Filomat 30:13 (2016), 3403–3409
DOI 10.2298/FIL1613403D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Singular Acyclic Matrices of Even Order with a P-Set of Maximum Size

Zhibin Du[a], Carlos M. da Fonseca[b]

[a]*School of Mathematics and Statistics, Zhaoqing University, Zhaoqing 526061, China*
[b]*Department of Mathematics, Kuwait University, Safat 13060, Kuwait*

Abstract. Let $m_A(0)$ denote the nullity of a given n-by-n symmetric matrix A. Set $A(\alpha)$ for the principal submatrix of A obtained after deleting the rows and columns indexed by the nonempty subset α of $\{1,\dots,n\}$. When $m_{A(\alpha)}(0) = m_A(0) + |\alpha|$, we call α a P-set of A. The maximum size of a P-set of A is denoted by $P_s(A)$. It is known that $P_s(A) \leqslant \left\lfloor \frac{n}{2} \right\rfloor$ and this bound is not sharp for singular acyclic matrices of even order. In this paper, we find the bound for this case and classify all of the underlying trees. Some illustrative examples are provided.

1. Introduction

At the 2013 ILAS Meeting held in Providence, RI, USA, we first presented the full classification of all of the trees for which there exists a matrix containing a P-set of maximum size. Our characterization did not depend on whether the acyclic matrices were singular or nonsingular. The complete analysis can be found in [4].

Recall that for any symmetric matrix A, of order n, we have always $P_s(A) \leqslant \left\lfloor \frac{n}{2} \right\rfloor$. As we pointed out, interestingly there is no singular acyclic matrix of even order reaching the bound and this was the only case where such situation occurred. At the end of the communication, there was some appealing discussion on what can be said about the trees in that case. In this paper, we answer to this question. With the solution to this problem, we are able to complete the study initiated few years ago of the acyclic matrices with maximal number of P-vertices and maximal size of a P-set.

In the next section, we recollect some results which one can find in [4] and define two particular families of trees. Section 3 is devoted to some definitions and to the development of new tools and considerations essential to our goals. The full characterization comes next and, in the end, we provide three examples of our results.

For the basic notation, which somehow has become standard, the reader is referred to the recent literature in this topic, namely [1–7, 9].

2010 *Mathematics Subject Classification*. Primary 15A18; Secondary 15A48, 05C50
Keywords. trees, singular acyclic matrices, multiplicity, eigenvalues, P-set, P-vertices
Received: 28 September 2014; Accepted: 08 July 2015
Communicated by Francesco Belardo
This work was supported by the National Natural Science Foundation of China (Grant No. 11426199), Guangdong Provincial Natural Science Foundation of China (Grant No. 2014A030310277), and the Department of Education of Guangdong Province Natural Science Foundation of China (Grant No. 2014KQNCX224).
Email addresses: zhibindu@126.com (Zhibin Du), carlos@sci.kuniv.edu.kw (Carlos M. da Fonseca)

2. The Maximum Size of a P-Set

For general symmetric matrices of order n, we have certified in [4] that the maximum size of a P-set of such matrices is $\left\lfloor \frac{n}{2} \right\rfloor$.

Proposition 2.1. *[4, 9] For any symmetric matrix A of order n, we have*

$$P_s(A) \leqslant \left\lfloor \frac{n}{2} \right\rfloor .$$

Furthermore, we also classified all of the trees T for which there exists a matrix $A \in \mathcal{S}(T)$ containing a P-set of maximum size. Recall that $\mathcal{S}(G)$ denotes the set of all symmetric matrices sharing the same graph G. From this definition and the references, loops are allowed. However, what defines the acyclic structure of the graph is incidence of the edges in distinct vertices.

Attaching the path P_2 to a vertex of a tree T is called *adding a pendant P_2 to T.*

Theorem 2.2. *[4, Theorem 4.2, Theorem 5.4] Let T be a tree on $n \geqslant 1$ vertices. The following two conditions are equivalent:*

(a) There exists a matrix $A \in \mathcal{S}(T)$ such that $P_s(A) = \lfloor \frac{n}{2} \rfloor$.

