Filomat 28:7 (2014), 1363–1380
DOI 10.2298/FIL1407363H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Common Fixed Point Results in Complex Valued Metric Spaces with Application to Integral Equations

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Abstract. In this paper, common fixed point of six mappings satisfying a contractive condition involving rational inequality in the framework of complex valued metric space are obtained. Moreover, some examples and applications to integral equations are given here to illustrate the usability of the obtained results.

1. Introduction and Preliminaries

There exist a number of generalizations of metric spaces, and one of them is the cone metric space initiated by Huang and Zhang [9]. They described the convergence in cone metric spaces, introduced the notion of completeness and proved some fixed point theorems of contractive mappings on these spaces. Then several authors [2, 4–8, 12, 14–16] obtained fixed points in different generalized metric spaces.

The problem of existence of common fixed points to a pair of nonlinear mappings is now a classical theme. The applications to differential and integral equations made it more interesting. A considerable importance has been attached to common fixed point theorems in ordered spaces [1, 11].

Azam et al.[3] introduced the concept of complex valued metric space and obtained the existence and uniqueness of common fixed points involving rational expressions. Then Rouzkard et al. [17] and Sintunavarat et al. [18] generalized the concept of Azam et al [3].

The aim of this paper is to extend and generalize common fixed point theorems for six self-maps of Jankovic et al. [13] from cone metric space to complex valued metric space of contractive type mappings involving rational inequality. We will illustrate this fact by proving the existence of nonnegative integrable solutions for an implicit integral equation in complex valued metric spaces.

Consistent with Azam, Fisher and Khan [3], the following definitions and results will be needed in what follows. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \operatorname{Re}(z_1) \leqslant \operatorname{Re}(z_2),\ \operatorname{Im}(z_1) \leqslant \operatorname{Im}(z_2).$$

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 46S40
Keywords. Compatible and weakly compatible maps, common fixed point, complex valued metric space
Received: 20 June 2013; Accepted: 29 January 2014
Communicated by Ljubomir Ćirić
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It follows that

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied:

$$\begin{aligned}
\text{(i) } \operatorname{Re}(z_1) &= \operatorname{Re}(z_2),\ \operatorname{Im}(z_1) < \operatorname{Im}(z_2),\\
\text{(ii) } \operatorname{Re}(z_1) &< \operatorname{Re}(z_2),\ \operatorname{Im}(z_1) = \operatorname{Im}(z_2),\\
\text{(iii) } \operatorname{Re}(z_1) &< \operatorname{Re}(z_2),\ \operatorname{Im}(z_1) < \operatorname{Im}(z_2),\\
\text{(iv) } \operatorname{Re}(z_1) &= \operatorname{Re}(z_2),\ \operatorname{Im}(z_1) = \operatorname{Im}(z_2).
\end{aligned}$$

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$\begin{aligned}
0 &\precsim z_1 \precnsim z_2 \implies |z_1| < |z_2|,\\
z_1 &\precsim z_2, z_2 \prec z_3 \implies z_1 \prec z_3.
\end{aligned}$$

Definition 1.1. *Let X be a nonempty set. Suppose that the self-mapping $d : X \times X \to \mathbb{C}$ satisfies:*

1. $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space. A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.$$

A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A \setminus \{x\}) \neq \phi.$$

A is called open whenever each element of A is an interior point of A. Moreover, a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{B(x, r) : x \in X, 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ on X.

Let x_n be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \longrightarrow x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space. Let X be a complete complex valued metric space and $T, f : X \to X$. The mappings T, f are said to be compatible if, for for arbitrary $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = t \in X$, and for arbitrary $c \in \mathbb{C}$ with $0 \prec c$, there exists $n_0 \in \mathbb{N}$ such that $d(Tfx_n, fTx_n) \prec c$, whenever $n > n_0$. The mappings T, f are said to be weakly compatible if they commute at their coincidence point (i. e. $Tfx = fTx$ whenever $Tx = fx$). A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$.We require the following lemmas:

Lemma 1.2. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.3. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Lemma 1.4. *If the pair (f, g) of self-mappings on the complex valued metric space (X, d) is compatible, then it is weakly compatible but the converse does not holds.*

Proof. *Let $fu = gu$ for some $u \in X$. We have to prove that $fgu = gfu$. Put $x_n = u$ for every $n \in \mathbb{N}$. We have fx_n, $gx_n \to fu = gu$. If $c \in \mathbb{C}$ with $0 \prec c$ then since the pair (f, g) is compatible, so we have $d(gfx_n, fgx_n) = d(gfu, fgu) \precsim c$ implies that $fgu = gfu$ as required.* □

2. Main Result

Hussain et al. [10] proved six mappings fixed point theorem for generalized (ψ, φ) contractions. Recently Jankovic et al. [13] proved a common fixed point theorem for six self-mappings satisfying generalized contraction in a cone metric space. Here we improve and generalize the result of Jankovic et al. to a complex valued metric space involving a rational type inequality.

Theorem 2.1. *Let (X, d) be a complete complex valued metric space and let $A, B, S, T, L, M : X \to X$ be a self-mappings satisfying the conditions:*

$$(cvms_1) \qquad d(Lx, My)) \precsim \lambda R(x, y) \tag{1}$$

for all $x, y \in X$ and $\lambda \in [0, 1)$, where

$$\begin{aligned} R(x, y) \in \; & \{d(ABx, STy), d(ABx, Lx), d(STy, My), \frac{1}{2}(d(STy, Lx) + d(ABx, My)), \\ & \frac{d(ABx, Lx)d(STy, My)}{1 + d(ABx, STy)}\}; \end{aligned}$$

$(cvms_2)$ $L(X) \subset ST(X); M(X) \subset AB(X)$;

$(cvms_3)$ $AB = BA; ST = TS; LB = BL; MT = TM$;
$(cvms_4)$ the pair (L, AB) is compatible and the pair (M, ST) is weakly compatible;
$(cvms_5)$ either AB or L is continuous.
Then A, B, S, T, L and M have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. From the condition $(cvms_2)$, there exist $x_1, x_2 \in X$ such that $Lx_0 = STx_1 = y_0$ and $Mx_1 = ABx_2 = y_1$. We can construct successively the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$y_{2n} = STx_{2n+1} = Lx_{2n} \text{ and } y_{2n+1} = ABx_{2n+2} = Mx_{2n+1} \tag{2}$$

for $n = 0, 1, 2, \cdots$. We prove that $|d(y_{n+1}, y_n)| \leq \lambda |d(y_n, y_{n-1})|$, for $n = 1, 2, \cdots$. Now from $(cvms_1)$, we get

$$d(y_{2n}, y_{2n+1}) = d(Lx_{2n}, Mx_{2n+1}) \precsim \lambda R(x_{2n}, x_{2n+1}), \tag{3}$$

where

$$\begin{aligned} R(x_{2n}, x_{2n+1}) \in \; & \{d(ABx_{2n}, STx_{2n+1}), d(ABx_{2n}, Lx_{2n}), d(STx_{2n+1}, Mx_{2n+1}), \\ & \frac{1}{2}(d(STx_{2n+1}, Lx_{2n}) + d(ABx_{2n}, Mx_{2n+1})), \\ & \frac{d(ABx_{2n}, Lx_{2n})d(STx_{2n+1}, Mx_{2n+1})}{1 + d(ABx_{2n}, STx_{2n+1})}\} \\ = \; & \{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{1}{2}((d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n+1})), \\ & \frac{d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n})}\} \\ = \; & \{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{1}{2}d(y_{2n-1}, y_{2n+1}), \frac{d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n})}\}. \end{aligned} \tag{4}$$

