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Generalized Bi-Quasi-Variational Inequalities for Quasi-Pseudo-Monotone Type II Operators in Non-Compact Settings

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Abstract. In this paper, we introduce a new class of generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type II operators in non-compact settings of locally convex Hausdorff topological vector spaces and show the existence results of solutions for generalized bi-quasi-variational inequalities. Our results improve, extend and generalized the corresponding results given by some authors.

1. Introduction

In 1985, Border [3] introduced the concept of escaping sequences in the book: "Fixed Point Theorems with Applications to Economics and Game Theory". Using this concept of escaping sequences, we obtain our results on generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type II operators in non-compact settings. But the main tools that we apply in obtaining our results are Chowdhury and Tan's result on generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type II operators on compact sets [14]. As applications, we show the existence theorem on generalized bi-complementarity problem for quasi-pseudo-monotone type II operators in non-compact settings.

The generalized bi-quasi-variational inequality problem was first introduced by Shih and Tan [19] in 1989. The following is the definition due to Shih and Tan [19].

Definition 1.1. Let E and F be vector spaces over Φ, $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional and be X a nonempty subset of E. If $S : X \to 2^X$ and $M, T : X \to 2^F$ are set-valued mappings, then the *generalized bi-quasi variational inequality problem* for the triple (S, M, T) is to find $\hat{y} \in X$ satisfying the following properties:
(1) $\hat{y} \in S(\hat{y})$;
(2) $\inf_{w\in T(\hat{y})} Re\langle f - w, \hat{y} - x\rangle \leq 0$ for all $x \in S(\hat{y})$ and $f \in M(\hat{y})$.

In this definition, for any nonempty set X, 2^X denote the class or family of all nonempty subsets of X. Also, we use $\mathcal{F}(X)$ to denote the family of all nonempty finite subsets of X. Moreover, throughout this paper, Φ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

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When T is single-valued, a generalized bi-quasi variational inequality problem reduces to a bi-quasi variational inequality problem. Note that the generalized bi-quasi variational inequality problem include the following generally known variational type inequality problems:

Suppose that E is a topological vector space, $F = E^*$ (: the vector space of all continuous linear functionals on E) and $\langle \cdot, \cdot \rangle$ is the usual duality pairing between E^* and E. Then we have the following:

(1) if $T \equiv 0$, then a generalized bi-quasi-variational inequality problem for $(S, M, 0)$ becomes a generalized quasi-variational inequality problem. Chan and Pang [6] first studied generalized bi-quasi-variational inequality problems in finite dimensional case and Shih and Tan [20] first studied them in infinite dimensional case;

(2) if $T \equiv 0$ and M is single-valued, then a generalized bi-quasi-variational inequality problem for $(S, M, 0)$ becomes a quasi-variational inequality problem which was introduced by Bensoussan and Lions [2];

(3) if $S(x) \equiv X$ for each $x \in X$ and $M \equiv 0$, a generalized bi-quasi-variational inequality problem becomes a generalized variational inequality problem which was studied by Browder [5] and Yen [21] among many others.

The following definition of generalized bi-quasi-variational inequality problem is a slight modification of Definition 1.1.

Definition 1.2. Let E and F be vector spaces over Φ, $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional and X be a non empty subset of E. If $S : X \to 2^X$ and $M, T : X \to 2^F$ are set-valued mappings, then the *generalized bi-quasi variational inequality problem* for the triple (S, M, T) is:

(1) to find a point $\hat{y} \in X$ and a point $\hat{w} \in T(\hat{y})$ such that

$$\hat{y} \in S(\hat{y}), \quad Re\langle f - \hat{w}, \hat{y} - x \rangle \leq 0$$

for all $x \in S(\hat{y})$ and $f \in M(\hat{y})$ or

(2) to find a point $\hat{y} \in X$, a point $\hat{w} \in T(\hat{y})$ and a point $\hat{f} \in M(\hat{y})$ such that

$$\hat{y} \in S(\hat{y}), \quad Re\langle \hat{f} - \hat{w}, \hat{y} - x \rangle \leq 0$$

for all $x \in S(\hat{y})$.

In this paper, we obtain our main results on generalized bi-quasi-variational inequalities in non-compact settings using Chowdhury and Tan's following definition of quasi-pseudo-monotone type II and strongly quasi-pseudo-monotone type II operators given in [13]:

Definition 1.3. Let E be a topological vector space, X be a nonempty subset of E and F be a topological vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional. Let $h : X \to \mathbb{R}$ be a mapping, $M : X \to 2^F$ and $T : X \to 2^F$ be set-valued mappings.