(b) T is a tree obtained from P_2 by sequentially adding pendant P_2's when n is even, while T is a tree obtained from P_1 by sequentially adding pendant P_2's when n is odd.

However, it is worth mentioning that, from [4, Theorem 2.2(i)], for the singular acyclic matrices of even order, the inequality in Proposition 2.1 is strict. As we revealed in the introduction, this time, as a sequel of [4], we will be focus on the singular acyclic matrices of even order, investigating as a consequence the maximum size of a P-set of such matrices as well.

So, first, we establish the upper bound of the maximum size of any P-set of singular acyclic matrices of even order, and later on we will present the corresponding properties for such underlying trees. The proof runs similarly to the proof of [4, Proposition 1.1, Theorem 2.2] and here we skip it to avoid repetitions.

Theorem 2.3. *Let T be a tree on $n \geqslant 2$ vertices, where n is even, and let A be a singular matrix in $\mathcal{S}(T)$. Then*

$$P_s(A) \leqslant \frac{n-2}{2} .$$

Furthermore, if α is a maximal P-set of A, i.e., $|\alpha| = \frac{n-2}{2}$, then one of the following conditions must be satisfied:

(i) $m_A(0) = 2$, and each component of $T(\alpha)$ is trivial and of nullity 1.

(ii) $m_A(0) = 1$, and each component of $T(\alpha)$ is trivial and, with one exception, of nullity 1.

(iii) $m_A(0) = 1$, $T(\alpha)$ contains exactly one edge, and each component of $T(\alpha)$ is of nullity 1.

Next, for even n, we define two sets of trees: $\mathbb{T}_n^1$ and $\mathbb{T}_n^2$. We start with $\mathbb{T}_n^1$. It is the set of trees on n vertices, each of which is a tree obtained from P_2 by sequentially adding pendant P_2's. Observe that each tree in $\mathbb{T}_n^1$ satisfies the condition (b) in Theorem 2.2. Now, let $\mathbb{T}_n^2$ be the set of trees on n vertices, each of which is a tree obtained under the following algorithm:

Step 1: Start from P_1 by adding sequentially pendant P_2's, the resulting tree is denoted by $\hat{T}$.

Step 2: Add an edge between a vertex of $\hat{T}$ and a vertex of P_3.

Step 3: Start from the resulting graph in Step 2 by adding sequentially pendant P_2's.

As we will see soon, the trees in the two sets $\mathbb{T}_n^1$ and $\mathbb{T}_n^2$ are the unique trees of even order for which there exists a singular matrix containing a P-set of maximum size. Furthermore, we explicitly show how to construct these matrices.

3. Old and New Technical Lemmas

We begin this section with two known results.

Lemma 3.1. *[8, Theorem 8] Let T be a tree on $n \geqslant 2$ vertices, and $A \in \mathcal{S}(T)$. For a vertex v in T, if there exists a neighbor, say u, of v in T, such that $m_{A(v,u)}(0) = m_{A(v)}(0) - 1$, then v is a P-vertex of A.*

Lemma 3.2. *[4, Lemma 2.4] Let T be a tree on $n \geqslant 2$ vertices, and $A \in \mathcal{S}(T)$. Suppose that u is a terminal vertex in T with unique neighbor v. If $A[u] = (0)$ or u is a P-vertex of A, then $m_{A(u,v)}(0) = m_A(0)$.*

Given a tree T, if u is a terminal vertex being adjacent to a vertex v of degree 2, and w is the neighbor of v in T different from u, then the subgraph of T induced by the vertices u, v, w is said to be a *pendant P_3 of T*. In this case, we write $P_3 = uvw$.

Lemma 3.3. *Suppose that T is a tree on $n \geqslant 4$ vertices, where n is even, containing a pendant $P_3 = uvw$. Let $\bar{T}$ be the tree obtained from T by deleting the vertices u and v. Suppose that A is a singular matrix in $\mathcal{S}(T)$ such that $P_s(A) = \frac{n-2}{2}$. If $P_s(A[\bar{T}]) < \frac{n-4}{2}$, then $T \in \mathbb{T}_n^2$.*

Proof. Let α be a P-set of A with $|\alpha| = \frac{n-2}{2}$.