By (3) and (4), we have possible four cases that are

$$|d(y_{2n}, y_{2n+1})| \leq \lambda |d(y_{2n-1}, y_{2n})|, \tag{5}$$

and

$$|d(y_{2n}, y_{2n+1})| \leq \lambda |d(y_{2n}, y_{2n+1})| < |d(y_{2n}, y_{2n+1})| \tag{6}$$

which is a contradiction. And

$$d(y_{2n}, y_{2n+1}) \precsim \frac{\lambda}{2} d(y_{2n-1}, y_{2n+1}) \precsim \frac{\lambda}{2} d(y_{2n-1}, y_{2n}) + \frac{\lambda}{2} d(y_{2n}, y_{2n+1}).$$

As $0 \leq \lambda < 1$, so

$$|d(y_{2n}, y_{2n+1})| \leq \frac{\lambda}{2} |d(y_{2n-1}, y_{2n})| + \frac{1}{2} |d(y_{2n}, y_{2n+1})|,$$

which implies that

$$|d(y_{2n}, y_{2n+1})| \leq \lambda |d(y_{2n-1}, y_{2n})|.$$

And

$$\begin{aligned} |d(y_{2n}, y_{2n+1})| &\leq \lambda \left| \frac{d(y_{2n-1}, y_{2n}) d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n})} \right| \\ &\leq \lambda \frac{|d(y_{2n-1}, y_{2n})| |d(y_{2n}, y_{2n+1})|}{|1 + d(y_{2n-1}, y_{2n})|}, \end{aligned}$$

since $|1 + d(y_{2n-1}, y_{2n})| > |d(y_{2n-1}, y_{2n})|$, so we

$$|d(y_{2n}, y_{2n+1})| \leq \lambda |d(y_{2n}, y_{2n+1})| < |d(y_{2n}, y_{2n+1})| \tag{7}$$

which is a contradiction. Thus

$$|d(y_{2n}, y_{2n+1})| \leq \lambda |d(y_{2n-1}, y_{2n})|. \tag{8}$$

Similarly from $(cvms_1)$, we get

$$d(y_{2n+1}, y_{2n+2}) = d(Mx_{2n+1}, Lx_{2n+2}) = d(Lx_{2n+2}, Mx_{2n+1}) \precsim \lambda R(x_{2n+2}, x_{2n+1}), \tag{9}$$

where

$$\begin{aligned} R(x_{2n+2}, x_{2n+1}) &\in \{d(ABx_{2n+2}, STx_{2n+1}), d(ABx_{2n+2}, Lx_{2n+2}), d(STx_{2n+1}, Mx_{2n+1}), \\ &\quad \frac{1}{2}(d(STx_{2n+1}, Lx_{2n+2}) + d(ABx_{2n+2}, Mx_{2n+1})), \\ &\quad \frac{d(ABx_{2n+2}, Lx_{2n+2}) d(STx_{2n+1}, Mx_{2n+1})}{1 + d(ABx_{2n+2}, STx_{2n+1})}\} \\ &\in \{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \frac{1}{2}(d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})), \\ &\quad \frac{d(y_{2n+1}, y_{2n+2}) d(y_{2n}, y_{2n+1})}{1 + d(y_{2n+1}, y_{2n})}\} \\ &\in \{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \frac{1}{2} d(y_{2n}, y_{2n+2}), \frac{d(y_{2n+1}, y_{2n+2}) d(y_{2n}, y_{2n+1})}{1 + d(y_{2n+1}, y_{2n})}\}. \end{aligned}$$

By (9) and (10), we have possible four cases that are

$$|d(y_{2n+1}, y_{2n+2})| \leq \lambda |d(y_{2n}, y_{2n+1}). \tag{11}$$

and

$$|d(y_{2n+1}, y_{2n+2})| \leq \lambda |d(y_{2n+1}, y_{2n+2})| < |d(y_{2n+1}, y_{2n+2})|. \tag{12}$$

which is a contradiction. And

$$d(y_{2n+1}, y_{2n+2}) \precsim \frac{\lambda}{2} d(y_{2n}, y_{2n+2}) \precsim \frac{\lambda}{2} d(y_{2n}, y_{2n+1}) + \frac{\lambda}{2} d(y_{2n+1}, y_{2n+2}).$$

As $0 \leq \lambda < 1$, so

$$\begin{aligned} |d(y_{2n+1}, y_{2n+2})| &\leq \frac{\lambda}{2} |d(y_{2n}, y_{2n+1})| + \frac{1}{2} |d(y_{2n+1}, y_{2n+2})| \\ &\leq \lambda |d(y_{2n}, y_{2n+1})|. \end{aligned}$$

And

$$|d(y_{2n+1}, y_{2n+2})| \leq \lambda \frac{|d(y_{2n+1}, y_{2n+2})||d(y_{2n}, y_{2n+1})|}{|1 + d(y_{2n}, y_{2n+1})|},$$

since $|1 + d(y_{2n}, y_{2n+1})| > |d(y_{2n}, y_{2n+1})|$, so we have

$$|d(y_{2n+1}, y_{2n+2})| \leq \lambda |d(y_{2n+1}, y_{2n+2})| < |d(y_{2n+1}, y_{2n+2})| \tag{13}$$

which is a contradiction. Thus

$$|d(y_{2n+1}, y_{2n+2})| \leq \lambda |d(y_{2n}, y_{2n+1})|. \tag{14}$$

It follows that

$$|d(y_n, y_{n+1})| \leq \lambda |d(y_{n-1}, y_n)| \leq \cdots \leq \lambda^n |d(y_0, y_1)|. \tag{15}$$

Using (15) and triangle inequality, for $m > n$, we have:

$$\begin{aligned} |d(y_n, y_m)| &\leq |d(y_n, y_{n+1})| + |d(y_{n+1}, y_{n+2})| + \cdots + |d(y_{m-1}, y_m)| \\ &\leq [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}] |d(y_0, y_1)| \\ &\leq \left[\frac{\lambda^n}{1-\lambda}\right] |d(y_0, y_1)| \to 0 \text{ as } n \to \infty. \end{aligned}$$

By lemmas 1.2 and 1.3, it follows that $\{y_n\}$ is a Cauchy sequence. Since X is complete, so there exists some $z \in X$ such that $y_n \to z$ as $n \to \infty$. For its subsequences we also have $Mx_{2n+1} \to z$, $STx_{2n+1} \to z$, $Lx_{2n} \to z$ and $ABx_{2n} \to z$. From the condition ($cvms_5$), we have two cases.