(1) Then T is said to be an *h-quasi-pseudo-monotone* (*resp., strongly h-quasi-pseudo-monotone*) *type II operator* if, for each $y \in X$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y (resp., weakly to y) with

$$\limsup_{\alpha} [\inf_{f \in M(y_\alpha)} \inf_{u \in T(y_\alpha)} Re\langle f - u, y_\alpha - y \rangle + h(y_\alpha) - h(y)] \leq 0,$$

we have

$$\begin{aligned} &\limsup_{\alpha} [\inf_{f \in M(y_\alpha)} \inf_{u \in T(y_\alpha)} Re\langle f - u, y_\alpha - x \rangle + h(y_\alpha) - h(x)] \\ &\geq \inf_{f \in M(y)} \inf_{w \in T(y)} Re\langle f - w, y - x \rangle + h(y) - h(x) \end{aligned}$$

for all $x \in X$;

(2) T is said to be a *quasi-pseudo-monotone* (*resp., strongly quasi-pseudo-monotone*) *type II operator* if T is an h-quasi-pseudo-monotone (resp., strongly h-quasi-pseudo-monotone) type II operator with $h \equiv 0$.

Note that an h-quasi-pseudo-monotone type II operator is an extension of the following h-pseudo-monotone type II operator (resp., strongly h-pseudo-monotone type II operator) defined in [10] or to an h-demi (resp., strong h-demi) operator defined in [11]) with slight modifications.

Definition 1.4. Let X be a nonempty subset of a topological vector space E and $T : X \to 2^{E^*}$ be a set-valued mapping. If $h : X \to \mathbb{R}$, then T is said to be

(1) an *h-pseudo-monotone type II operator* (*resp., a strongly h-pseudo-monotone type II*) *operator* if, for each $y \in X$ and every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y (resp., weakly to y) with

$$\limsup_{\alpha}[\inf_{u\in T(y)} Re\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y)] \le 0,$$

we have

$$\begin{aligned}&\limsup_{\alpha}[\inf_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)]\\ &\ge \inf_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)\end{aligned}$$

for all $x \in X$;

(2) a *pseudo-monotone type II operator* (resp., *strongly pseudo-monotone type II operator*) if T is an h-pseudo-monotone type II operator (resp., strongly h-pseudo-monotone type II operator) with $h \equiv 0$.

Note that, in [11], the above operator was called an h-demi or demi (resp., a strong h-demi or a strong demi) operator. Later, these operators were re-named as pseudo-monotone type II operators in [10].

The quasi-pseudo-monotone type II operators given in Definition 1.3 [14] above is an extension of pseudo-monotone type II operators [10, 11]. We use these operators to obtain some general theorems on solutions for a new class of generalized bi-quasi-variational inequalities of quasi-pseudo-monotone type II operators defined in non compact settings in topological vector spaces.

2. Preliminaries

Let E be a topological vector space over Φ, F be a vector space over Φ and $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional. For each $x_0 \in E$, each nonempty subset A of E and $\epsilon > 0$, let

$$W(x_0;\epsilon) := \{y \in F : |\langle y, x_0\rangle| < \epsilon\}$$

and

$$U(A;\epsilon) := \{y \in F : \sup_{x\in A} |\langle y, x\rangle| < \epsilon\}.$$

Let $\sigma\langle F, E\rangle$ be the (weak) topology on F generated by the family $\{W(x;\epsilon) : x \in E \text{ and } \epsilon > 0\}$ as a subbase for the neighbourhood system at 0 and $\delta\langle F, E\rangle$ be the (strong) topology on F generated by the family $\{U(A;\epsilon) : A \text{ is a nonempty bounded subset of } E \text{ and } \epsilon > 0\}$ as a base for the neighbourhood system at 0.

We note then that F, when equipped with the (weak) topology $\sigma\langle F, E\rangle$ or the (strong) topology $\delta\langle F, E\rangle$, becomes a locally convex topological vector space which is not necessarily Hausdorff. But, if the bilinear functional $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ separates points in F, i.e., for each $y \in F$ with $y \neq 0$, there exists $x \in E$ such that $\langle y, x\rangle \neq 0$, then F also becomes Hausdorff.

Furthermore, for any net $\{y_\alpha\}_{\alpha\in\Gamma}$ in F and $y \in F$, we have the following:

(1) $y_\alpha \to y$ in $\sigma\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for each $x \in E$;

(2) $y_\alpha \to y$ in $\delta\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ uniformly for $x \in A$ for each nonempty bounded subset A of E.

If X and Y are topological spaces and $T : X \to 2^Y$, then the graph of T is defined to be the set $G(T) := \{(x, y) \in X \times Y : y \in T(x)\}$. If X and Y are sets and f maps X into Y, the graph of f is the set of all points $(x, f(x))$ in the cartesian product $X \times Y$. If X and Y are topological spaces, then $X \times Y$ is given the usual product topology (the smallest topology that contains all sets $U \times V$ with U and V open in X and Y, respectively), and, if $f : X \to Y$ is continuous and Y is Hausdorff, then the graph G of f is closed.