Denote by $w_1, \ldots, w_r$ the neighbors of w in T different from v, and T_i, for $1 \leqslant i \leqslant r$, the component containing w_i, among the components of the forest obtained from T by deleting the vertex w. Let n_i denote the number of vertices in T_i, for $1 \leqslant i \leqslant r$.

Observe that $P_s(A[\bar{T}]) < \frac{n-4}{2}$, from [4, Lemma 3.1], implies that exactly one of u, v, w is in α.

Furthermore, from a similar reasoning as in the proof of [4, Lemma 5.3], we can deduce that: exactly one of A_i's, say A_1, has the property $P_s(A[T_1]) = \frac{n_1-1}{2}$, and the others $A[T_i]$'s (i.e., $A[T_i]$, for $2 \leqslant i \leqslant r$) have the property $P_s(A[T_i]) = \frac{n_i}{2}$.

Clearly, $A[T_i] \in \mathcal{S}(T_i)$, for $1 \leqslant i \leqslant r$. From Theorem 2.2, T_1 is a tree obtained from P_1 by sequentially adding pendant P_2's, and T_i, for $2 \leqslant i \leqslant r$, is a tree obtained from P_2 by sequentially adding pendant P_2's. Consequently, we have $T \in \mathbb{T}_n^2$. □

If x and y are two terminal vertices sharing a common vertex z in a tree T, then the path of T induced by the vertices x, y, z is said to be an *outer P_3 of T*. In this case, we write $P_3 = xzy$.

Lemma 3.4. *Suppose that T is a tree on $n \geqslant 4$ vertices, where n is even, containing an outer P_3. If there exists a singular matrix $A \in \mathcal{S}(T)$ such that $P_s(A) = \frac{n-2}{2}$, then $T \in \mathbb{T}_n^2$.*

Proof. Suppose that A is a singular matrix in $\mathcal{S}(T)$ such that $P_s(A) = \frac{n-2}{2}$.

Let $P_3 = xzy$ be an outer P_3 in T, where x, y are both terminal vertices. Denote by $z_1, \ldots, z_r$ the neighbors of z in T different from x, y, and T_i, for $1 \leqslant i \leqslant r$, the component containing z_i, among the components of the forest obtained from T by deleting the vertex z. Let n_i denote the number of vertices in T_i, for $1 \leqslant i \leqslant r$.

Similarly to the proof of [4, Lemma 5.2], we can deduce that: exactly one of A_i's, say A_1, satisfies the equality $P_s(A[T_1]) = \frac{n_1-1}{2}$, and the others $A[T_i]$'s (i.e., $A[T_i]$, for $2 \leqslant i \leqslant r$) satisfy $P_s(A[T_i]) = \frac{n_i}{2}$.

As in the proof of Lemma 3.3, we may conclude now that $T \in \mathbb{T}_n^2$. □

Lemma 3.5. *Suppose that A is a singular matrix of order n, and let*

$$B = \left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & & & & \vdots & \vdots \\ & & A & & 0 & 0 \\ & & & & 1 & 0 \\ \hline 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{array}\right).$$

Then

(i) B is a singular matrix of order $n+2$;

(ii) if n is even and $P_s(A) = \frac{n-2}{2}$, then $P_s(B) = \frac{n}{2}$.

Proof. The first statement is clear and (ii) can be easily deduced from the proof of [4, Lemma 4.1]. □

Assume that T is a tree on n vertices and let $\tilde{T}$ be the tree of order $n+2$ obtained under the adding pendant P_2 operation on T at the vertex n. For the matrices A and B in Lemma 3.5, if $A \in \mathcal{S}(T)$, then $B \in \mathcal{S}(\tilde{T})$. In particular, if A is the adjacency matrix of T, then B is obviously the adjacency matrix of $\tilde{T}$.

Naturally we can reverse the operation of adding pendant P_2: deleting a path P_2 of a given tree T, with degrees 1 and 2 in T, is called *deleting a pendant P_2 of T*.

It is readily verified that if there is no outer P_3 in a tree T on $n \geqslant 3$ vertices, then some deleting pendant P_2 operation can be applied to T.

4. The Characterization

The sole result of this section contains the main characterization of all the trees of even order for which there exists a singular matrix with a P-set of maximum size, which is the main aim of this paper.

