^b P(x,t)dt = 0.$$

On the other hand, we have

$$\left|\frac{1}{b-a}\int_a^b P(x,t)\left[f'(t) - C\right]dt\right| \le \frac{1}{b-a}.\max_{t\in[a,b]}\left|f'(t) - C\right|\int_a^b |P(x,t)|\,dt. \tag{14}$$

Since

$$\max_{t\in[a,b]}\left|f'(t) - C\right| \le \frac{\Gamma + \gamma}{2} \tag{15}$$

and

$$\frac{1}{b-a}\int_a^b |P(x,t)|\,dt = \frac{1}{b-a}\left[\left(\frac{x-a}{2}\right)^2 - x(a+b) - \left(\frac{a+b-2x}{4}\right)^2 + \left(x - \frac{3a+b}{4}\right)^2\right]. \tag{16}$$

From (14)-(16), we get (13). □

**Corollary 2.9.** *If we substitute $x = a$, in (13), we get*

$$\left|\frac{1}{2}\left[f(a) + f(b)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{2(b-a)}\left[-a(a+b)\right](\Gamma + \gamma). \tag{17}$$

**Corollary 2.10.** *If we substitute $x = \frac{a+b}{2}$, in (13), we get*

$$\left|\frac{1}{4}\left[2f\left(\frac{a+b}{2}\right) + f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{18}$$
$$\le \frac{1}{16(b-a)}\left[(b-a)^2 - 4(a+b)^2\right](\Gamma + \gamma).$$

**Corollary 2.11.** *If we substitute $x = \frac{3a+b}{4}$, in (13), we get*

$$\left|\frac{1}{4}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{19}$$
$$\leq \frac{-1}{8(b-a)}\left[(3a+b)(a+b)\right](\Gamma+\gamma).$$

**Corollary 2.12.** *If we substitute $x = \frac{a+3b}{4}$, in (13), we get*

$$\left|\frac{1}{4}\left[f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)+f\left(\frac{5a+3b}{8}\right)+f\left(\frac{3a+5b}{8}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{20}$$
$$\leq \frac{-1}{16(b-a)}\left[(b-a)^2+2(a+b)(a+3b)\right](\Gamma+\gamma).$$

By using (3), we can prove another interesting theorem.

#### 2.1.3. ***Case.1(c):***

**Theorem 2.13.** *Let $f:[a,b]\to R$ be a differentiable mapping in $(a,b)$. If $f' \in L^1[a,b]$ and $\gamma \leq f'(x) \leq \Gamma$, then we have*

$$\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \Omega.(S-\gamma) \tag{21}$$

*and*

$$\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \Omega.(\Gamma-S), \tag{22}$$

*for all $x \in [a;\frac{a+b}{2}]$, where*

$$\Omega = \max_{t\in[a,b]}|P(x,t)|, S = \frac{f(b)-f(a)}{b-a}, \gamma = \inf_{t\in[a,b]} f'(t), \Gamma = \sup_{t\in[a,b]} f'(t).$$

*Proof.* As we know that

$$\frac{1}{b-a}\int_a^b P(x,t)f'(t)dt - \frac{1}{(b-a)^2}\int_a^b P(x,t)dt.\int_a^b f'(t)dt \tag{23}$$
$$= \frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]$$

We denote

$$R_n(x) = \frac{1}{b-a}\int_a^b P(x,t)f'(t)dt - \frac{1}{(b-a)^2}\int_a^b P(x,t)dt.\int_a^b f'(t)dt. \tag{24}$$

If $C \in R$ is an arbitrary constant, then we have

$$R_n(x) = \frac{1}{b-a}\int_a^b (f'(t)-C)\left[P(x,t)-\frac{1}{b-a}\int_a^b P(x,s)ds\right]dt. \tag{25}$$