Let X be a non-empty subset of a topological vector space E. Then X is called a *cone* in E if X is convex and $\lambda X \subset X$ for all $\lambda \geq 0$. If X is a cone in E and $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ is a bilinear functional, then $\widehat{X} = \{w \in F : Re\langle w, x\rangle \geq 0 \text{ for all } x \in X\}$ is also a cone in F, which is called the *dual cone* of X (with respect to the bilinear functional $\langle \cdot, \cdot \rangle$).

Let X be a convex set in a topological vector space E. Then $f : X \to \mathbb{R}$ is called

(1) *lower semi-continuous* if, for all $\lambda \in \mathbb{R}$, $\{x \in X | f(x) \leq \lambda\}$ is closed in X;

(2) *upper semi-continuous* if $-f$ is lower semi-continuous, i.e., for all $\lambda \in \mathbb{R}$, $\{x \in X : f(x) \geq \lambda\}$ is closed in X.

Let X and Y be topological spaces and $T : X \to 2^Y$ be a set-valued mapping. Then T is said to be *upper* (*resp., lower*) *semi-continuous* at $x_0 \in X$ if, for each open set G in Y with $T(x_0) \subset G$ (resp., $T(x_0) \cap G \neq \emptyset$), there exists an open neighborhood U of x_0 in X such that $T(x) \subset G$ (resp., $T(x) \cap G \neq \emptyset$) for all $x \in U$. Moreover, T is said to be *continuous* at the point $x_0 \in X$ if T is both upper semi-continuous and lower semi-continuous at $x_0 \in X$. T is said to be *continuous* on X if T is continuous at each point x_0 of X.

Let X be a convex set in a vector space E. Then a mapping $f : X \to \mathbb{R}$ is *convex* if, for all $x, y \in X$ and $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

The following definition was given by Border [3]:

Definition 2.1. (Escaping Sequences) Let X be a topological space such that $X = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact subsets of X. Then a sequence $\{x_n\}_{n=1}^{\infty}$ is said to be *escaping* from X relative to $\{C_n\}_{n=1}^{\infty}$ if, for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $x_k \notin C_n$ for all $k \geq m$.

In this paper, we obtain some general theorems on solutions for a new class of generalized bi-quasi-variational inequality problems for quasi-pseudo-monotone type II operators defined in non-compact settings in topological vector spaces.

To obtain these results on GBQVI for quasi-pseudo-monotone type II operators in non-compact settings, we use the concept of escaping sequences introduced by Border [3] with applications of Chowdhury and Tan's result [Theorem 2.2 below] on generalized bi-quasi-variational inequality problems for quasi-pseudo-monotone type II operators on compact sets ([14]).

First, we state the following result of Chowdhury and Tan in [14] (Theorem 3.1):

Theorem 2.2. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty compact convex subset of E and F be a Hausdorff topological vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a continuous bilinear functional. Suppose that*

(a) *$S : X \to 2^X$ is upper semi-continuous such that each $S(x)$ is closed and convex;*

(b) *$h : X \to \mathbb{R}$ is convex and continuous;*

(c) *$T : X \to 2^F$ is an h-quasi-pseudo-monotone type II (resp., strongly h-quasi-pseudo-monotone type II) operator and is upper semi-continuous such that each $T(x)$ is compact (resp., weakly compact) and convex and $T(X)$ is strongly bounded, i.e., bounded in the strong topology of F;*

(d) *$M : X \to 2^F$ is an upper semi-continuous mapping such that each $M(x)$ is weakly compact and convex;*

(e) *the set $\Sigma = \{y \in X : \sup_{x \in S(y)}(\inf_{f \in M(y)} \inf_{u \in T(y)} Re\langle f - u, y - x\rangle + h(y) - h(x)) > 0\}$ is open in X.*

Then there exists a point $\hat{y} \in X$ such that

(1) *$\hat{y} \in S(\hat{y})$;*

(2) *there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{f} - \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Moreover, if $S(x) = X$ for all $x \in X$, then E is not required to be locally convex and, if $T \equiv 0$, then the continuity assumption on $\langle \cdot, \cdot \rangle$ can be weakened to the assumption that, for each $f \in F$, the mapping $x \mapsto \langle f, x \rangle$ is continuous (resp., weakly continuous) on X.