Case 01. If AB is continuous.
As AB is continuous, then $ABABx_{2n} \to ABz$ and $ABLx_{2n} \to ABz$, as $n \to \infty$. Also, since the pair (L, AB) is compatible, this implies that $LABx_{2n} \to ABz$. Indeed

$$d(LABx_{2n}, ABz) \precsim d(LABx_{2n}, ABLx_{2n}) + d(ABLx_{2n}, ABz).$$

Now

$$|d(LABx_{2n}, ABz)| \leq |d(LABx_{2n}, ABLx_{2n})| + |d(ABLx_{2n}, ABz)| \to 0 \text{ as } n \to \infty.$$

(a) We first prove that $ABz = z$. We suppose on the contrary that $ABz \neq z$. Then $d(ABz, z) \succ 0$. Now from the triangular inequality, we get

$$d(ABz, z) \precsim d(ABz, LABx_{2n}) + d(LABx_{2n}, Mx_{2n+1}) + d(Mx_{2n+1}, z). \tag{16}$$

Applying the condition ($cvms_3$) to $x = ABx_{2n}$, $y = x_{2n+1}$, we get

$$d(LABx_{2n}, Mx_{2n+1}) \precsim \lambda R(ABx_{2n}, x_{2n+1}),$$

where

$$\begin{aligned} R(ABx_{2n}, x_{2n+1}) \in\ & \{d(ABABx_{2n}, STx_{2n+1}), d(ABABx_{2n}, LABx_{2n}), d(STx_{2n+1}, Mx_{2n+1}), \\ & \frac{1}{2}(d(STx_{2n+1}, LABx_{2n}) + d(ABABx_{2n}, Mx_{2n+1})), \\ & \frac{d(ABABx_{2n}, LABx_{2n})d(STx_{2n+1}, Mx_{2n+1})}{1 + d(ABABx_{2n}, STx_{2n+1})}\}. \end{aligned}$$

Now, we have the following five cases:
(i)

$$\begin{aligned} d(LABx_{2n}, Mx_{2n+1}) &\precsim \lambda d(ABABx_{2n}, STx_{2n+1}) \\ &\precsim \lambda d(ABABx_{2n}, ABz) + \lambda d(ABz, z) + \lambda d(z, STx_{2n+1}), \end{aligned}$$

from (16), we get

$$\begin{aligned} |d(ABz, z)| \leq\ & \frac{1}{1-\lambda}|d(ABz, LABx_{2n})| + \frac{\lambda}{1-\lambda}|d(ABABx_{2n}, ABz)| \\ & + \frac{\lambda}{1-\lambda}|d(z, STx_{2n+1})| + \frac{1}{1-\lambda}|d(Mx_{2n+1}, z)|. \end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$|d(ABz, z)| \leq 0.$$

That is $|d(ABz, z)| = 0$, a contradiction. Thus by lemma 1.2, we get $ABz = z$.

(ii)

$$\begin{aligned} d(LABx_{2n}, Mx_{2n+1}) &\precsim \lambda d(ABABx_{2n}, LABx_{2n}) \\ &\precsim \lambda d(ABABx_{2n}, ABz) + \lambda d(ABz, LABx_{2n}), \end{aligned}$$

using (16), we get

$$|d(ABz, z)| \leq (1+\lambda)|d(ABz, LABx_{2n})| + \lambda|d(ABABx_{2n}, ABz)| + \lambda|d(Mx_{2n+1}, z)|.$$

Now taking the limit as $n \to \infty$, we get

$$|d(ABz, z)| \leq 0.$$

That is $|d(ABz, z)| = 0$, a contradiction. Thus by lemma 1.2, we get $ABz = z$.
(iii)

$$\begin{aligned} d(LABx_{2n}, Mx_{2n+1}) &\precsim \lambda d(STx_{2n+1}, Mx_{2n+1}) \\ &\precsim \lambda d(STx_{2n+1}, z) + \lambda d(z, Mx_{2n+1}), \end{aligned}$$

from (16), we get

$$|d(ABz, z)| \leq |d(ABz, LABx_{2n})| + (1+\lambda)|d(Mx_{2n+1}, z)| + \lambda|d(STx_{2n+1}, z)|.$$

Now taking the limit as $n \to \infty$ and since the pair (L, AB) is compatible, so we get

$$|d(ABz, z)| \leq 0.$$

That is $|d(ABz, z)| = 0$, a contradiction. Thus by lemma 1.2, we get $ABz = z$.

(iv)

$$d(LABx_{2n}, Mx_{2n+1}) \precsim \frac{\lambda}{2}(d(STx_{2n+1}, LABx_{2n}) + d(ABABx_{2n}, Mx_{2n+1})),$$

$$\begin{aligned} d(LABx_{2n}, Mx_{2n+1}) &\precsim \frac{\lambda}{2}((d(STx_{2n+1}, z) + d(z, ABz) + d(ABz, LABx_{2n})) \\ &\quad + \frac{\lambda}{2}(d(ABABx_{2n}, ABz) + d(ABz, z) + d(z, Mx_{2n+1})) \\ &\precsim \frac{\lambda}{2}((d(STx_{2n+1}, z) + d(z, Mx_{2n+1})) + \frac{\lambda}{2}(d(ABABx_{2n}, ABz) \\ &\quad + d(ABz, LABx_{2n})) + \lambda d(ABz, z). \end{aligned}$$

From (16), we get

$$\begin{aligned} |d(ABz, z)\} &\leq \frac{1}{1-\lambda}|d(ABz, LABx_{2n})| + \frac{\lambda}{2(1-\lambda)}(|(d(STx_{2n+1}, z)| + |d(z, Mx_{2n+1})|) \\ &\quad + \frac{\lambda}{2(1-\lambda)}(|d(ABABx_{2n}, ABz)| + |d(ABz, LABx_{2n})|) + \frac{1}{1-\lambda}|d(Mx_{2n+1}, z)|. \end{aligned}$$

That is $|d(ABz, z)| = 0$, a contradiction. Thus $ABz = z$.

