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Korovkin Type Approximation Theorems via Lacunary Equistatistical Convergence

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Abstract. H. Aktuğlu and H. Gezer [Central European J. Math. **7** (2009), 558–567] introduced the concepts of lacunary equistatistical convergence, lacunary statistical pointwise convergence and lacunary statistical uniform convergence for sequences of functions. In this paper, we apply the notion of lacunary equistatistical convergence to prove a Korovkin type approximation theorem by using test functions $1, \frac{x}{1-x}, (\frac{x}{1-x})^2$.

1. Introduction and Preliminaries

The following concept of statistical convergence for sequences of real numbers was introduced by Fast [7]. Let $\mathbb{N}$ be the set of positive integers and $K \subseteq \mathbb{N}$. Let $K_n = \{j : j \leq n \text{ and } j \in K\}$ .Then the *natural density* of K is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{|K_n|}{n}$$

if the limit exists, where $|K_n|$ denotes the *cardinality* of the set K_n.
A sequence $x = (x_j)$ of real numbers is said to be *statistically convergent* to the number L if, for every $\epsilon > 0$, the set $\{j : j \in \mathbb{N} \quad \text{and} \quad |x_j - L| \geq \epsilon\}$ has natural density zero, that is, if, for each $\epsilon > 0$, we have

$$\lim_n \frac{1}{n}\Big|\{j : j \leq n \quad \text{and} \quad |x_j - L| \geq \epsilon\}\Big| = 0.$$

By a lacunary sequence we means an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be abbreviated by q_r.
Fridy and Orhan [8] defined the notion of lacunary statistical convergence as follows:
Let θ be a lacunary sequence; the number sequence x is S_θ-convergent to L provided that for every $\epsilon > 0$,

$$\lim_r \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0. \tag{1}$$

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In this case we write S_θ-limit $x = L$ or $x_k \to L(S_\theta)$.
Note that after the paper of Gadjiev and Orhan [9], the concept of statistcal convergence and its generalizations and variants have been used in proving several approximation theorems, e.g. [4], [5], [6], [18], [19], [20], [21], [22] and [24]. Further, we refere to some useful papers on approximating positive linear operators, e.g. [2], [15], [17] and [23].
The concept of equistatistical convergence was introduced by Balcerzak *et al.* [3] and was subsequently applied for establishing approximation theorems in [1], [10], [11] and [12]. In [1], Aktuglu and Gezer [1] generalized the idea of statistical convergence to lacunary equistatistical convergence. Recently, Y. Kaya and N. Gönül [13] established some analogues of the Korovkin approximation theorem via lacunary equistatistical convergence. In this paper, we prove such type of theorem via lacunary equistatistical convergence by using the test functions 1, $\frac{x}{1-x}$ and $(\frac{x}{1-x})^2$.
Let $C[a,b]$ be the linear space of all real-valued continuous functions f on $[a,b]$. We know that $C[a,b]$ is a Banach space with the norm given by

$$\|f\|_{C[a,b]} := \sup_{x\in[a,b]} |f(x)| \qquad (f \in C[a,b]).$$

Let f and f_n $(n \in \mathbb{N})$ be real-valued functions defined on a subset X of the set $\mathbb{N}$ of positive integers.

Definition 1.1. *A sequence (f_k) of real-valued functions is said to be lacunary equistatistically convergent to f on X if, for every $\epsilon > 0$, the sequence $\big(S_r(\epsilon, x)\big)_{r\in\mathbb{N}}$ of real-valued functions converges uniformly to the zero function on X, that is, if, for every $\epsilon > 0$, we have*

$$\lim_{r\to\infty} \|S_r(\epsilon, x)\|_{C(X)} = 0,$$

where

$$S_r(\epsilon, x) := \frac{1}{h_r}\Big|\{k : k \in I_r \quad \text{and} \quad |f_k(x) - f(x)| \geq \epsilon\}\Big|$$

and $C(X)$ denotes the space of all continuous functions on X. In this case, we write

$$f_k \rightsquigarrow f \ \ (\theta\text{-equistat}).$$

Definition 1.2. *A sequence (f_k) is said to be lacunary statistically pointwise convergent to f on X if, for every $\epsilon > 0$ and for each $x \in X$, we have*

$$\lim_r \frac{1}{h_r}\Big|\{k : k \in I_r \quad \text{and} \quad | f_k(x) - f(x) | \geq \epsilon\}\Big| = 0.$$