Applying Theorem 2.1, Chowdhury and Tan obtained the following result in [14] (Theorem 3.2):

Theorem 2.3. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty compact convex subset of E and F be a vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and, for each $f \in F$, the mapping $x \mapsto \langle f, x \rangle$ is continuous on E. Equip F with the strong topology $\delta\langle F, E \rangle$. Suppose that*

(a) *$S : X \to 2^X$ is a continuous map such that each $S(x)$ is closed and convex;*

(b) *$h : X \to \mathbb{R}$ is convex and continuous;*

(c) *$T : X \to 2^F$ is an h-quasi-pseudo-monotone type II (resp., strongly h-quasi-pseudo-monotone type II) operator and is an upper semi-continuous mapping such that each $T(x)$ is strongly, i.e., $\delta\langle F, E \rangle$-compact and convex (resp., weakly, i.e., $\sigma\langle F, E \rangle$-compact and convex);*

(d) *$M : X \to 2^F$ is an upper semi-continuous mapping such that each $M(x)$ is weakly, i.e., $\sigma\langle F, E \rangle$-compact and convex and, also, for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)}[\inf_{f \in M(y)} \inf_{u \in T(y)} Re\langle f - u, y - x \rangle + h(y) - h(x)] > 0\}$, there exists a point x in $S(y)$ with $\inf_{f \in M(y)} \inf_{u \in T(y)} Re\langle f - u, y - x \rangle + h(y) - h(x) > 0$.*

Then there exists a point $\hat{y} \in X$ such that

(1) *$\hat{y} \in S(\hat{y})$;*

(2) *there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with*

$$Re\langle \hat{f} - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$$

for all $x \in S(\hat{y})$.

Moreover, if $S(x) = X$ for all $x \in X$, then E is not required to be locally convex.

For the proof of Theorem 2.3, we refer to [14].

3. Existence Theorems for Generalized Bi-Quasi-Variational Inequalities of Quasi-Pseudo-Monotone Type II Operators

In this section, we obtain our main results for existence theorems on non-compact generalized bi-quasi-variational inequalities of quasi-pseudo-monotone type II operators. To obtain these results, we mainly use the concept of escaping sequences given in Definition 2.1 and apply Theorem 2.2.

Now, we establish our main result as follows:

Theorem 3.1. *Let E be a locally convex Hausdorff topological vector space over Φ, X a non-empty (convex) subset of E such that $X = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of X and let F be a vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and for each $f \in F$, the mapping $x \mapsto \langle f, x \rangle$ is continuous on X. Equip F with the strong topology $\delta\langle F, E \rangle$. Suppose that*

(a) *$S : X \to 2^X$ is a continuous map such that, for each $x \in X$, $S(x)$ is a closed and convex subset of X and, for each $n \in \mathbb{N}$, $S(x) \subset C_n$ for all $x \in C_n$;*

(b) *$h : X \to \mathbb{R}$ is convex and continuous;*

(c) *$T : X \to 2^F$ is an h-quasi-pseudo-monotone type II (resp., strongly h-quasi-pseudo-monotone type II) operator and is an upper semi-continuous map such that each $T(x)$ is strongly, i.e., $\delta\langle F, E \rangle$-compact and convex (resp., weakly, i.e., $\sigma\langle F, E \rangle$-compact and convex);*

(d) $M : X \to 2^F$ *is an upper semi-continuous mapping such that each* $M(x)$ *is weakly, i.e.,* $\sigma\langle F, E\rangle$*-compact and convex; also for each* $y \in \Sigma = \{y \in X : \sup_{x\in S(y)}[\inf_{f\in M(y)} \inf_{u\in T(y)} Re\langle f - u, y - x\rangle + h(y) - h(x)] > 0\}$, *there exists a point* x *in* $S(y)$ *with* $\inf_{f\in M(y)} \inf_{u\in T(y)} Re\langle f - u, y - x\rangle + h(y) - h(x) > 0$;

(e) *for each sequence* $\{y_n\}_{n=1}^{\infty}$ *in* X, *with* $y_n \in C_n$ *for each* $n \in \mathbb{N}$, *which is escaping from* X *relative to* $\{C_n\}_{n=1}^{\infty}$, *either there exists* $n_0 \in \mathbb{N}$ *such that* $y_{n_0} \notin S(y_{n_0})$ *or there exist* $n_0 \in \mathbb{N}$ *and* $x_{n_0} \in S(y_{n_0})$ *such that*

$$\min_{f\in M(y_{n_0})} \min_{w\in T(y_{n_0})} Re\langle f - w, y_{n_0} - x_{n_0}\rangle + h(y_{n_0}) - h(x_{n_0}) > 0 \tag{Ω}$$

holds.