(v)

$$d(LABx_{2n}, Mx_{2n+1}) \precsim \lambda \frac{d(ABABx_{2n}, LABx_{2n}) d(STx_{2n+1}, Mx_{2n+1})}{1 + d(ABABx_{2n}, STx_{2n+1})},$$

from (16), we get

$$|d(ABz, z)| \leq |d(ABz, LABx_{2n})| + \lambda \frac{|d(ABABx_{2n}, LABx_{2n})||d(STx_{2n+1}, Mx_{2n+1})|}{|1 + d(ABABx_{2n}, STx_{2n+1})|} + |d(Mx_{2n+1}, z)|.$$

Now taking the limit as $n \to \infty$, we get

$$|d(ABz, z)| \leq 0.$$

That is $|d(ABz, z)| = 0$, a contradiction. Thus $ABz = z$. Hence, in all cases $ABz = z$.

(b) Now we prove that $Lz = z$. We suppose on the contrary that $Lz \neq z$. Then $d(Lz, z) \succ 0$. From the triangular inequality, we get

$$d(Lz, z) \precsim d(Lz, Mx_{2n+1}) + d(Mx_{2n+1}, z). \tag{17}$$

Applying the condition (*iii*) to $x = z, y = x_{2n+1}$, we get

$$d(Lz, Mx_{2n+1}) \precsim \lambda R(ABx_{2n}, x_{2n+1}),$$

where

$$\begin{aligned} R(z, x_{2n+1}) \in &\{d(ABz, STx_{2n+1}), d(ABz, Lz), d(STx_{2n+1}, Mx_{2n+1}), \frac{1}{2}(d(STx_{2n+1}, Lz) + d(ABz, Mx_{2n+1})), \\ &\frac{d(ABz, Lz) d(STx_{2n+1}, Mx_{2n+1})}{1 + d(ABz, STx_{2n+1})}\}. \end{aligned}$$

We have the following five cases: (i)

$$d(Lz, Mx_{2n+1}) \precsim \lambda d(ABz, STx_{2n+1}) = \lambda d(z, STx_{2n+1}).$$

from (17), we get

$$|d(Lz, z)| \leq \lambda |d(z, STx_{2n+1})| + |d(Mx_{2n+1}, z)|,$$

Now taking the limit as $n \to \infty$, we get

$$|d(Lz, z)| \leq 0.$$

That is $|d(Lz, z)| = 0$, a contradiction. Thus $Lz = z$.

(ii)

$$d(Lz, Mx_{2n+1}) \precsim \lambda d(ABz, Lz) = \lambda d(z, Lz),$$

from (17), we get

$$\begin{aligned} |d(Lz, z)| &\leq \lambda |d(z, Lz)| + |d(Mx_{2n+1}, z)| \\ &\leq \frac{1}{1-\lambda} |d(Mx_{2n+1}, z)| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Which implies that $|d(Lz, z)| \leq 0$, a contradiction. Thus $Lz = z$.

(iii)

$$\begin{aligned} d(Lz, Mx_{2n+1}) &\precsim \lambda d(STx_{2n+1}, Mx_{2n+1}) \\ &\precsim \lambda d(STx_{2n+1}, z) + \lambda d(z, Mx_{2n+1}), \end{aligned}$$

from (17), we get

$$|d(Lz, z)| \leq (1+\lambda)|d(Mx_{2n+1}, z)| + \lambda |d(STx_{2n+1}, z)| \to 0 \text{ as } n \to \infty.$$

Which implies that $|d(Lz, z)| = 0$, a contradiction. Thus $Lz = z$.

(iv)

$$d(Lz, Mx_{2n+1}) \precsim \frac{\lambda}{2}(d(STx_{2n+1}, Lz) + d(ABz, Mx_{2n+1}))$$

$$d(Lz, Mx_{2n+1}) \precsim \frac{\lambda}{2}((d(STx_{2n+1}, z) + d(z, Lz)) + \frac{\lambda}{2} d(z, Mx_{2n+1}),$$

from (17), we get

$$d(Lz, z) \precsim \frac{\lambda}{2}((d(STx_{2n+1}, z) + d(z, Lz)) + (\frac{\lambda}{2} + 1) d(z, Mx_{2n+1}).$$

Which implies that

$$|d(Lz, z)| \leq \frac{\lambda}{2}(|(d(STx_{2n+1}, z)| + |d(z, Lz)|) + (\frac{\lambda}{2} + 1)|d(z, Mx_{2n+1})|.$$

Now taking the limit as $n \to \infty$, we get $|d(Lz, z)| = 0$, a contradiction. Thus $Lz = z$.

(v)

$$d(Lz, Mx_{2n+1}) \precsim \lambda \frac{d(ABz, Lz) d(STx_{2n+1}, Mx_{2n+1})}{1 + d(ABz, STx_{2n+1})},$$

from (17), we get

$$|d(Lz, z)| \leq \lambda \frac{|d(ABz, Lz)||d(STx_{2n+1}, Mx_{2n+1})|}{|1 + d(ABz, STx_{2n+1})|} + |d(Mx_{2n+1}, z)|,$$

Taking the limit as $n \to \infty$, we get

$$|d(Lz, z)| = 0, \text{ a contradiction.}$$

Thus $Lz = z$. Thus in all cases we have $Lz = z$.

(c) Now we prove that $Bz = z$. We suppose on the contrary that $Bz \neq z$. Then $d(Bz, z) \succ 0$. Now using the triangular inequality, we get

$$d(Bz, z) = d(BLz, z) = d(LBz, z) \precsim d(LBz, Mx_{2n+1}) + d(Mx_{2n+1}, z), \tag{18}$$

From (1), we get

$$d(LBz, Mx_{2n+1}) \precsim \lambda R(Bz, x_{2n+1}),$$

where

$$\begin{aligned}
R(Bz, x_{2n+1}) &\in \{d(ABBz, STx_{2n+1}), d(ABBz, LBz), d(STx_{2n+1}, Mx_{2n+1}),\\
&\quad \frac{1}{2}(d(STx_{2n+1}, Lz) + d(ABz, Mx_{2n+1})), \frac{d(ABBz, LBz)d(STx_{2n+1}, Mx_{2n+1})}{1 + d(ABBz, STx_{2n+1})}\}\\
R(Bz, x_{2n+1}) &\in \{d(BABz, STx_{2n+1}), d(BABz, BLz), d(STx_{2n+1}, Mx_{2n+1}),\\
&\quad \frac{1}{2}(d(STx_{2n+1}, Lz) + d(ABz, Mx_{2n+1})), \frac{d(BABz, BLz)d(STx_{2n+1}, Mx_{2n+1})}{1 + d(BABz, STx_{2n+1})}\}\\
R(Bz, x_{2n+1}) &\in \{d(Bz, STx_{2n+1}), d(Bz, Bz), d(STx_{2n+1}, Mx_{2n+1}),\\
&\quad \frac{1}{2}(d(STx_{2n+1}, z) + d(z, Mx_{2n+1})), 1\frac{d(Bz, Bz)d(STx_{2n+1}, Mx_{2n+1})}{1 + d(Bz, STx_{2n+1})}\}.
\end{aligned}$$

Now, we have the following five cases:

(i)

$$\begin{aligned}
d(LBz, Mx_{2n+1}) &\precsim \lambda d(Bz, STx_{2n+1})\\
&\precsim \lambda d(Bz, z) + \lambda d(z, STx_{2n+1}),
\end{aligned}$$

from (18), we get

$$\begin{aligned}
|d(Bz, z)| &\leq \lambda |d(Bz, z)| + \lambda |d(z, STx_{2n+1})| + |d(Mx_{2n+1}, z)|\\
|d(Bz, z)| &\leq \frac{\lambda}{1-\lambda}|d(z, STx_{2n+1})| + \frac{\lambda}{1-\lambda}|d(Mx_{2n+1}, z)|.
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we get $|d(Bz, z)| = 0$, a contradiction. Thus $Bz = z$.

