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Convex Sets in Proximal Relator Spaces

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Abstract. This article introduces convex sets in finite-dimensional normed linear spaces equipped with a proximal relator. A proximal relator is a nonvoid family of proximity relations $\mathcal{R}_\delta$ (called a proximal relator) on a nonempty set. A normed linear space endowed with $\mathcal{R}_\delta$ is an extension of the Száz relator space. This leads to a basis for the study of the nearness of convex sets in proximal linear spaces.

1. Introduction

This article introduces convex sets [9, 13] in finite-dimensional normed linear spaces equipped with a proximal relator (briefly, proximal linear spaces). A *proximal relator* is a nonvoid family of proximity relations $\mathcal{R}_\delta$ (called a proximal relator) on a nonempty set. This form of relator is an extension of a Száz relator [10–12]. For simplicity, we consider only two proximity relations, namely, the Efremovič proximity δ [4] and the proximity δ_S between convex sets in defining $\mathcal{R}_\delta$ [7, 8]. The assumption made here is that each proximal linear space is a topological space that provides the structure needed to define proximity relations. The proximity relation δ_S defines a nearness relation between convex sets useful in many applications.

2. Preliminaries

Let E be a finite-dimensional real normed linear space, $A \subset E$. The Hausdorff distance $D(x, A)$ is defined by $D(x, A) = inf\left\{\left\|x - y\right\| : y \in A\right\}$ and $\left\|x - y\right\|$ is the distance between vectors x and y. The Čech closure [1] of A (denoted by cA) is defined by c$A = \{x \in V : D(x, A) = 0\}$. The sets A and B are proximal (near) (denoted $A\ \delta\ B$) if and only if c$A\ \cap$ c$B \neq \emptyset$.

A subset S in E is convex, provided S is the set of all points in E that are nearest to a point z in S [5, 9]. Let S_z (called Klee-Phelphs convex set) be the set of all points in E having $z \in S$ as the nearest point in S, defined by

$$S_z = \left\{x \in E : \|x - z\| = \underset{y \in S}{inf} \left\|x - y\right\|\right\}.$$

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In effect, S_z is a convex cone with vertex z. This leads to the following useful Lemma.

Lemma 2.1. (Phelps [9, §4]) *If E is an inner product space and $z \in S \subset E$, then S_z is convex.*

Let $S \subset E, x, y \in S$, and S_x, S_y are nonempty Klee-Phelps nearest point sets. From Lemma 2.1, S_x, S_y are convex sets. Next, consider the proximity relation δ_S between convex sets. Sets S_x, S_y are proximal (denoted by $S_x\ \delta_S\ S_y$) if and only if $\|a - b\| = 0$ for some $a \in \text{cl}S_x, b \in \text{cl}S_y$. That is, convex sets S_x, S_y are near, provided the convex set S_x has at least one point a that matches some point b in S_y. In effect, S_x, S_y are proximal if and only if $cl(S_x) \cap cl(S_y) \neq \emptyset$.

In general, a subset $K \subset E$ that contains every segment whose endpoints belong to K is convex [6]. Convex sets A, B are near, provided A and B have at least one common point. Convex sets A, B are remote (denoted by $A\ \underline{\delta}_S\ B$), provided $\|a - b\| \neq 0$ for all $a \in A, b \in B$.

3. Main Results

In a real normed linear space endowed with a proximal relator $\mathcal{R}_\delta$ (briefly, *proximal linear space*), we obtain the following results.

Theorem 3.1. *Let $(E, \|\cdot\|, \mathcal{R}_\delta)$ be a proximal linear space, $S_x, S_y \subset E$. Then*
1^o $z \in S_x \cap S_y$ implies $S_x\ \delta_S\ S_y$.
2^o Let S_x, S_y be convex sets in E. $S_x\ \delta\ S_y$, if and only if, $S_x\ \delta_S\ S_y$.
3^o Let A, B be convex sets in E. $A\ \delta\ B$, if and only if, $A\ \delta_S\ B$.
4^o Let A, B, C be convex sets in E. $(A \cup B)\ \delta\ C$ implies $(A \cup B)\ \delta_S\ C$.
5^o $clA\ \delta\ clB$ implies $clA\ \delta_S\ clB$.