Then there exists a point $\hat{y} \in X$ *such that*

(1) $\hat{y} \in S(\hat{y})$;

(2) *there exist a point* $\hat{f} \in M(\hat{y})$ *and a point* $\hat{w} \in T(\hat{y})$ *such that*

$$Re\langle \hat{f} - \hat{w}, \hat{y} - x\rangle \le h(x) - h(\hat{y})$$

for all $x \in S(\hat{y})$.

Moreover, if $S(x) = X$ *for all* $x \in X$, E *is not required to be locally convex.*

Proof. Let us fix an arbitrary $n \in \mathbb{N}$. We note that C_n is a nonempty compact and convex subset of E. Let us define the mappings $S_n : C_n \to 2^{C_n}$, $h_n : C_n \to \mathbb{R}$ and M_n, $T_n : C_n \to 2^F$ by

$$S_n(x) = S(x), \quad h_n(x) = h(x), \quad M_n(x) = M(x), \quad T_n(x) = T(x),$$

respectively, for all $x \in C_n$, i.e.,

$$S_n = S|_{C_n}, \quad h_n = h|_{C_n}, \quad M_n = M|_{C_n}, \quad T_n = T|_{C_n},$$

respectively. Then, by Theorem 2.2, there exist a point $\hat{y}_n \in C_n$, a point $\hat{f}_n \in M(\hat{y}_n) = M_n(\hat{y}_n)$ and a point $\hat{w}_n \in T(\hat{y}_n) = T_n(\hat{y}_n)$ such that

(1)′ $\hat{y}_n \in S_n(\hat{y}_n)$;

(2)′ $Re\langle \hat{f}_n - \hat{w}_n, \hat{y}_n - x\rangle \le h(x) - h(\hat{y}_n)$ for all $x \in S_n(\hat{y}_n)$.

Note that $\{\hat{y}_n\}_{n=1}^{\infty}$ is a sequence in $X = \bigcup_{n=1}^{\infty} C_n$ with $\hat{y}_n \in C_n$ for each $n \in \mathbb{N}$.

We consider two cases as follows:

Case 1: $\{\hat{y}_n\}_{n=1}^{\infty}$ is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$. Then, by the hypothesis (e), there exists $n_0 \in \mathbb{N}$ such that $\hat{y}_{n_0} \notin S(\hat{y}_{n_0}) = S_{n_0}(\hat{y}_{n_0})$, which contradicts (1)′ or there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in S(\hat{y}_{n_0}) = S_{n_0}(\hat{y}_{n_0})$ such that

$$\min_{f\in M(\hat{y}_{n_0})} \min_{w\in T(\hat{y}_{n_0})} Re\langle f - w, \hat{y}_{n_0} - x_{n_0}\rangle + h(\hat{y}_{n_0}) - h(x_{n_0}) > 0,$$

which contradicts (2)′.

Case 2: $\{\hat{y}_n\}_{n=1}^{\infty}$ is not escaping from X relative to $\{C_n\}_{n=1}^{\infty}$. Then there exist $n_1 \in \mathbb{N}$ and a subsequence $\{\hat{y}_{n_j}\}_{j=1}^{\infty}$ of $\{\hat{y}_n\}_{n=1}^{\infty}$ such that $\hat{y}_{n_j} \in C_{n_1}$ for all $j = 1, 2, \cdots$. Since C_{n_1} is compact, there exist a subnet $\{\hat{z}_\alpha\}_{\alpha\in\Gamma}$ of $\{\hat{y}_{n_j}\}_{j=1}^{\infty}$ and $\hat{y} \in C_{n_1} \subset X$ such that $\hat{z}_\alpha \to \hat{y}$. For each $\alpha \in \Gamma$, let $\hat{z}_\alpha = \hat{y}_{n_\alpha}$, where $n_\alpha \to \infty$. Then, according to our choice of $\hat{y}_{n_\alpha}$ in C_{n_α}, there exist a point $\hat{f}_{n_\alpha} \in M_{n_\alpha}(\hat{y}_{n_\alpha}) = M(\hat{y}_{n_\alpha})$ and a point $\hat{w}_{n_\alpha} \in T_{n_\alpha}(\hat{y}_{n_\alpha}) = T(\hat{y}_{n_\alpha})$ such that

(1)″ $\hat{y}_{n_\alpha} \in S_{n_\alpha}(\hat{y}_{n_\alpha}) = S(\hat{y}_{n_\alpha})$;

(2)″ $Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x\rangle + h(\hat{y}_{n_\alpha}) - h(x) \le 0$ for all $x \in S_{n_\alpha}(\hat{y}_{n_\alpha}) = S(\hat{y}_{n_\alpha})$.