In this case, we write

$$f_r \longrightarrow f \ \ (\theta\text{-stat}).$$

Definition 1.3. *A sequence (f_r) is said to be lacunary statistically uniformly convergent to f on X if (for every $\epsilon > 0$), we have*

$$\lim_r \frac{1}{h_r}\Big|\{k : k \in I_r \quad \text{and} \quad \|f_k - f\|_{C(X)} \geq \epsilon\}\Big| = 0.$$

In this case, we write

$$f_r \rightrightarrows f \ \ (\theta\text{-stat})$$

Definition 1.4. *(see [12]). A sequence (f_r) of real-valued functions is said to be equistatistically convergent to f on X if, for every $\epsilon > 0$, the sequence $\left(P_{n,\epsilon}(x)\right)_{n\in\mathbb{N}}$ of real-valued functions converges uniformly to the zero function on X, that is, if (for every $\epsilon > 0$) we have*

$$\lim_{n\to\infty} \|P_{n,\epsilon}(x)\|_{C(X)} = 0,$$

where

$$P_{n,\epsilon}(x) = \frac{1}{n}\Big|\{k : k \leqq n \quad \textit{and} \quad |f_k(x) - f(x)| \geq \epsilon\}\Big| = 0.$$

In this case, we write

$$f_k \rightsquigarrow f \ \ (\text{equistat}).$$

The following implications of the above definitions and concepts are trivial.

$$f_k \rightrightarrows f \ \ (\theta\text{-stat}) \Longrightarrow f_k \rightsquigarrow f \ \ (\theta\text{-equistat}) \Longrightarrow f_k \to f \ \ (\theta\text{-stat}).$$

Furthermore, in general, the reverse implications do not hold true.

2. Main Result

Let $I = [0, A]$, $J = [0, B]$, $A, B \in (0, 1)$ and $K = I \times J$. We denote by $C_B(K)$ the space of all bounded and continuous real valued functions on K. This space is a equipped with norm

$$\|f\|_{C_B(K)} := \sup_{x\in K} |f(x)|, \ f \in C_B(K),$$

where $x = (u, v)$, $u \in I, v \in J$. Let $H_\omega(K)$ denote the space of all real valued functions f on K such that

$$| f(s) - f(x) |\leq \omega(f;\delta),$$

where ω is the modulus of continuity, i.e.

$$\omega(f;\delta) = \sup_{s,x\in K} \{|f(s) - f(x)| : |s - x |\leq \delta\} \ (\delta > 0).$$

It is to be noted that any function $f \in H_\omega(K)$ is continuous and bounded on K, and a necessary and sufficient condition for $f \in H_\omega(K)$ is that

$$\lim_{\delta\to 0} \omega(f;\delta) = 0.$$

In [1], Aktuğlu and Gezer proved the Korovkin theorem for lacunary eqistatistiacal convergence by using the test functions $1, x$ and x^2; while we use here the test functions $1, \frac{x}{1-x}$ and $(\frac{x}{1-x})^2$.

Let T be a linear operator which maps $C[a, b]$ into itself. We say that T is *positive* if, for every non-negative $f \in C[a, b]$, we have

$$T(f, x) \geqq 0 \qquad (x \in [a, b]).$$

We prove the following result:

Theorem 2.1. Let (L_r) be a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$. Then for all $f \in H_\omega(K)$

$$L_r(f) \rightsquigarrow f \ (\theta\text{-equistat}) \tag{2}$$

if and only if

$$L_r(g_i) \rightsquigarrow g_i \ (\theta\text{-equistat}) \ (i = 0, 1, 2). \tag{3}$$

with

$$g_0(x) = 1, \ \ g_1(x) = \frac{x}{1-x} \quad \textit{and} \quad g_2(x) = (\frac{x}{1-x})^2.$$