Since

$$\int_a^b \left[ P(x,t) - \frac{1}{b-a} \int_a^b P(x,s)dt \right] dt = 0.$$

Furthermore, we have

$$|R_n(x)| \leq \frac{1}{b-a} \max_{t\in[a,b]} |P(x,t) - 0| \int_a^b \left| f'(t) - C \right| dt$$

and

$$\max_{t\in[a,b]} |P(x,t)| = \Omega. \tag{26}$$

We also have [11]

$$\int_a^b \left| f'(t) - \gamma \right| dt = (S - \gamma)(b-a). \tag{27}$$

$$\int_a^b \left| f'(t) - \Gamma \right| dt = (\Gamma - S)(b-a). \tag{28}$$

By using (7), (8), (23), (26), (27) and (28), we get (21) and (22). □

### 2.2. *Case.2: When $f' \in L^2[a,b]$*

**Theorem 2.14.** *Let $f : [a,b] \to R$ be an absolutely continuous mapping in $(a,b)$ with $f' \in L^2[a,b]$. Then, we have*

$$\begin{aligned} &\left| \frac{1}{4} \left[ f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\ &\leq \sqrt{\frac{\sigma(f')}{b-a}} \left[ \frac{1}{48} \left( 13a^2 + 4b^2 + 13a(b-3x) - 21bx + 30x^2 \right) \right]^{\frac{1}{2}} \end{aligned} \tag{29}$$

*for all $x \in \left[a, \frac{a+b}{2}\right]$, where*

$$\sigma(f') = \left\| f'' \right\|_2^2 - \frac{(f(b) - f(a))^2}{b-a} = \left\| f'' \right\|_2^2 - S^2(b-a).$$

*Proof.* Let $R_n(x)$ is defined as in (24). Then from (23), we get

$$R_n(x) = \frac{1}{4} \left[ f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt$$

If we choose

$$C = \frac{1}{b-a} \int_a^b f'(s)\,ds$$

in (25) and using the cauchy inequality, we get

$$\begin{aligned} |R_n(x)| &\leq \frac{1}{b-a}\int_a^b \left| f'(t) - \frac{1}{b-a}\int_a^b f'(s)\,ds \right| \left| P(x,t) - \frac{1}{b-a}\int_a^b P(x,s)ds \right| dt \\ &\leq \frac{1}{b-a}\left[\int_a^b \left( f'(t) - \frac{1}{b-a}\int_a^b f'(s)\,ds \right)^2 dt\right]^{\frac{1}{2}} \left[\int_a^b \left( P(x,t) - \frac{1}{b-a}P(x,s)ds \right)^2 dt\right]^{\frac{1}{2}} \\ &\leq \sqrt{\sigma(f')}\left[\frac{1}{48}\left(13a^2+4b^2+13a(b-3x)-21bx+30x^2\right)\right]^{\frac{1}{2}}(b-a)^{\frac{-1}{2}}. \end{aligned}$$

The sharpness of the constant $\frac{1}{48}$ in (29) can be obtained for $x = a$ or $x = \frac{a+b}{2}$ which is already proven in [6]. □

**Corollary 2.15.** *If we substitute $x = a$, in (29), we get*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \sqrt{\frac{\sigma(f')}{b-a}}\left[\frac{(a+b)^2}{12}\right]^{\frac{1}{2}} \tag{30}$$

**Corollary 2.16.** *If we substitute $x = \frac{a+b}{2}$, in (29), we get*

$$\left| \frac{1}{4}\left[2f\left(\frac{a+b}{2}\right)+f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{4}\left(40a^2-2ab+b^2\right)\sqrt{\frac{\sigma(f')}{3(b-a)}}. \tag{31}$$

**Corollary 2.17.** *If we substitute $x = \frac{3a+b}{4}$, in (29), we get*

$$\left| \frac{1}{4}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq -\frac{1}{8}\sqrt{\frac{5(b-a)\sigma(f')}{6}}. \tag{32}$$

**Corollary 2.18.** *If we substitute $x = \frac{a+3b}{4}$, in (29), we get*

$$\left| \frac{1}{4}\left[f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)+f\left(\frac{5a+3b}{8}\right)+f\left(\frac{3a+5b}{8}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq -\frac{1}{8}\sqrt{\frac{41(b-a)\sigma(f')}{6}}. \tag{33}$$