Since $n_\alpha \to \infty$, there exists $\alpha_1 \in \Gamma$ such that $n_\alpha \ge n_1$ for all $\alpha \ge \alpha_1$. Thus $C_{n_1} \subset C_{n_\alpha}$ for all $\alpha \ge \alpha_1$. From (1)″, we have $(\hat{y}_{n_\alpha}, \hat{y}_{n_\alpha}) \in G(S)$ for all $\alpha \in \Gamma$. Since S is upper semicontinuous with closed values, $G(S)$ is closed in $X \times X$ and so it follows that $\hat{y} \in S(\hat{y})$.

Moreover, since $\{\hat{f}_{n_\alpha}\}_{\alpha\geq\alpha_1}$ and $\{\hat{w}_{n_\alpha}\}_{\alpha\geq\alpha_1}$ are nets in the compact sets $\cup_{x\in C_{n_1}} M(x)$ and $\cup_{x\in C_{n_1}} T(x)$, respectively, without loss of generality, we may assume that the nets $\{\hat{f}_{n_\alpha}\}_{\alpha\in\Gamma}$ and $\{\hat{w}_{n_\alpha}\}_{\alpha\in\Gamma}$ converge to a point $\hat{f} \in \cup_{x\in C_{n_1}} M(x)$ and a $\hat{w} \in \cup_{x\in C_{n_1}} T(x)$, respectively. Note that M has a closed graph. Also, since T has a closed graph on C_{n_1}, $\hat{f} \in M(\hat{y})$ and $\hat{w} \in T(\hat{y})$.

Let $x \in S(\hat{y})$ be arbitrarily fixed. Let $n_2 \geq n_1$ be such that $x \in C_{n_2}$. Since S is lower semi-continuous at $\hat{y}$, without loss of generality, we may assume that, for each $\alpha \in \Gamma$, there exists $x_{n_\alpha} \in S(\hat{y}_{n_\alpha})$ such that $x_{n_\alpha} \to x$. By (2)′′, we have

$$Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle + h(\hat{y}_{n_\alpha}) - h(x_{n_\alpha}) \leq 0$$

for all $\alpha \in \Gamma$. Note that $\hat{f}_{n_\alpha} - \hat{w}_{n_\alpha} \to \hat{f} - \hat{w}$ in $\delta\langle F, E\rangle$ and $\{\hat{y}_{n_\alpha} - x_{n_\alpha}\}_{\alpha\in\Gamma}$ is a net in the compact (and hence bounded) set $C_{n_2} - \cup_{y\in C_{n_2}} S(y)$. Thus, for each $\epsilon > 0$, there exists $\alpha_2 \geq \alpha_1$ such that

$$|Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha} - \hat{f} - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle| < \epsilon/2$$

for all $\alpha \geq \alpha_2$. Since $\langle \hat{f} - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle \to \langle \hat{f} - \hat{w}, \hat{y} - x\rangle$, there exists $\alpha_3 \geq \alpha_2$ such that

$$|Re\langle \hat{f} - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle - Re\langle \hat{f} - \hat{w}, \hat{y} - x\rangle| < \epsilon/2$$

for all $\alpha \geq \alpha_3$. Thus it follows that, for $\alpha \geq \alpha_3$,

$$\begin{aligned} &|Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle - Re\langle \hat{f} - \hat{w}, \hat{y} - x\rangle| \\ &\leq |Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha} - \hat{f} - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle| + |Re\langle \hat{f} - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha} - (\hat{y} - x)\rangle| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Thus we have

$$\lim_\alpha Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle = Re\langle \hat{f} - \hat{w}, \hat{y} - x\rangle.$$

Since h is continuous, we have

$$\begin{aligned} &Re\langle \hat{f} - \hat{w}, \hat{y} - x\rangle + h(\hat{y}) - h(x) \\ &= \lim_\alpha [Re\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle + h(\hat{y}_{n_\alpha}) - h(x_{n_\alpha})] \\ &\leq 0. \end{aligned}$$

This completes the proof. □

Corollary 3.2. *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a nonempty closed convex subset of E and F be a vector space over Φ. Let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle\cdot,\cdot\rangle$ separates points in F and, for each $f \in F$, the mapping $x \mapsto \langle f, x\rangle$ is continuous on X. Equip F with the strong topology $\delta\langle F, E\rangle$. Suppose that*

(a) *$S : X \to 2^X$ is a weakly continuous mapping such that, for each $x \in X$, $S(x)$ is a closed and convex subset of X;*

(b) *$h : X \to \mathbb{R}$ is convex and (weakly) continuous;*

(c) *$T : X \to 2^F$ is a strongly h-quasi-pseudo-monotone type II operator and is weakly upper semi-continuous such that each $T(x)$ is $\sigma\langle F, E\rangle$-compact and convex;*