Proof. Since each of the functions g_i belongs to $H_\omega(K)$, conditions (3) follow immediately. Let $g \in H_\omega(K)$ and $x \in K$ be fixed. Then for $\varepsilon > 0$ there exist $\delta > 0$ such that $| f(s) - f(x) |< \varepsilon$ holds for all $s \in K$ satisfying $|\frac{s}{1-s} - \frac{x}{1-x}| < \delta$. Let

$$K(\delta) := \{s \in K : |\frac{s}{1-s} - \frac{x}{1-x} |< \delta\}.$$

Hence

$$| f(s) - f(x) |=| f(s) - f(x) |_{\chi_{K(\delta)}(s)} + | f(s) - f(x) |_{\chi_{K\setminus K(\delta)}(s)} \leq \varepsilon + 2N_{\chi_{K\setminus K(\delta)}(s)} \tag{4}$$

where χ_D denotes the characteristic function of the set D and $N = ||f||_{C_B(K)}$. Further we get

$$\chi_{K\setminus K(\delta)}(s) \leq \frac{1}{\delta^2}(\frac{s}{1-s} - \frac{x}{1-x})^2. \tag{5}$$

Combining (4) and (5), we get

$$| f(s) - f(x) |\leq \varepsilon + \frac{2N}{\delta^2}(\frac{s}{1-s} - \frac{x}{1-x})^2. \tag{6}$$

After using the linearity and positivity of operators $\{L_r\}$, we get

$$| L_r(f; x) - f(x) |\leq \varepsilon + M\{ \ | L_r(g_0; x) - g_0(x) | + | L_r(g_1; x) - g_1(x) |$$

$$+ | L_r(g_2; x) - g_2(x) | + | L_r(g_3; x) - g_3(x) | \ \} \tag{7}$$

which implies that

$$|L_r(f; x) - f(x)| \leq \epsilon + M \sum_{i=o}^{2} |L_r(g_i; x) - g_i(x)|, \tag{8}$$

where $M := \varepsilon + N + \frac{4N}{\delta^2}$. Now for a given $\rho > 0$, choose $\epsilon > 0$ such that $\epsilon < \rho$. Then, for each $i = 0, 1, 2$, set $\psi_\rho(x) := |\{k \in I_r : |L_k(f; x) - f(x)| \geq \rho\}|$ and $\psi_{i,\rho}(x) := |\{k \in I_r : |L_k(g_i; x) - g_i(x)| \geq \frac{\rho-\epsilon}{3K}\}|$ for $(i = 0, 1, 2)$, it follows from (8) that $\psi_\rho(x) \subseteq \cup_{i=0}^{2} \psi_{i,\rho}(x)$. Hence

$$\frac{||\psi_\rho(x)||_{C_B(K)}}{h_r} \leq \sum_{i=o}^{2} \Big(\frac{||\psi_{i,\rho}(x)||_{C_B(K)}}{h_r} \Big). \tag{9}$$

Now using the hypothesis (3) and the Definition 1.1, the right hand side of (9) tends to zero as $r \to \infty$. Therefore, we have

$$\lim_{r\to\infty} \frac{||\psi_\rho(x)||_{C_B(K)}}{h_r} = 0 \text{ for every } \rho > 0,$$

i.e. (2) holds.

This completes the proof of the theorem.

Example 2.1. Consider the following Meyer-König and Zeller [16] operators:

$$B_n(f;x) := (1-x)^{n+1} \sum_{k=0}^{\infty} f\Big(\frac{k}{k+n+1}\Big)\binom{n+k}{k} x^k, \tag{10}$$

where $f \in H_\omega(K)$, and $K = [0, A]$, $A \in (0, 1)$.
Since, for $x \in [0, A]$, $A \in (0, 1)$,

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k,$$

it is easy to see that

$$B_n(g_0; x) = g_0(x).$$

Also, we obtain

$$B_n(g_1; x) = (1-x)^{n+1} \sum_{k=0}^{\infty} \frac{k}{n+1} x^k$$

$$= (1-x)^{n+1} x \sum_{k=0}^{\infty} \frac{1}{n+1} \frac{(n+k)!}{n!(k-1)!} x^{k-1}$$

$$= (1-x)^{n+1} x \frac{1}{(1-x)^{n+2}} = \frac{x}{(1-x)}.$$

Finally, we get

$$B_n(g_2; x) = (1-x)^{n+1} \sum_{k=0}^{\infty} (\frac{k}{n+1})^2 x^k$$

$$= (1-x)^{n+1} \frac{x}{n+1} \sum_{k=0}^{\infty} \frac{k}{n+1} \frac{(n+k)!}{n!(k-1)!} x^{k-1}$$

$$= (1-x)^{n+1} \frac{x}{n+1} x \sum_{k=0}^{\infty} \frac{(n+k+1)!}{(n+1)!(k-1)!} x^{k-1}$$

$$= \frac{n+2}{n+1}(\frac{x}{1-x})^2 + \frac{1}{n+1}\frac{x}{1-x} \to (\frac{x}{1-x})^2.$$