We can state ostrowski inequality in an other way also:

### 2.3. *Case.3: When $f'' \in L^2[a,b]$*

**Theorem 2.19.** *Let $f : [a,b] \to R$ be a twice continuously differentiable mapping in $(a,b)$ with $f'' \in L^2[a,b]$. Then*

$$\begin{aligned} &\left| \frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \left[\frac{1}{48\pi}\left(13a^2+4b^2+13a(b-3x)-21bx+30x^2\right)\right]^{\frac{1}{2}}(b-a)^{\frac{3}{2}}\left\|f''\right\|_2, \end{aligned} \tag{34}$$

*for all $x \in \left[a, \frac{a+b}{2}\right]$.*

*Proof.* Let $R_n(x)$ be defined by (24). From ( 23),

$$R_n(x) = \frac{1}{4}\left[f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt.$$

If we choose $C = f'\left(\frac{a+b}{2}\right)$ in (25) and use the Cauchy Inequality, we get

$$\begin{aligned} |R_n(x)| &\leq \frac{1}{b-a}\int_a^b \left|f'(t) - f'\left(\frac{a+b}{2}\right)\right| \left|P(x,t) - \frac{1}{b-a}\int_a^b P(x,s)ds\right| dt \\ &\leq \frac{1}{b-a}\left[\int_a^b \left(f'(t) - f'\left(\frac{a+b}{2}\right)\right)^2 dt\right]^{\frac{1}{2}} \left[\int_a^b \left(P(x,t) - \frac{1}{b-a}P(x,s)ds\right)^2 dt\right]^{\frac{1}{2}}. \end{aligned}$$

We can use the Diaz-Metcalf inequality [8] or [11], to get

$$\int_a^b \left(f'(t) - f'\left(\frac{a+b}{2}\right)\right)^2 dt \leq \frac{(b-a)^2}{\pi^2}\left\|f''\right\|_2^2.$$

We also have

$$\begin{aligned} \int_a^b \left(P(x,t) - \frac{1}{b-a}P(x,s)ds\right)^2 dt &= \int_a^b (P(x,t)^2\,dt \qquad (35) \\ &= \frac{1}{48}(b-a)\left(13a^2 + 4b^2 + 13a(b-3x) - 21bx + 30x^2\right). \end{aligned}$$

Therefore, using the above relations, we obtain (34). □

**Corollary 2.20.** *If we substitute $x = a$, in (34), we get*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{12}\left[(a+b)^2 - 2a^2\right]\right]^{\frac{1}{2}} \frac{(b-a)^{\frac{3}{2}}}{\pi}\left\|f''\right\|_2. \qquad (36)$$

**Corollary 2.21.** *If we substitute $x = \frac{a+b}{2}$, in (34), we get*

$$\left|\frac{1}{4}\left[2f\left(\frac{a+b}{2}\right) + f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{4\sqrt{3}}\frac{(b-a)^2}{\pi}\left\|f''\right\|_2. \qquad (37)$$

**Corollary 2.22.** *If we substitute $x = \frac{3a+b}{4}$, in (34), we get*

$$\begin{aligned} &\left|\frac{1}{4}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{7a+b}{8}\right) + f\left(\frac{a+7b}{8}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \qquad (38) \\ &\leq \frac{b-a}{8\pi\sqrt{6}}\sqrt{5a^2 + 5b^2 - 62ab}\left\|f''\right\|_2. \end{aligned}$$

**Corollary 2.23.** *If we substitute $x = \frac{a+3b}{4}$, in (34), we get*

$$\left|\frac{1}{4}\left[f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)+f\left(\frac{5a+3b}{8}\right)+f\left(\frac{3a+5b}{8}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{39}$$
$$\leq \frac{1}{8\pi}\sqrt{\frac{41}{6}}(b-a)^2\left\|f''\right\|_2 .$$