(ii)

$$d(LBz, Mx_{2n+1}) \precsim \lambda d(Bz, Bz),$$

from (18), we get

$$|d(Bz, z)| \leq |d(Mx_{2n+1}, z)|.$$

Taking limit as $n \to \infty$ in the above inequality, we get $|d(Bz, z)| = 0$, a contradiction. Thus $Bz = z$.

(iii)

$$\begin{aligned}
d(LBz, Mx_{2n+1}) &\precsim d(STx_{2n+1}, Mx_{2n+1})\\
&\precsim \lambda d(STx_{2n+1}, z) + \lambda d(z, Mx_{2n+1})
\end{aligned}$$

from (18), we get

$$|d(Bz, z)| \leq \lambda |d(STx_{2n+1}, z)| + \lambda |d(z, Mx_{2n+1})| + |d(Mx_{2n+1}, z)|.$$

Taking limit as $n \to \infty$ in the above inequality, we get $|d(Bz, z)| = 0$, a contradiction. Thus $Bz = z$.

(iv)

$$d(LBz, Mx_{2n+1}) \precsim \frac{\lambda}{2}(d(STx_{2n+1}, z) + d(z, Mx_{2n+1})),$$

from (18), we get

$$d(LBz, z) \precsim \frac{\lambda}{2}d(STx_{2n+1}, z) + (\frac{\lambda}{2} + 1)d(Mx_{2n+1}, z),$$

which implies that

$$|d(LBz, z)| \leq \frac{\lambda}{2}|d(STx_{2n+1}, z)| + (\frac{\lambda}{2} + 1)|d(Mx_{2n+1}, z)|.$$

Taking limit as $n \to \infty$ in the above inequality, we get $|d(Bz, z)| = 0$, a contradiction. Thus $Bz = z$.

(v)

$$d(LBz, Mx_{2n+1}) \precsim \frac{d(Bz, Bz)d(STx_{2n+1}, Mx_{2n+1})}{1 + d(Bz, STx_{2n+1})},$$

from (18), we get

$$\begin{aligned} |d(Bz, z)| &\leq \lambda \frac{|d(Bz, Bz)||d(STx_{2n+1}, Mx_{2n+1})|}{|1 + d(Bz, STx_{2n+1})|} + |d(Mx_{2n+1}, z)| \\ &\leq |d(Mx_{2n+1}, z)|, \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we get $|d(Bz, z)| = 0$, a contradiction. Thus $Bz = z$. Thus in all cases we have $Bz = z$.

(d) As $L(X) \subset ST(X)$, so there exists $v \in X$ such that $z = Lz = STv$. First, we shall show that $STv = Mv$. For this we have

$$d(STv, Mv) = d(Lz, Mv) \precsim \lambda R(z, v), \tag{19}$$

where

$$\begin{aligned} R(z, v) \in \; & \{d(ABz, STv), d(ABz, Lz), d(STv, Mv), \frac{1}{2}(d(STv, Lz) + d(ABz, Mv)), \\ & \frac{d(ABz, Lz)d(STv, Mv)}{1 + d(ABz, STv)}\}; \end{aligned}$$

$$\begin{aligned} R(z, v) \in \; & \{d(z, z), d(z, z), d(STv, Mv), \frac{1}{2}(d(z, z) + d(z, Mv)), \\ & \frac{d(z, z)d(STv, Mv)}{1 + d(z, z)}\}. \end{aligned}$$

This implies that

$$R(z, v) \in \{0, d(STv, Mv), \frac{1}{2}d(STv, Mv)\}. \tag{20}$$

From (19) and (20), it follows that

$$|d(STv, Mv)| = 0.$$

That is $STv = Mv = z$. As the pair (M, ST) is weakly compatible, so we have $STMv = MSTv$. Thus

$$STz = Mz.$$

(e) Now we prove that $Mz = z$. Now we have

$$d(z, Mz) = d(Lz, Mz) \precsim \lambda R(z, z), \tag{21}$$

where

$$\begin{aligned} R(z, z) \;\; &\in \;\; \{d(ABz, STz), d(ABz, Lz), d(STz, Mz), \frac{1}{2}(d(STz, Lz) + d(ABz, Mz)), \\ &\quad \frac{d(ABz, Lz)d(STz, Mz)}{1 + d(ABz, STz)}\}; \\ R(z, z) &\in \{d(z, Mz), d(z, z), d(Mz, Mz), \frac{1}{2}(d(Mz, z) + d(z, Mz)), \frac{d(z, z)d(Mz, Mz)}{1 + d(z, Mz)}\} \\ R(z, z) &\in \{0, d(z, Mz)\}. \end{aligned} \tag{22}$$

From (21) and (22), we get

$$|d(z, Mz)| = 0,$$

that is

$$Mz = z.$$

(f) Now we prove that $Tz = z$. Now we have

$$d(z, Tz) = d(Lz, TMz) = d(Lz, MTz).$$

From ($cvms_1$), we get

$$d(z, Tz) = d(Lz, MTz) \precsim \lambda R(z, Tz), \tag{23}$$

where

$$\begin{aligned} R(z, Tz) \;\; &\in \;\; \{d(ABz, STTz), d(ABz, Lz), d(STTz, MTz), \frac{1}{2}(d(STTz, Lz) + d(ABz, MTz)), \\ &\quad \frac{d(ABz, Lz)d(STz, Mz)}{1 + d(ABz, STTz)}\} \\ R(z, Tz) \;\; &\in \;\; \{d(z, TSTz), d(z, z), d(TSTz, TMz), \frac{1}{2}(d(TSTz, Lz) + d(ABz, TMz)), \\ &\quad \frac{d(z, z)d(TSTz, TMz)}{1 + d(z, TSTz)}\} \\ R(z, Tz) &\in \{d(z, Tz), d(z, z), d(Tz, Tz), \frac{1}{2}(d(Tz, z) + d(z, Tz)), \frac{d(z, z)d(Tz, Tz)}{1 + d(z, Tz)}\} \\ R(z, Tz) &\in \{0, d(z, Tz)\}, \end{aligned} \tag{24}$$

from (23) and (24), we get

$$|d(z, Tz)| = 0,$$

that is

$$Tz = z.$$

Since

$$STz = z,$$

it follows that

$$Sz = z.$$

Thus if AB is continuous then we proved that

$$Az = Bz = Sz = Tz = Lz = Mz = z.$$

Hence, the six self mappings have a common fixed point in the case when AB is continuous.