(d) *$M : X \to 2^F$ is an upper semi-continuous mapping such that each $M(x)$ is weakly, i.e., $\sigma\langle F, E\rangle$-compact and convex and, for each $y \in \Sigma$, there exists a point x in $S(y)$ with $\inf_{f\in M(y)} \inf_{w\in T(y)} Re\langle f - w, y - x\rangle + h(y) - h(x) > 0$, where*

$$\Sigma = \{y \in X : \sup_{x\in S(y)} [\inf_{f\in M(y)} [\inf_{w\in T(y)} Re\langle f - w, y - x\rangle + h(y) - h(x)] > 0\}.$$

Suppose, further, that

(e) *there exists an increasing sequence $\{r_n\}_{n=1}^\infty$ of positive numbers with $r_n \to \infty$ such that $S(x) \subset C_n$ for each $x \in C_n$ and each $n \in \mathbb{N}$ where $C_n = \{x \in X : \|x\| \leq r_n\}$;*

(f) *for each sequence $\{y_n\}_{n=1}^\infty$ in X, with $\|y_n\| \to \infty$, either there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in S(y_{n_0})$ such that (Ω) holds.*

Then there exist a point $\hat{y} \in X$ such that

(1) $\hat{y} \in S(\hat{y})$;

(2) *there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that*

$$Re\langle \hat{f} - \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y})$$

for all $x \in S(\hat{y})$.

Proof. In proving this corollary, we follow the similar method of proof of Corollary 1 in [12]. We equip E with the weak topology. Then C_n is weakly compact convex for each $n \in \mathbb{N}$ such that $X = \cup_{n=1}^{\infty} C_n$. Now, if $\{y_n\}_{n=1}^{\infty}$ is a sequence in X with $y_n \in C_n$ for each $n = 1, 2, \cdots$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, then $\|y_n\| \to \infty$. By (b), either there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in S(y_{n_0})$ such that (Ω) holds. Thus all the hypotheses of Theorem 3.1 are satisfied and so the conclusion follows. This completes the proof. □

By taking $M \equiv 0$ and replacing T by $-T$ in Theorem 3.1, we obtain the following result:

Corollary 3.3. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty* (*convex*) *subset of E such that $X = \cup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of X, and F be a vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle \cdot, \cdot \rangle$ separates points in F and, for each $f \in F$, the mapping $x \mapsto \langle f, x\rangle$ be continuous on X. Equip F with the strong topology $\delta\langle F, E\rangle$. Suppose that* (a)–(c) *and* (e) *of Theorem 3.1 hold.*
Then there exist a point $\hat{y} \in X$ and a point $\hat{w} \in T(\hat{y})$ such that

(1) $\hat{y} \in S(\hat{y})$;

(2) $Re\langle \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y})$ *for all $x \in S(\hat{y})$.*

Moreover, if $S(x) = X$ for all $x \in X$, then E is not required to be locally convex.

4. Existence Theorems for Generalized Bi-Complementarity Problems for Quasi-Pseudo-Monotone Type II Operators

In this section, as an application of Theorem 3.1, we obtain the existence theorem on generalized bi-complementarity problem for quasi-pseudo-monotone type II operators in non-compact settings.

By modifying the proof of the result observed by Fang (for example, see [6], pp. 213, and [18], pp. 59), the following result was obtained in Chowdhury [15], Lemma 4.4.10. Note that E is not required to be Hausdorff.

Lemma 4.1. *Let X be a cone in a topological vector space E over Φ and F be a vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional. Let M, $T : X \to 2^F$ be two mappings. Then the following are equivalent:*

(1) *There exist $\hat{y} \in X$, $\hat{f} \in M(\hat{y})$ and $\hat{w} \in T(\hat{y})$ such that*

$$Re\langle \hat{f} - \hat{w}, \hat{y} - x\rangle \leq 0$$

for all $x \in X$.

(2) *There exist $\hat{y} \in X$, $\hat{f} \in M(\hat{y})$ and $\hat{w} \in T(\hat{y})$ such that*

$$Re\langle \hat{f} - \hat{w}, \hat{y}\rangle = 0, \quad \hat{f} - \hat{w} \in \widehat{X}.$$

When X is a cone in E, by applying Lemma 4.1 and Theorem 3.1 with $h \equiv 0$ and $S(x) = X$ for all $x \in X$, we have immediately the following existence theorem for a generalized bi-complementarity problem of quasi-pseudo-monotone type II operators:

Theorem 4.2. *Let E be a Hausdorff topological vector space over Φ, X be a cone in E such that $X = \cup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of X, and F be a vector space over Φ. Let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle\cdot,\cdot\rangle$ separates points in F and, for each $f \in F$, the mapping $x \mapsto \langle f, x\rangle$ is continuous on X. Equip F with the strong topology $\delta\langle F, E\rangle$. Suppose that*