Theorem 4.1. *Let T be a tree on $n \geqslant 2$ vertices, where n is even. Then there exists a singular matrix $A \in \mathcal{S}(T)$ such that $P_s(A) = \frac{n-2}{2}$ if and only if $T \in \mathbb{T}_n^1 \cup \mathbb{T}_n^2$.*

Proof. Necessity. Suppose that A is a singular matrix in $\mathcal{S}(T)$ such that $P_s(A) = \frac{n-2}{2}$.

Let us start from T, deleting pendant P_2's repeatedly, until there exists an outer P_3 or a path P_2 is left. The deletion process is listed as follows:

$$T = T_1 \to \cdots \to T_k\,,$$

where T_{i+1} is the tree obtained from T_i by deleting a pendant P_2, for $1 \leqslant i \leqslant k-1$.

Let $A_i = A[T_i]$ and n_i be the order of A_i, for $1 \leqslant i \leqslant k$. Clearly, $A_i \in \mathcal{S}(T_i)$. By Proposition 2.1, $P_s(A_i) \leqslant \frac{n_i}{2}$, for $1 \leqslant i \leqslant k$.

If $P_s(A_i) = \frac{n_i}{2}$, for some $1 \leqslant i \leqslant k$, then from Theorem 2.2, $T_i \in \mathbb{T}_{n_i}^1$. Hence $T \in \mathbb{T}_n^1$.

So in the following we may assume that $P_s(A_i) \leqslant \frac{n_i-2}{2}$, for all $1 \leqslant i \leqslant k$.

Case 1. A_i is singular, for all $2 \leqslant i \leqslant k$.

Subcase 1.1. $P_s(A_k) = \frac{n_k-2}{2}$.

If $T_k \cong P_2$, then $T \in \mathbb{T}_n^1$ follows clearly. Otherwise, there exists an outer P_3 in T_k, from Lemma 3.4, $T_k \in \mathbb{T}_{n_k}^2$, and thus $T \in \mathbb{T}_n^2$.

Subcase 1.2. $P_s(A_k) < \frac{n_k-2}{2}$.

Recall that $P_s(A) = \frac{n-2}{2}$, i.e., $P_s(A_1) = \frac{n_1-2}{2}$. So there exists an index, say j, such that $P_s(A_j) = \frac{n_j-2}{2}$ and $P_s(A_{j+1}) < \frac{n_{j+1}-2}{2}$. Note that A_j is singular. From Lemma 3.3, $T_j \in \mathbb{T}_{n_j}^2$, and consequently $T \in \mathbb{T}_n^2$.

Case 2. A_i is nonsingular, for some $2 \leqslant i \leqslant k$.

Recall that $A_1 = A$ is singular and $P_s(A_1) = \frac{n_1-2}{2}$. Thus there exists an index, say ℓ, such that A_ℓ is singular and $P_s(A_\ell) = \frac{n_\ell-2}{2}$, and either $A_{\ell+1}$ is nonsingular or $P_s(A_{\ell+1}) < \frac{n_{\ell+1}-2}{2}$. Assume that the index ℓ is chosen as small as possible.

If $P_s(A_{\ell+1}) < \frac{n_{\ell+1}-2}{2}$, then from Lemma 3.3, we can get that $T_\ell \in \mathbb{T}_{n_\ell}^2$, and thus $T \in \mathbb{T}_n^2$.

Suppose in the following that $A_{\ell+1}$ is nonsingular. Assume that $T_{\ell+1}$ is the tree obtained from T_ℓ by deleting the pendant $P_2 = uv$, where u is a terminal vertex, and w is the neighbor of v in T_ℓ different from u.

Subcase 2.1. $A_\ell[u] = (0)$ or u is a P-vertex of A_ℓ.

From Lemma 3.2, $m_{A_{\ell+1}}(0) = m_{A_\ell}(0)$, so the singularity of $A_{\ell+1}$ follows from the same property of A_ℓ, which is a contradiction.

Subcase 2.2. $A_\ell[u] \neq 0$ and u is not a P-vertex of A_ℓ.