Case 02. If L is continuous.

As L is continuous, then $L^2x_{2n} \to Lz$ and $LABx_{2n} \to Lz$, as $n \to \infty$. As the pair (L, AB) is compatible, we have $ABLx_{2n} \to Lz$, as $n \to \infty$. Indeed

$$d(ABLx_{2n}, Lz) \precsim d(ABLx_{2n}, LABx_{2n}) + d(LABx_{2n}, Lz),$$

Now

$$|d(ABLx_{2n}, Lz)| \le |d(ABLx_{2n}, LABx_{2n})| + |d(LABx_{2n}, Lz)| \to 0, \text{ as } n \to \infty.$$

(a) First we prove that $Lz = z$. By triangular inequality, we get

$$d(Lz, z) \precsim d(Lz, L^2x_{2n}) + d(L^2x_{2n}, Mx_{2n+1}) + d(Mx_{2n+1}, z). \tag{25}$$

Now putting $x = Lx_{2n}$ and $y = x_{2n+1}$, in $(cvms_1)$, we get

$$d(L^2x_{2n}, Mx_{2n+1}) \precsim \lambda R(Lx_{2n}, x_{2n+1}),$$

where

$$\begin{aligned} R(Lx_{2n}, x_{2n+1}) \in \; & \{d(ABLx_{2n}, STx_{2n+1}), d(ABLx_{2n}, L^2x_{2n}), d(STx_{2n+1}, Mx_{2n+1}), \\ & \frac{1}{2}(d(STx_{2n+1}, L^2x_{2n}) + d(ABLx_{2n}, Mx_{2n+1})), \\ & \frac{d(ABLx_{2n}, LLx_{2n})d(STx_{2n+1}, Mx_{2n+1})}{1 + d(ABLx_{2n}, STx_{2n+1})}\}; \end{aligned}$$

Now, we have the following five cases:
(i)

$$\begin{aligned} d(L^2x_{2n}, Mx_{2n+1}) &\precsim \lambda d(ABLx_{2n}, STx_{2n+1}) \\ &\precsim \lambda d(ABLx_{2n}, Lz) + \lambda d(Lz, z) + \lambda d(z, STx_{2n+1}), \end{aligned}$$

from (25), we get

$$d(Lz, z) \precsim d(Lz, L^2x_{2n}) + \lambda d(ABLx_{2n}, Lz) + \lambda d(Lz, z) + \lambda d(z, STx_{2n+1}) + d(Mx_{2n+1}, z).$$

Now

$$|d(Lz, z)| \le |d(Lz, L^2x_{2n})| + \lambda|d(ABLx_{2n}, Lz)| + \lambda|d(Lz, z)| + \lambda|d(z, STx_{2n+1})| + |d(Mx_{2n+1}, z)|,$$

which implies that

$$|d(Lz, z)| \le \frac{1}{1-\lambda}|d(Lz, L^2x_{2n})| + \frac{\lambda}{1-\lambda}|d(ABLx_{2n}, Lz)| + \frac{\lambda}{1-\lambda}|d(z, STx_{2n+1})| + \frac{1}{1-\lambda}|d(Mx_{2n+1}, z)|.$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$|d(Lz, z)| = 0.$$

Thus $Lz = z$.

(ii)

$$\begin{aligned} d(L^2x_{2n}, Mx_{2n+1}) &\precsim \lambda d(ABLx_{2n}, L^2x_{2n}) \\ &\precsim \lambda d(ABLx_{2n}, Lz) + \lambda d(Lz, L^2x_{2n}), \end{aligned}$$

from (25), we get

$$d(Lz, z) \precsim d(Lz, L^2x_{2n}) + \lambda d(ABLx_{2n}, Lz) + \lambda d(Lz, L^2x_{2n}) + d(Mx_{2n+1}, z).$$

Now

$$|d(Lz, z)| \leq |d(Lz, L^2x_{2n})| + \lambda|d(ABLx_{2n}, Lz)| + \lambda|d(Lz, L^2x_{2n})| + |d(Mx_{2n+1}, z)|.$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$|d(Lz, z)| = 0.$$

Thus $Lz = z$.

(iii)

$$\begin{aligned} d(L^2x_{2n}, Mx_{2n+1}) &\precsim \lambda d(STx_{2n+1}, Mx_{2n+1}) \\ &\precsim \lambda d(STx_{2n+1}, z) + \lambda d(z, Mx_{2n+1}), \end{aligned}$$

from (25), we get

$$\begin{aligned} d(Lz, z) &\precsim d(Lz, L^2x_{2n}) + \lambda d(STx_{2n+1}, z) + \lambda d(z, Mx_{2n+1}) + d(Mx_{2n+1}, z) \\ &\precsim d(Lz, L^2x_{2n}) + \lambda d(STx_{2n+1}, z) + (1 + \lambda)d(z, Mx_{2n+1}). \end{aligned}$$

Now

$$|d(Lz, z)| \leq |d(Lz, L^2x_{2n})| + \lambda|d(STx_{2n+1}, z)| + (1 + \lambda)|d(z, Mx_{2n+1})|.$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$|d(Lz, z)| = 0.$$

Thus $Lz = z$.

(iv)

$$d(L^2x_{2n}, Mx_{2n+1}) \precsim \frac{\lambda}{2}(d(STx_{2n+1}, L^2x_{2n}) + d(ABLx_{2n}, Mx_{2n+1}))$$

$$\begin{aligned} d(L^2x_{2n}, Mx_{2n+1}) &\precsim \frac{\lambda}{2}(d(STx_{2n+1}, z) + d(z, Lz) + d(Lz, L^2x_{2n})) \\ &\quad + \frac{\lambda}{2}(d(ABLx_{2n}, Lz) + d(Lz, z) + d(z, Mx_{2n+1})), \end{aligned}$$

from (25), we get

$$\begin{aligned} d(Lz, z) &\precsim d(Lz, L^2x_{2n}) + \frac{\lambda}{2}(d(STx_{2n+1}, z) + d(z, Lz) + d(Lz, L^2x_{2n})) \\ &\quad + \frac{\lambda}{2}(d(ABLx_{2n}, Lz) + d(Lz, z) + d(z, Mx_{2n+1})) + d(Mx_{2n+1}, z) \end{aligned}$$

$$\begin{aligned} d(Lz, z) &\precsim \frac{1}{1-\lambda}d(Lz, L^2x_{2n}) + \frac{\lambda}{2(1-\lambda)}d(STx_{2n+1}, z) + \frac{\lambda}{2(1-\lambda)}d(Lz, L^2x_{2n}) \\ &\quad + \frac{\lambda}{2(1-\lambda)}d(ABLx_{2n}, Lz) + \frac{\lambda}{2(1-\lambda)}d(z, Mx_{2n+1}) + \frac{1}{1-\lambda}d(Mx_{2n+1}, z). \end{aligned}$$