## 3. An application to Composite Quadrature Rules

Let $I_n : a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, ...., n-1)$; a sequence of intermediate points $h_i = x_{i+1} - x_i$ $(i = 0, 1, .....n-1)$. We have the following quadrature formula:

**Theorem 3.1.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0$, the interior of the interval $I$, and let $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$ and $\gamma \leq f'(t) \leq \Gamma$ for all $x \in [a,b]$, then we have the following quadrature formula:*

$$\int_a^b f(t)dt = A(f, I_n) + R(f, I_n), \tag{40}$$

*where*

$$A(f, I_n) = \frac{1}{4}\sum_{i=0}^{n-1} h_i\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)+f\left(\frac{7x_i+x_{i+1}}{8}\right)+f\left(\frac{x_i+7x_{i+1}}{8}\right)\right] \tag{41}$$

*and remainder satisfies the estimation*

$$\left|R(f, I_n)\right| \leq \frac{1}{16}(\Gamma-\gamma)\sum_{i=0}^{n-1} h_i, \tag{42}$$

*for all $\xi_i \in [x_i, x_{i+1}]$, where $h_i := x_{i+1} - x_i$, $(i = 1, 2, ....n-1)$.*

*Proof.* Apply (11) on the interval $[x_i, x_{i+1}]$, $\xi_i \in [x_i, x_{i+1}]$, where $h_i := x_{i+1} - x_i$ $(i = 1, 2, ....n-1)$, to get

$$R(f, I_n) = \int_{x_i}^{x_{i+1}} f(t)dt - \frac{1}{4}\sum_{i=0}^{n-1} h_i\left[\ f\left(\tfrac{3x_i+x_{i+1}}{4}\right)+f\left(\tfrac{x_i+3x_{i+1}}{4}\right)+f\left(\tfrac{7x_i+x_{i+1}}{8}\right)+f\left(\tfrac{x_i+7x_{i+1}}{8}\right)\ \right].$$

Summing over $i$ from 0 to $n-1$, we get

$$\begin{aligned} R(f, I_n) &= \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(t)dt - \frac{1}{4}\sum_{i=0}^{n-1} h_i\left[\ f\left(\tfrac{3x_i+x_{i+1}}{4}\right)+f\left(\tfrac{x_i+3x_{i+1}}{4}\right)+f\left(\tfrac{7x_i+x_{i+1}}{8}\right)+f\left(\tfrac{x_i+7x_{i+1}}{8}\right)\ \right] \\ &= \int_a^b f(t)dt - \frac{1}{4}\sum_{i=0}^{n-1} h_i\left[\ f\left(\tfrac{3x_i+x_{i+1}}{4}\right)+f\left(\tfrac{x_i+3x_{i+1}}{4}\right)+f\left(\tfrac{7x_i+x_{i+1}}{8}\right)+f\left(\tfrac{x_i+7x_{i+1}}{8}\right)\ \right]. \end{aligned}$$

From (11), it follows that

$$\begin{aligned} \left|R(f, I_n)\right| &= \left|\int_a^b f(t)dt - \frac{1}{4}\sum_{i=0}^{n-1} h_i\left[\ f\left(\tfrac{3x_i+x_{i+1}}{4}\right)+f\left(\tfrac{x_i+3x_{i+1}}{4}\right)+f\left(\tfrac{7x_i+x_{i+1}}{8}\right)+f\left(\tfrac{x_i+7x_{i+1}}{8}\right)\ \right]\right| \\ &\leq \frac{1}{16}h_i(\Gamma-\gamma). \end{aligned}$$

which completes the proof. □

**Theorem 3.2.** *Let $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ $(i = 0, 1, ...., n-1)$ and let $f : [a, b] \to R$ be an absolutely continuous mapping in $(a, b)$ with $f' \in L^2[a, b]$. Then we have*