Let α^* be a P-set of A_ℓ with $|\alpha^*| = \frac{n_\ell-2}{2}$. Clearly, $A_\ell(v) = A_{\ell+1} \oplus A_\ell[u]$, where $\oplus$ represents the direct sum of matrices. So $m_{A_\ell(v)}(0) = 0$, i.e., v is not a P-vertex of A_ℓ, which implies that $v \notin \alpha^*$.

On the other hand, $u \notin \alpha^*$ follows from the hypothesis that u is not a P-vertex of A_ℓ.

From Theorem 2.3, each component of $T_\ell(\alpha^*)$ contains at most one edge, and thus $w \in \alpha^*$. Now similarly to the proof of [4, Lemma 5.3(ii)], $T_\ell \in \mathbb{T}^2_{n_\ell}$ follows easily, and so $T \in \mathbb{T}^2_n$.

Therefore, $T \in \mathbb{T}^1_n \cup \mathbb{T}^2_n$.

Sufficiency. Suppose that $T \in \mathbb{T}^1_n \cup \mathbb{T}^2_n$.

Case 1. $T \in \mathbb{T}^1_n$.

Let us set $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{S}(P_2)$. Clearly M is singular and $P_s(M) = 0$. Thus applying Lemma 3.5 repeatedly, we can construct a singular matrix $A \in \mathcal{S}(T)$ such that

$$P_s(A) = P_s(M) + \frac{n-2}{2} = \frac{n-2}{2}\,.$$

Case 2. $T \in \mathbb{T}^2_n$.

We recall now the construction of $T \in \mathbb{T}^2_n$:

(1) Start from P_1, adding sequentially pendant P_2's.

(2) Add a pendant $P_2 = uv$ to some vertex, where v is of degree 2. Attach a terminal vertex to v, and the resulting graph is denoted by T^v, or attach a terminal vertex to u, and the resulting graph is denoted by T^u.

(3) Sequentially add pendant P_2's.

Denote by T_i the resulting graph in the ith step, and let $n_i = |V(T_i)|$, where $i = 1, 2, 3$. Let $A_i = A(T_i)$, i.e., the adjacency matrix of T_i, for $i = 1, 2, 3$.

It was shown in [4, Theorem 5.4] that A_1 is a singular matrix in $\mathcal{S}(T_1)$ such that $P_s(A_1) = \frac{n_1-1}{2}$. Let α^* be a P-set of A_1 with $|\alpha^*| = \frac{n_1-1}{2}$, i.e., $m_{A_1(\alpha^*)}(0) = m_{A_1}(0) + |\alpha^*|$.

Subcase 1. $T_2 = T^v$.

Clearly,

$$A_2(u,v) = A_1 \oplus \begin{pmatrix} 0 \end{pmatrix}.$$

Since $A_2[u] = (0)$, from Lemma 3.2,

$$m_{A_2}(0) = m_{A_2(u,v)}(0) = m_{A_1}(0) + 1\,,$$

which implies that A_2 is singular.

Moreover, noting that

$$A_2(v) = A_1 \oplus \begin{pmatrix} 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \end{pmatrix},$$

we get

$$\begin{aligned} m_{A_2(\alpha^* \cup \{v\})}(0) &= m_{A_1(\alpha^*)}(0) + 2 \\ &= m_{A_1}(0) + |\alpha^*| + 2 \\ &= m_{A_1}(0) + 1 + |\alpha^* \cup \{v\}| \\ &= m_{A_2}(0) + |\alpha^* \cup \{v\}|\,, \end{aligned}$$

i.e., $\alpha^* \cup \{v\}$ is a P-set of A_2. Clearly, $|\alpha^* \cup \{v\}| = \frac{n_1+1}{2} = \frac{n_2-2}{2}$, so $P_s(A_2) = \frac{n_2-2}{2}$, since A_2 is singular and n_2 is even.

Applying Lemma 3.5 repeatedly, we can deduce that $A(T) \in \mathcal{S}(T)$ is a singular matrix such that

$$P_s(A(T)) = P_s(A_2) + \frac{n-n_2}{2} = \frac{n-2}{2}\,.$$

Subcase 2. $T_2 = T^u$.

Denote by u^* the terminal vertex attached to u in the second step.