(a) *$T : X \to 2^F$ is a quasi-pseudo-monotone type II (resp., strongly quasi-pseudo-monotone type II) operator and is upper semi-continuous such that each $T(x)$ is strongly, i.e., $\delta\langle F, E\rangle$-compact and convex (resp., weakly, i.e., $\sigma\langle F, E\rangle$-compact and convex);*

(b) *$M : X \to 2^F$ is an upper semi-continuous mapping such that each $M(x)$ is weakly, i.e., $\sigma\langle F, E\rangle$-compact and convex and, for each $y \in \Sigma$, there exists a point x in X with $\inf_{f\in M(y)} \inf_{w\in T(y)} Re\langle f - w, y - x\rangle > 0$, where*

$$\Sigma = \{y \in X : \sup_{x\in X}[\inf_{f\in M(y)}[\inf_{w\in T(y)} Re\langle f - w, y - x\rangle] > 0\};$$

(c) *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X with $y_n \in C_n$ for each $n \in \mathbb{N}$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in X$ such that*

$$\min_{f\in M(y_{n_0})} \min_{w\in T(y_{n_0})} Re\langle f - w, y_{n_0} - x_{n_0}\rangle > 0. \qquad (\Omega')$$

Then there exist a point $\hat{y} \in X$, a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{f} - \hat{w}, \hat{y}\rangle = 0, \quad \hat{f} - \hat{w} \in \widehat{X}.$$

Corollary 4.3. *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a closed cone in E and F be a vector space over Φ. Let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle\cdot,\cdot\rangle$ separates points in F and, for each $f \in F$, the mapping $x \mapsto \langle f, x\rangle$ is continuous on X. Equip F with the strong topology $\delta\langle F, E\rangle$. Suppose that*

(a) *$T : X \to 2^F$ is a strongly quasi-pseudo-monotone type II operator and is weakly upper semi-continuous such that each $T(x)$ is $\sigma\langle F, E\rangle$-compact and convex;*

(b) *$M : X \to 2^F$ is an upper semi-continuous mapping such that each $M(x)$ is weakly, i.e., $\sigma\langle F, E\rangle$-compact and convex and, for each $y \in \Sigma$, there exists a point x in X with $\inf_{f\in M(y)} \inf_{w\in T(y)} Re\langle f - w, y - x\rangle > 0$, where*

$$\Sigma = \{y \in X : \sup_{x\in X}[\inf_{f\in M(y)}[\inf_{w\in T(y)} Re\langle f - w, y - x\rangle] > 0\}.$$

Suppose, further, that

(c) *there exists an increasing sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers with $r_n \to \infty$ and $C_n = \{x \in X : \|x\| \le r_n\}$ for each $n \in \mathbb{N}$;*

(d) *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X, with $\|y_n\| \to \infty$, there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in X$ such that*

$$\min_{f\in M(y_{n_0})} \min_{w\in T(y_{n_0})} Re\langle f - w, y_{n_0} - x_{n_0}\rangle > 0.$$

Then there exist $\hat{y} \in X$, $\hat{f} \in M(\hat{y})$ and $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{f} - \hat{w}, \hat{y}\rangle = 0, \quad \hat{f} - \hat{w} \in \widehat{X}.$$

Proof. In proving this corollary, we follow the similar method of proof of Corollary 3 in [15]. We equip E with the weak topology. Then C_n is weakly compact convex for each $n \in \mathbb{N}$ such that $X = \cup_{n=1}^{\infty} C_n$. Now, if $\{y_n\}_{n=1}^{\infty}$ is a sequence in X with $y_n \in C_n$ for each $n = 1, 2, \cdots$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, then $\|y_n\| \to \infty$. Hence, by the hypothesis, there exists $n_0 \in \mathbb{N}$ and $x_{n_0} \in X$ such that (Ω') holds. Thus all the hypotheses of Theorem 4.2 are satisfied and so the conclusion follows. This completes the proof. $\square$

Remark 4.4.

(1) Theorems 3.1 of this paper is a further extension of the results obtained in [19] into generalized bi-quasi-variational inequalities of quasi-pseudo-monotone type II operators and strongly quasi-pseudo-monotone type II operators on non-compact sets.

(2) In 1989, Shih and Tan [19] obtained results on generalized bi-quasi-variational inequalities in locally convex topological vector spaces and their results were obtained on compact sets where the set-valued mappings were either lower semi-continuous or upper semi-continuous.

(3) For more details on variational inequalities and quasi-variational inequalities, completely generalized multi-valued co-variational inequalities, generalized nonlinear vector quasi-variational-like inequalities with set-valued mappings and others, refer to [1], [16] and [17].

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