$$\int_a^b f(x)dx = A(f, I_n) + R(f, I_n),$$

*and remainder satisfies the estimation*

$$\left|R(f, I_n)\right| \leq \frac{1}{4}\sqrt{\frac{b-a}{3}\sigma(f')}$$

*Proof.* Applying (31) to the interval $[x_i, x_{i+1}]$, then we get

$$\left|\frac{h}{4}\left[2f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\frac{3x_i, x_{i+1}}{4}\right) + f\left(\frac{x_i, 3x_{i+1}}{4}\right)\right] - \int_{x_i}^{x_{i+1}} f(t)\,dt\right|$$
$$\leq \frac{\sqrt{h}}{4\sqrt{3}}\left(40x_i^2 - 2x_i x_{i+1} + x_{i+1}^2\right)\left[\int_{x_i}^{x_{i+1}} (f(t))^2\,dt - \frac{(f(x_{i+1}) - f(x_i))^2}{h}\right]^{\frac{1}{2}}$$

for $i = 0, 1, ...., n-1$.

Now summing over $i$ from 0 to $n-1$, using the triangle inequality and Cauchy inequality twice, we get

$$\left|\frac{h}{4}\sum_{i=0}^{n-1}\left[2f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\frac{3x_i, x_{i+1}}{4}\right) + f\left(\frac{x_i, 3x_{i+1}}{4}\right)\right] - \int_a^b f(t)\,dt\right|$$
$$\leq \frac{\sqrt{h}}{4\sqrt{3}}\sum_{i=0}^{n-1}\left(\left(40x_i^2 - 2x_i x_{i+1} + x_{i+1}^2\right)\left[\int_{x_i}^{x_{i+1}} (f(t))^2\,dt - \frac{(f(x_{i+1}) - f(x_i))^2}{h}\right]^{\frac{1}{2}}\right)$$
$$\leq \frac{\sqrt{h}}{4\sqrt{3}}\sqrt{n}\left[\left\|f'\right\|_2^2 - \frac{n}{b-a}\sum_{i=0}^{n-1}(f(x_{i+1}) - f(x_i))^2\right]^{\frac{1}{2}}$$
$$\leq \frac{\sqrt{h}}{4\sqrt{3}}\sqrt{n}\left[\left(40x_i^2 - 2x_i x_{i+1} + x_{i+1}^2\right)\left(\left\|f'\right\|_2^2 - \frac{(f(b) - f(a))^2}{b-a}\right)\right]^{\frac{1}{2}}$$
$$= \frac{1}{4}\sqrt{\frac{b-a}{3}\left(40x_i^2 - 2x_i x_{i+1} + x_{i+1}^2\right)\sigma(f')}.$$

□

**Theorem 3.3.** *Let $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ $(i = 0, 1, 2, \cdots, n-1)$ and let $f : [a; b] \to R$ be a twice continuously differentiable mapping in $(a, b)$ with $f'' \in L^2[a, b]$. Then we have*

$$\int_a^b f(x)dx = A(f, I_n) + R(f, I_n),$$

*where the remainder satisfies the estimation*

$$\left|R(f, I_n)\right| \leq \sqrt{\frac{41}{6}}\frac{(b-a)^3}{8\pi n^{\frac{5}{2}}}\left\|f''\right\|_2.$$

*Proof.* Applying (39) to the interval $[x_i, x_{i+1}]$, we get

$$\left|\frac{h}{4}\left[f\left(\frac{x_i+3x_{i+1}}{4}\right)+f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{5x_i+3x_{i+1}}{8}\right)+f\left(\frac{3x_i+5x_{i+1}}{8}\right)\right]-\int_{x_i}^{x_{i+1}} f(t)\,dt\right| \le \frac{1}{8\pi}\sqrt{\frac{41}{6}h^3}\left[\int_{x_i}^{x_{i+1}} (f''(t))^2\,dt\right]^{\frac{1}{2}},$$

for $i = 0, 1, ...., n-1$.