Therefore

$$B_n(g_i; x) \to g_i(x) \ \ (n \to \infty) \ (i = 0, 1, 2),$$

and cosequently, we have

$$B_n(g_i) \rightsquigarrow g_i(\theta\text{-equistat}) \ (i = 0, 1, 2).$$

Hence by Theorem 2.1, we have

$$B_n(f) \rightsquigarrow f(\theta\text{-equistat}).$$

3. Rate of Lacunary Equistatistical Convergence

In this section we study the rate of lacunary equistatistical convergence of a sequence of positive linear operators as given in [24].

Definition 3.1. *Let (a_n) be a positive non-increasing sequence. A sequence (f_r) is lacunary equistatistically convergent to a function f with the rate $o(a_r)$ if for every $\epsilon > 0$,*

$$\lim_{r\to\infty} \frac{\Lambda_r(x,\epsilon)}{a_r} = 0$$

uniformly with respect to $x \in K$ or equivalently, for every $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{\|\Lambda_r(.,\epsilon)\|_{C_B(X)}}{a_r} = 0,$$

where

$$\Lambda_r(x,\epsilon) := \frac{1}{h_r} \mid \{k \in I_r :\mid f_k(x) - f(x) \mid\geq \varepsilon\} \mid .$$

In this case, it is denoted by $f_r - f = o(a_r))$ (θ-equistat) on K.

We have the following basic lemma.

Lemma 3.1. *Let (f_r) and (g_r) be sequences of functions belonging to $C(K)$. Assume that $f_r - f = o(a_r)$ (θ-equistat) on K and $g_r - g = o(b_r)$ (θ-equistat) on K . Let $c_r = \max\{a_r, b_r\}$. Then the following statement holds:*
(i) $(f_r + g_r) - (f + g) = o(c_r)$ *(θ-equistat) on K ,*
(ii) $(f_r - f)(g_r - g) = o(a_r b_r)$ *(θ-equistat) on K,*
(iii) $\mu(f_r - f) = o(a_r)$ *(θ-equistat) on K for any real number μ,*
(iv) $\sqrt{|f_r - f|} = o(a_r)$ *(θ-equistat) on K.*
We recall that the modulus of continuity of a function $f \in H_\omega(K)$ is defined by

$$\omega(f;\delta) = \sup_{s,x\in K} \{|f(s) - f(x)| : |s - x |\leq \delta\} \ (\delta > 0).$$

Now we prove the following result.

Theorem 3.2. *Let $\{L_r\}$ be a sequence of positive linear operators from $H_\omega(K)$ into $C_B(K)$. Assume that the following conditions hold:*
(a) $L_r(g_0;x) - g_0 = o(a_r)$ *(θ-equistat) on K,*
(b) $\omega(f,\delta_r) = o(b_r)$ *(θ-equistat) on K , where* $\delta_r(x) = \sqrt{L_r(\phi^2;x)}$
with $\phi(x) = \left(\frac{s}{1-s} - \frac{x}{1-x}\right)$. *Then for all $f \in H_\omega(K)$, we have*

$$L_r(f) - f = o(c_r) \ (\theta\text{-equistat) on } K,$$

where $c_r = \max\{a_r, b_r\}$.

Proof. Let $f \in H_\omega(K)$ and $x \in K$. Then it is well known that,

$$|L_r(f;x) - f(x)| \leq M|L_r(g_0;x) - g_0(x)| + (L_r(g_0;x) + \sqrt{L_r(g_0;x)})\omega(f,\delta_r),$$

where $M = \|f\|_{H_\omega(K)}$. This yields that

$$|L_r(f;x) - f(x)| \quad\leq\quad M|(L_r(g_0;x) - g_0(x)| + 2\omega(f,\delta_r) + \omega(f,\delta_r)|(L_r(g_0;x) - g_0(x)| + \omega(f,\delta_r)\sqrt{|(L_r(g_0;x) - g_0(x)|}.$$

Now using the conditions (a), (b) and Lemma 3.1 in the above inequality, we get $L_r(f) - f = o(c_r)$ (θ-equistat) on K.

This completes the proof of the theorem.

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