This implies that

$$\begin{aligned}|d(Lz,z)| &\leq \frac{1}{1-\lambda}|d(Lz,L^2x_{2n})| + \frac{\lambda}{2(1-\lambda)}|d(STx_{2n+1},z)| + \frac{\lambda}{2(1-\lambda)}|d(Lz,L^2x_{2n})| \\ &+\frac{\lambda}{2(1-\lambda)}|d(ABLx_{2n},Lz)| + \frac{\lambda}{2(1-\lambda)}|d(z,Mx_{2n+1})| + \frac{1}{1-\lambda}|d(Mx_{2n+1},z)|.\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$|d(Lz,z)| \leq 0, \text{ a contradiction.}$$

Thus $|d(Lz,z)| = 0$ implies $Lz = z$.

(v)

$$d(L^2x_{2n},Mx_{2n+1}) \precsim \lambda\frac{d(ABLx_{2n},LLx_{2n})d(STx_{2n+1},Mx_{2n+1})}{1+d(ABLx_{2n},STx_{2n+1})},$$

from (25), we get

$$d(Lz,z) \precsim d(Lz,L^2x_{2n}) + \lambda\frac{d(ABLx_{2n},LLx_{2n})d(STx_{2n+1},Mx_{2n+1})}{1+d(ABLx_{2n},STx_{2n+1})} + d(Mx_{2n+1},z),$$

Now

$$|d(Lz,z)| \leq |d(Lz,L^2x_{2n})| + \lambda\frac{|d(ABLx_{2n},LLx_{2n})||d(STx_{2n+1},Mx_{2n+1})|}{|1+d(ABLx_{2n},STx_{2n+1})|} + |d(Mx_{2n+1},z)|.$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$|d(Lz,z)| = 0.$$

Thus $Lz = z$. Now, using steps (d),(e) and (f), and continuing the step (f) give us

$$Mz = Sz = Tz = z.$$

(b) As $M(X) \subset AB(X)$, so there exists $w \in X$ such that

$$z = Mz = ABw.$$

We shall show that

$$Lw = ABw = z.$$

For this we have

$$d(Lw,ABw) = d(Lw,Mz) \precsim \lambda R(w,z),$$

where

$$\begin{aligned}R(w,z) &\in \{d(ABw,STz),d(ABw,Lw),d(STz,Mz),\frac{1}{2}(d(STz,Lw) \\ &+d(ABw,Mz)),\frac{d(ABw,Lw)d(STz,Mz)}{1+d(ABw,STz)}\}\end{aligned}$$

that is

$$R(w,z) \in \{0,\frac{1}{2}(d(z,Lw)\},$$

which implies that

$$d(Lw, ABw) = d(Lw, Mz) = d(Lw, z) \precsim \frac{\lambda}{2}(d(z, Lw) \text{ i.e } Lw = ABw = z.$$

Which implies that

$$Lw = ABw = z.$$

As the pair (L, AB) is compatible, so it must be weakly compatible, we have

$$Lz = ABz.$$

Further, $Bz = z$ follows from step (c). Thus, $Az = Bz = Lz = z$ and we obtain that z is the common fixed point of six mappings in this case too. Now we prove the uniqueness of these six mappings. Let z^* be another common fixed point of A, B, S, T, L and M; then

$$Az^* = Bz^* = Sz^* = Tz^* = Lz^* = Mz^* = z^*.$$

Putting $x = z, y = z^*$ in $(cvms_1)$, we get

$$d(z, z^*) = d(Lz, Mz^*)) \precsim \lambda R(z, z^*), \tag{26}$$

where

$$\begin{aligned} R(z, z^*) \quad &\in \quad \{d(ABz, STz^*), d(ABz, Lz), d(STz^*, Mz^*), \frac{1}{2}(d(STz^*, Lz) \\ &\quad +d(ABz, Mz^*)), \frac{d(ABz, Lz)d(STz^*, Mz^*)}{1 + d(ABz, STz^*)}\} \end{aligned}$$

$$R(z, z^*) \in \{0, d(z, z^*)\}. \tag{27}$$

From (26) and (27), we get

$$|d(z, z^*)| = 0.$$

Which implies that $z = z^*$. Thus z is the unique common fixed point of A, B, S, T, L and M. □

In Theorem 2.1, put $B = T = I_X$, the identity mapping on X, to obtain the following result:

Corollary 2.2. *Let (X, d) be a complete complex valued metric space and let $A, S, L, M : X \to X$ be a self-mappings satisfying the conditions:*

$$(cvms_6) \qquad d(Lx, My)) \precsim \lambda R(x, y)$$

for all $x, y \in X$ and $\lambda \in [0, 1)$,where

$$R(x, y) \in \{d(Ax, Sy), d(Ax, Lx), d(Sy, My), \frac{1}{2}\{d(Ax, My) + d(Sy, Lx)\}, \frac{d(Ax, Lx)d(Sy, My)}{1 + d(Ax, Sy)}\};$$

$(cvms_7)$ $L(X) \subset S(X); M(X) \subset A(X);$
$(cvms_8)$ *the pair (L, A) is compatible and the pair (M, S) is weakly compatible;*
$(cvms_9)$ *either A or L is continuous.*
Then A, S, L and M have a unique common fixed point.

Putting $L = M = F$ and $A = B = S = T = I_X$ in Theorem 2.1, we get the following corollary.

Corollary 2.3. *Let (X, d) be a complete complex valued metric space and let $F : X \to X$ be a self-mappings satisfying the conditions:*

$$(cvms_{10}) \qquad d(Fx, Fy)) \precsim \lambda R(x, y)$$

for all $x, y \in X$ and $\lambda \in [0, 1)$,where

$$R(x,y) \in \{d(x,y), d(x,Fx), d(y,Fy), \frac{1}{2}\{d(x,Fy) + d(y,Fx)\}, \frac{d(x,Fx)d(y,Fy)}{1 + d(x,y)}\}.$$

Then F has a unique fixed point.

By setting $L, M = F$ and $A, B, S, T = g$ in Theorem 2.1, following example illustrates of our main result.