Now summing over $i$ from 0 to $n-1$, using the triangle inequality and Cauchy inequality twice, we get

$$\begin{aligned}
&\left|\frac{h}{4}\sum_{i=0}^{n-1}\left[f\left(\frac{x_i+3x_{i+1}}{4}\right)+f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{5x_i+3x_{i+1}}{8}\right)+f\left(\frac{3x_i+5x_{i+1}}{8}\right)\right]-\int_{a}^{b} f(t)\,dt\right| \\
&\le \frac{1}{8\pi}\sqrt{\frac{41}{6}h^3}\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}} (f''(t))^2\,dt\right]^{\frac{1}{2}} \\
&\le \frac{1}{8\pi}\sqrt{\frac{41n}{6}h^3}\left[\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} (f''(t))^2\,dt\right]^{\frac{1}{2}} \\
&= \sqrt{\frac{41}{6}}\frac{(b-a)^3}{8\pi n^{\frac{5}{2}}}\left\|f''\right\|_2 .
\end{aligned}$$

□

## 4. An Application to Cumulative Distribution Function

Let $X$ be a random variable taking values in the finite interval $[a, b]$ with the probability density function $f : [a, b] \to [0, 1]$ and cumulative distributive function

$$F(x) = \Pr(X \le x) = \int_a^x f(t)\,dt. \tag{43}$$

$$F(b) = \Pr(X \le b) = \int_a^b f(u)\,du = 1. \tag{44}$$

**Theorem 4.1.** *With the assumptions of Theorem 2.2, we have the following inequality which holds*

$$\left|\frac{1}{4}\left[F(x)+F(a+b-x)+F\left(\frac{a+x}{2}\right)+F\left(\frac{a+2b-x}{2}\right)\right]-\frac{b-E(X)}{b-a}\right| \le \frac{1}{16}(b-a)(\Gamma-\gamma), \tag{45}$$

*for all* $x \in \left[a, \frac{a+b}{2}\right]$. *Where* $E(X)$ *is the expectation of* $X$.

*Proof.* In the proof of Theorem 2.2, let $f = F$ and using the fact that

$$E(X) = \int_a^b t\,dF(t) = b - \int_a^b F(t)\,dt.$$

Further details are left to the interested readers. □

**Theorem 4.2.** *With the assumptions of Theorem 2.8, we have the following inequality which holds*

$$\left|\frac{1}{4}\left[F(x)+F(a+b-x)+F\left(\frac{a+x}{2}\right)+F\left(\frac{a+2b-x}{2}\right)\right]-\frac{b-E(X)}{b-a}\right| \tag{46}$$
$$\leq \frac{1}{2(b-a)}\left[\left(\frac{x-a}{2}\right)^2-x(a+b)-\left(\frac{a+b-2x}{4}\right)^2+\left(x-\frac{3a+b}{4}\right)^2\right](\Gamma+\gamma),$$

*for all* $x\in\left[a,\frac{a+b}{2}\right]$, *where* $E(X)$ *is the expectation of* $X$.

*Proof.* Applying (43) and (44) on (13) and using the same conditions that we used in Theorem 4.1, we get the required inequality. □

**Corollary 4.3.** *Under the assumptions of Theorem 4.1, if we put* $x=a$ *in* **(**45**)**, *then we get*

$$\left|\frac{F(a)+F(b)}{2}-\frac{b-E(X)}{b-a}\right|\leq\frac{1}{16}(b-a)(\Gamma-\gamma).$$

**Corollary 4.4.** *Under the assumptions of Theorem 4.1, if we put* $x=\frac{a+b}{2}$ *in* **(**45**)**, *then we get*

$$\left|\frac{1}{4}\left[2F\left(\frac{a+b}{2}\right)+F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right|\leq\frac{1}{16}(b-a)(\Gamma-\gamma).$$

**Corollary 4.5.** *Under the assumptions of Theorem 4.1, if we put* $x=\frac{3a+b}{4}$ *in* **(**45**)**, *then we get*

$$\left|\frac{1}{4}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)+F\left(\frac{7a+b}{8}\right)+F\left(\frac{a+7b}{8}\right)\right]-\frac{b-E(X)}{b-a}\right|\leq\frac{1}{16}(b-a)(\Gamma-\gamma).$$

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