Let B be the matrix obtained from A_2 (i.e., the adjacency matrix of T_2) by replacing the two diagonal entries 0 corresponding to vertices u, u^* by 1. In particular, $B[u,u^*] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $B \in \mathcal{S}(T_2)$.

Clearly,

$$B(v) = A_1 \oplus B[u,u^*] = A_1 \oplus \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus $m_{B(v)}(0) = m_{A_1}(0) + 1$.

On the other hand, noting that

$$B(v,u) = A_1 \oplus B[u^*] = A_1 \oplus \begin{pmatrix} 1 \end{pmatrix}.$$

Thus $m_{B(v,u)}(0) = m_{A_1}(0)$.

So we have $m_{B(v,u)}(0) = m_{B(v)}(0) - 1$. From Lemma 3.1, v is a P-vertex of B, i.e., $m_{B(v)}(0) = m_B(0) + 1$. Now it follows that $m_{A_1}(0) = m_B(0)$, which implies that B is singular.

Moreover, we have

$$\begin{aligned} m_{B(\alpha^* \cup \{v\})}(0) &= m_{A_1(\alpha^*)}(0) + 1 \\ &= m_{A_1}(0) + |\alpha^*| + 1 \\ &= m_B(0) + |\alpha^* \cup \{v\}|, \end{aligned}$$

i.e., $\alpha^* \cup \{v\}$ is a P-set of B. Clearly, $|\alpha^* \cup \{v\}| = \frac{n_1+1}{2} = \frac{n_2-2}{2}$, so $P_s(B) = \frac{n_2-2}{2}$ because B is singular and n_2 is even.

Applying Lemma 3.5 repeatedly, we can construct a singular matrix $A \in \mathcal{S}(T)$ such that

$$P_s(A) = P_s(B) + \frac{n - n_2}{2} = \frac{n-2}{2}.$$

This completes the proof. □

5. Examples

We reserved this final section for some illustrative examples concerning Theorem 4.1.

Let us consider the following tree:

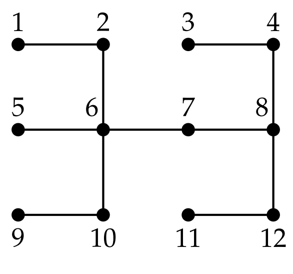

Fig. 1

For the tree as depicted in Fig. 1, if we consider its standard adjacency matrix where the first and the second diagonal entries are 1 instead of 0, with $\alpha = \{4, 6, 7, 10, 12\}$, we can see that the maximum size of a P-set is 5.

We consider now more elaborated graphs. We start with:

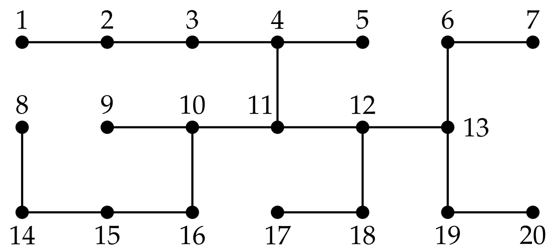

Fig. 2

This time, we turn to the tree in Fig. 2, if we take its adjacency matrix, with $\alpha = \{2, 4, 6, 10, 12, 14, 16, 18, 19\}$, we can get an example of a singular acyclic matrix with a P-set of maximum size. We point out that this matrix has nullity 2.

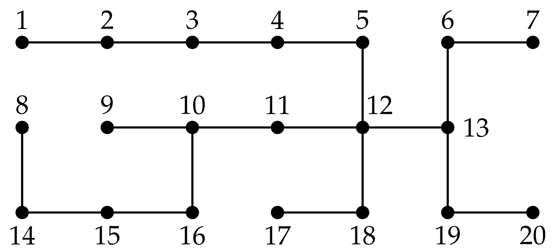

Fig. 3

For the tree we can find in Fig. 3, if we have $v = 5$, $u = 4$, and $u^* = 3$. Once we consider the adjacency matrix of this tree where the uth and u^*th diagonal entries are 1 instead of 0, and setting $\alpha = \{2, 5, 6, 10, 12, 14, 16, 18, 19\}$, we can easily get a desired example.

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