Proof.
2^o: $A\ \delta_S\ B$, *i.e.*, $\|a - b\| = 0$ for some $a \in A, b \in B \Leftrightarrow a \in \text{cl}A \cap \text{cl}B \Leftrightarrow A\ \delta\ B$.
$2^o \Rightarrow 1^o$.
3^o: $(A \cup B)\ \delta\ C$ provided $\text{cl}(A \cup B) \cap \text{cl}C \neq \emptyset$. Consequently, there is at least one point $z \in \text{cl}(A \cup B) \cap \text{cl}C$ such that $\|z - y\| = 0$ for some $z \in \text{cl}(A \cup B), y \in \text{cl}C$. Hence, $(A \cup B)\ \delta_S\ C$.
$3^o \Rightarrow 4^o$. □

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in S \subset E, x \cdot y = x_1y_1 + \cdots + x_ny_n$ (dot product), $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ (norm of x). The angle θ between x and y is defined by

$$\theta = cos^{-1}\left[\frac{x \cdot y}{\|x\|\,\|y\|}\right].$$

Each $\theta \in \mathbb{R}$. Let S_θ be the set of angles between points in E that are nearest to the angle $\theta \in S$, *i.e.*,

$$S_\theta = \left\{\theta' \in \mathbb{R} : \|\theta' - \theta\| = \inf_{y \in S_\theta} \|\theta' - y\|\right\}.$$

Theorem 3.2. *Let U, V be finite-dimensional proximal linear spaces, $A \subset U, B \subset V$. Let $A_\theta, B_{\theta^\angle}$ be sets of angles nearest $\theta, \theta^\angle$ between points in U, V, respectively. Then*

$$\|x - y\| = 0 \textit{ for some } x \in A_\theta, y \in B_{\theta^\angle} \textit{ if and only if } A_\theta\ \delta_S\ B_{\theta^\angle}.$$

Proof. From Lemma 2.1, $A_\theta, B_{\theta^\angle}$ are convex sets. $A_\theta\ \delta\ B_{\theta^\angle} \Leftrightarrow A_\theta\ \delta_S\ B_{\theta^\angle}$ (from Theorem 3.1) if and only if $\|x - y\| = 0$ for some $x \in A_\theta, y \in B_{\theta^\angle}$. □

Remark 3.3. *Since no assumption is made about the dimensions of the proximal linear spaces in Theorem 3.2, this means that subsets in spaces with unequal dimensions can be compared. Let $x, y \in U, x', y' \in V$ and let $\theta(x, y), \theta(x', y')$ be angles between points in U, V, respectively. Further, let $A \subset U, B \subset V$. By adding the condition that $0 < \|x - y\| < \varepsilon$ and $0 < \|x' - y'\| < \varepsilon$ and finding $A_{\theta(x,y)}\ \delta_S\ B_{\theta(x',y')}$, it is then possible to identify subsets with similar shapes in proximal linear spaces. In effect, we thereby obtain a means of classifying shapes in such spaces.* ∎

In a proximal real linear space E, the neighbourhood of a point $x \in X$ (denoted by $N_{x,\varepsilon}$), for $\varepsilon > 0$, is defined by

$$N_{x,\varepsilon} = \left\{ y \in X : \|x - y\| < \varepsilon \right\}.$$

Let $A \subset E$ be a convex set. The interior of a set A (denoted by $\text{int}(A)$) and boundary of A (denoted by $\text{bdy}(A)$) in E are defined by

$$\text{int}(A) = \{x \in X : N_{x,\varepsilon} \subseteq A\}.$$
$$\text{bdy}(A) = \text{cl}(A) \setminus \text{int}(A).$$

A set A has a *natural strong inclusion* in a set B associated with δ [2, 3] (denoted by $A \ll_\delta B$), provided $A \subset \text{int}(\text{cl}(\text{int}B))$, *i.e.*, $A\ \underline{\delta}\ (X \setminus \text{cl}(\text{int}B))$ (A is far from the complement of $\text{cl}(\text{int}B)$). This leads to the following results.

Theorem 3.4. *Let E be a proximal linear space and let $A, B \subset E$ be convex sets. Then*
1^o $A \ll_\delta B \Rightarrow A\ \delta_S\ B$.
2^o Let S_x, S_y be convex sets in E. $S_x \ll_\delta S_y \Rightarrow S_x\ \delta_S\ S_y$.
3^o $clA \ll_\delta clB \Leftrightarrow clA\ \delta_S\ clB$.
4^o Let S_x, S_y be convex sets in E. $clS_x \ll_\delta clS_y \Leftrightarrow clS_x\ \delta_S\ clS_y$.