Example 2.4. *Let $X = \{(1,2), (2,3), (3,4), (4,5), (5,6)\}$ and define a mapping $d : X \times X \to \mathbb{C}$ as*

$$d(z_1, z_2) = \frac{4}{3}|x_1 - x_2| + 2i|y_1 - y_2|$$

where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, then (X, d) is a complete complex valued metric space. Set $L = M = F, A = B = S = T = g$ and define the self mappings F and g on X (with $z = x + iy$) as

$$Fz = |x - y| + 2i|x - y| \text{ for all } z \in X$$

and

$$gz = \begin{cases} x + iy & \text{for } z = (1,2) \\ 2|x - y| + 3i|x - y| & \text{for } z \in \{(2,3), (3,4), (4,5), (5,6)\}. \end{cases}$$

By a routine calculation, one can easily verify that F and g satisfy the contraction condition (1). Notice that the point $(1, 2) \in X$ a unique common fixed point of F and g.

The following example illustrates our corollary 2.3.

Example 2.5. *Consider*

$$X_{1=}\{z \in \mathbb{C} : -1 \leq Rez \leq 1, Imz = 0\}$$

$$X_2 = \{z \in \mathbb{C} : -1 \leq Imz \leq 1, Rez = 0\}$$

and let $X = X_1 \cup X_2$. Then with $z = x + iy$. Set $L = M = F$ and $A = B = S = T = I$ (identity mapping). Define $F : X \to X$ as follows

$$Fz = \begin{cases} \frac{i}{2}x & \text{if } z \in X_1 \\ y & \text{if } z \in X_2 \end{cases}$$

If d_u is usual metric on X then F is not contractive as $d_u(Fz_1, Fz_2) = |y_1 - y_2| = d_u(z_1, z_2) \quad \forall\, z_1, z_2 \in X_2$. Therefore, the Banach contraction theorem is not valid to find the unique fixed point 0 of F. To apply the Theorem 2.1, consider a complex valued metric $d : X \times X \to \mathbb{C}$ as follows

$$d(z_1, z_2) = \begin{cases} \frac{1}{2}|x_1 - x_2| + \frac{2i}{3}|x_1 - x_2|, & \text{if } z_1, z_2 \in X_1 \\ \frac{2}{3}|y_1 - y_2| + \frac{i}{3}|y_1 - y_2|, & \text{if } z_1, z_2 \in X_1 \\ (\frac{1}{2}x_1 + \frac{2}{3}y_2) + i(\frac{2}{3}x_1 + \frac{1}{3}y_2), & \text{if } z_1 \in X_1, z_2 \in X_2 \\ (\frac{2}{3}y_1 + \frac{1}{2}x_2) + i(\frac{1}{3}y_1 + \frac{2}{3}x_2) & \text{if } z_1 \in X_2, z_2 \in X_1 \end{cases}$$

where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in X$. Then (X, d) is a complete complex valued metric space. By a routine calculation, one can easily verify that F satisfies the contraction condition (1). Notice that the point $0 \in X$ remains fixed under F is indeed unique.

3. Application

Fixed point theorems for operators in ordered Banach spaces are widely investigated and have found various applications in differential and integral equations (see [1, 11] and references therein).

Theorem 3.1. *Let $X = C([a,b],\mathbb{R}^n)$, $a > 0$ and $d : X \times X \to \mathbb{C}$ be defined as follows:*

$$d(x,y) = \max_{t\in[a,b]} \left\|x(t) - y(t)\right\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}.$$

Consider the Urysohn integral equations

$$x(t) = \int_a^b K_1(t,s,x(s))ds + g(t), \tag{28}$$

$$x(t) = \int_a^b K_2(t,s,x(s))ds + h(t), \tag{29}$$

where $t \in [a,b] \subset \mathbb{R}$, $x, g, h \in X$.

Suppose that $K_1, K_2 : [a,b]\times[a,b]\times\mathbb{R}^n \to \mathbb{R}^n$ are such that $F_x, G_x \in X$ for each $x \in X$, where,

$$F_x(t) = \int_a^b K_1(t,s,x(s))ds, \quad G_x(t) = \int_a^b K_2(t,s,x(s))ds \text{ for all } t \in [a,b].$$

If there exists $0 < h < 1$ such that for every $x, y \in X$

$$\left\|F_x(t) - G_y(t) + g(t) - h(t)\right\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a} \precsim hR(x,y)(t),$$

where,

$$R(x,y)(t) \in \{A(x,y)(t), B(x,y)(t), C(x,y)(t), D(x,y)(t), E(x,y)(t)\},$$

$$\begin{aligned}
A(x,y)(t) &= \left\|x(t) - y(t)\right\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}, \\
B(x,y)(t) &= \left\|F_x(t) + g(t) - x(t)\right\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a}, \\
C(x,y)(t) &= \left\|G_y(t) + h(t) - y(t)\right\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a} \\
D(x,y)(t) &= \frac{\left(\left\|G_y(t) + h(t) - x(t)\right\|_\infty + \left\|F_x(t) + g(t) - x(t)\right\|_\infty\right)\sqrt{1+a^2}e^{i\tan^{-1}a}}{2} \\
E(x,y)(t) &= \frac{\left\|F_x(t) + g(t) - x(t)\right\|_\infty \left\|G_y(t) + h(t) - y(t)\right\|_\infty}{1 + \max\limits_{t\in[a,b]} A(x,y)(t)} \sqrt{1+a^2}e^{i\tan^{-1}a}
\end{aligned}$$

then the system of integral equations (28) have a unique common solution.

Proof. Define $L, M : X \to X$ by

$$Lx = F_x + g, \quad Mx = G_x + h.$$

Then

$$d(Lx, My) = \max_{t\in[a,b]} \left\|F_x(t) - G_y(t) + g(t) - h(t)\right\|_\infty \sqrt{1+a^2}e^{i\tan^{-1}a},$$

$$d(x,y) = \max_{t\in[a,b]} A(x,y)(t),$$

$$d(x,Lx) = \max_{t\in[a,b]} B(x,y)(t),$$

$$d(y,My) = \max_{t\in[a,b]} C(x,y)(t)$$

$$\frac{d(x,Ly)+d(y,Mx)}{2} = \max_{t\in[a,b]} D(x,y)(t)$$

$$\frac{d(x,Lx)d(y,My)}{1+d(x,y)} = \max_{t\in[a,b]} E(x,y)(t).$$

It is easily seen that $d(Lx,My) \precsim hR(x,y)$, where

$$R(x,y) \in \left\{ d(x,y), d(x,Lx), d(y,My), \frac{1}{2}\{d(x,My)+d(y,Lx)\}, \frac{d(x,Lx)d(y,My)}{1+d(x,y)} \right\}$$

for every $x,y \in X$. By Theorem 2.1, the Urysohn integral equations (28) and (29) have a unique common solution. □

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