Proof.
1^o: $A \ll_\delta B \Leftrightarrow x \in \text{int}(\text{cl}(\text{int}B))$ for each $x \in A \Leftrightarrow A\ \delta_S\ B$.
$1^o \Rightarrow 2^o$.
3^o: Symmetric with the proof of 1^o.
$3^o \Rightarrow 4^o$. □

Theorem 3.5. *Let E be a proximal linear space, $A \subset X$. Then $A \subseteq cl(A)$.*

Proof. Let $x \in (X \setminus A)$ such that $x = a$ for some $a \in \text{cl}A$. Consequently, $x \in \text{cl}A$. Hence, $A \subseteq \text{cl}A$. □

Theorem 3.6. *Let E be a proximal linear space, $A \subset X$. Then*
1^0 $A \subseteq int(A) \subset cl(A)$.
2^0 $bdy(A) \subseteq cl(A)$.

Proof. Immediate from the definition of $\text{int}(A), \text{bdy}(A), \text{cl}(A)$. □

Theorem 3.7. *Let E be a proximal linear space, $A \subset X$. Then $int(A) \cup bdy(A) \subseteq cl(A)$.*

Proof. Immediate Theorem 3.6. □

Theorem 3.8. *Let E be a proximal linear space, $A \subset E$. Then $cl(A) = int(A) \cup bdy(A)$.*

Proof.

$$\begin{aligned} A \subseteq \text{cl}(A)[\text{Theorem 3.5}] &\Rightarrow \text{int}(A) \cup \text{bdy}(A) \subseteq \text{cl}(A)[\text{Theorem 3.7}] \\ &\Rightarrow \text{bdy}(A) \subset \text{cl}(A), \text{from Theorem 3.6, and} \\ &\quad\ \text{int}(A) \subset \text{cl}(A), \text{from Theorem 3.6} \\ &\Rightarrow \text{cl}(A) \subseteq \text{int}(A) \cup \text{bdy}(A). \end{aligned}$$

□

If E be a proximal linear space, members of the families

$$\mathcal{E} = \{A \subset E : \mathrm{int}(A) \neq \emptyset\} \text{ and}$$
$$\mathcal{D} = \{A \subset E : \mathrm{cl}(A) = E\}$$

are called *fat* and *dense* collections of subsets of E, respectively. Let $\mathrm{cx}\mathcal{E}, \mathrm{cx}\mathcal{D}$ denote fat and dense collections of convex subsets of E. Extensions of $\mathcal{E}, \mathcal{D}$ (denoted by $\mathrm{ext}\mathcal{E}, \mathrm{ext}\mathcal{D}$) are introduced in Theorem 3.9.

Theorem 3.9. *Let E be a proximal linear space. Let $S_x, S_y \subset E$ be convex sets. Then*
1^0 $ext\mathcal{E} = \{A \subset E : \forall B \in \mathcal{D},\ int(A)\ \delta\ cl(B)\}$.
2^0 $ext\mathcal{D} = \{A \subset E : \forall B \in ext\mathcal{E},\ clA\ \delta\ clB\}$.
3^0 $cx\mathcal{E} = \left\{S_x \subset E : \forall S_y \in \mathcal{D},\ int(S_x)\ \delta\ cl(S_y)\right\}$.
4^0 $cx\mathcal{D} = \left\{S_x \subset E : \forall S_y \in ext\mathcal{E},\ cl(S_x)\ \delta\ cl(S_y)\right\}$.

Proof.
1^o: Let $A \in \mathrm{ext}\mathcal{E}, B \in \mathcal{D}$. Then $\mathrm{cl}B = E$. Consequently, $\mathrm{int}(A) \cap \mathrm{cl}B \neq \emptyset$. Hence, $\mathrm{int}(A)\ \delta\ \mathrm{int}(B)$.
2^o: Symmetric with the proof of 1^o.
$1^o \Rightarrow 3^o$.
$2^o \Rightarrow 4^o$. □

Theorem 3.10. *Let E be a proximal linear space. Then*
1^0 $A \in cx\mathcal{E}, B \in \mathcal{D} \Rightarrow\ int(A)\ \delta_S\ cl(B)$.
2^0 $A \in cx\mathcal{D}, B \in ext\mathcal{E} \Rightarrow\ cl(A)\ \delta_S\ cl(B)$.

Proof.
1^o: Let $A \in \mathrm{ext}\mathcal{E}, B \in \mathcal{D}$. Consequently, $\mathrm{int}(A)\ \delta\ \mathrm{cl}(B)$ from Theorem 3.9. Hence, $\mathrm{int}(A)\ \delta_S\ \mathrm{cl}(B)$ from Theorem 3.1.
2^o: Symmetric with the proof of 1^o. □

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