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The Drazin Inverse of the Sum of Two Matrices and its Applications

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Abstract. In this paper, we give the results for the Drazin inverse of $P + Q$, then derive a representation for the Drazin inverse of a block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ under some conditions. Moreover, some alternative representations for the Drazin inverse of M^D where the generalized Schur complement $S = D - CA^DB$ is nonsingular. Finally, the numerical example is given to illustrate our results.

1. Introduction and preliminaries

Let $\mathbb{C}^{n\times n}$ denote the set of $n \times n$ complex matrix. By $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $rank(A)$ we denote the range, the null space and the rank of matrix A. The Drazin inverse of A is the unique matrix A^D satisfying

$$A^DAA^D = A^D, \qquad AA^D = A^DA, \qquad A^{k+1}A^D = A^k. \tag{1}$$

where $k = ind(A)$ is the index of A, the smallest nonnegative integer k which satisfies $rank(A^{k+1}) = rank(A^k)$. If $ind(A) = 0$, then we call A^D is the group inverse of A and denote it by $A^\sharp$. If $ind(A) = 0$, then $A^D = A^{-1}$. In addition, we denote $A^\pi = I - AA^D$, and define $A^0 = I$, where I is the identity matrix with proper sizes[1].

For $A \in \mathbb{C}^{n\times n}$, k is the index of A, there exists unique matrices C and N, such that $A = C + N$, $CN = NC = 0$, N is the nilpotent of index k, and $ind(C) = 0$ or 1.We shall always use C, N in this context and will refer to $A = C + N$ as the core-nilpotent decomposition of A, Note that $A^D = C^D$.

The Drazin inverse of a square matrix is widely applied in many fields, such as singular differential or difference equations, Markov chains, iterative method, cryptography and numerical analysis,which can be found in[2, 3]. The Drazin inverse in perturbation bounds for the relative eigenvalue problem has an important application value [4]. Accordingly, the Drazin inverse of 2×2 block matrix and its applications can be found in [3].

Suppose P, $Q \in \mathbb{C}^{n\times n}$ such that $PQ = QP = 0$, then $(P + Q)^D$=P^D+Q^D. This result was firstly proved by Drazin [5] in 1958. In 2001, Hartwig et al. [6] gave a formula for $(P+Q)^D$ under the one side condition $PQ = 0$. In 2005, Castro-González [7] derived a result under the conditions $P^DQ = 0$, $PQ^D = 0$ and $Q^\pi PQP^\pi = 0$. In 2009, Martínez-Serrano and Castro-González [8] extended these results to the case $P^2Q = 0$, $Q^2 = 0$ and gave the formula for $(P + Q)^D$. Hartwig and Patricio [9] under the condition $P^2Q + PQ^2 = 0$. In 2010, Wei and Deng [10] studied the additive result for generalized Drazin inverse under the commutative condition of $PQ = QP$ on a Banach space. Liu et al. [11] gave the representations of the Drazin inverse of $(P \pm Q)^D$ with $P^3Q = QP$ and $Q^3P = PQ$ satisfied. In 2011, Liu et al. [12] extended the results to the case $P^2Q = 0$, $QPQ = 0$. In 2012, Bu et al. [13] gave the representations of the Drazin inverse of $(P + Q)^D$ under the following conditions:

$$(i)P^2Q = 0,\ Q^2P = 0;\ (ii)QPQ = 0,\ QP^2Q = 0,\ P^3Q = 0.$$

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The results about the representation of $(P+Q)^D$ are useful in computing the representations of the Drazin inverse for block matrices, analyzing a class of perturbation and iteration theory. The general questions of how to express $(P+Q)^D$ by P, Q, P^D, Q^D and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^D$ by A, B, C, D without side condition are very difficult and have not been solved.

In this paper, we first give the formulas of $(P+Q)^D$ under the conditions $P^2Q = 0$, $PQ + QP = 0$ and $P^DQ = 0$, $PQ - QP = 0$, $\mathcal{N}(P) \bigcap \mathcal{N}(Q) = 0$. And similar reasoning is presented. In the second, we use the formulas of $(P+Q)^D$ to give some representations for the Drazin inverse of block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (A and D are square) under some conditions. Then we give the representation of M^D in which the generalized Schur complement $S = D - CA^DB$ is nonsingular under new conditions. Finally, we take some numerical examples to illustrate our results.

Before giving the main results, we first introduce several lemmas as follows.

Lemma 1.1. ([14]) *Let*

$$M_1 = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}, \quad M_2 = \begin{pmatrix} B & C \\ 0 & A \end{pmatrix},$$

where A and B are square matrices with ind(A)=r and ind(B)=s. Then

$$M_1^D = \begin{pmatrix} A^D & 0 \\ X & B^D \end{pmatrix}, \quad M_2^D = \begin{pmatrix} B^D & X \\ 0 & A^D \end{pmatrix},$$

where $X = \sum_{i=0}^{r-1} (B^D)^{i+2} CA^i A^\pi + B^\pi \sum_{i=0}^{s-1} B^i C (A^D)^{i+2} - B^D CA^D$.

Lemma 1.2. ([6]) *Let* $P, Q \in C^{n\times n}$ *be such that ind(P)=r, ind(Q)=s and* $PQ = 0$. *Then*

$$(P+Q)^D = Q^\pi \sum_{i=0}^{s-1} Q^i (P^D)^{i+1} + \sum_{i=0}^{r-1} (Q^D)^{i+1} P^i P^\pi.$$

Lemma 1.3. ([8]) *Let* $A, B \in C^{n\times n}$,

(i) *If R is nonsingular and* $B = RAR^{-1}$, *then* $B^D = RA^DR^{-1}$.

(ii) *If ind(A)=k* ≥ 0, *then exists a nonsingular matrix R such that* $A = R\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} R^{-1}$, *where* $A_1 \in C^{r\times r}$ *is nonsingular and* $A_2 \in C^{(n-r)\times(n-r)}$ *is k-nilpotent. Relative to the above form,* A^D *and* $A^\pi = I - AA^D$, *are given by*

$$A^D = R\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} R^{-1}, \quad A^\pi = R\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} R^{-1}.$$

Lemma 1.4. ([5]) *Let* $P, Q \in C^{n\times n}$ *be such that* $PQ = QP = 0$, *then* $(P+Q)^D = P^D + Q^D$.

Lemma 1.5. ([1]) *Let* $A \in C^{m\times n}$, $B \in C^{n\times m}$, *then* $(AB)^D = A\Big((BA)^2\Big)^D B$.

Lemma 1.6. ([1]) *Let* $A, B \in C^{n\times n}$, *if* $AB = BA$, *then*

(i) $(AB)^D = B^DA^D = A^DB^D$.

(ii) $A^DB = BA^D$ *and* $AB^D = B^DA$.

Lemma 1.7. ([20]) *Let* $A, B \in C^{n\times n}$, *suppose that c is such that* $(cA+B)$ *is invertible, then*

(i) $(cA+B)^{-1}A$ *and* $(cA+B)^{-1}B$ *commute;*

(ii) $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$ *and* $\mathcal{N}(A) = \mathcal{N}\Big((cA+B)^{-1}A\Big)$, $\mathcal{N}(B) = \mathcal{N}\Big((cA+B)^{-1}B\Big)$.

2. Additive Results

In [10], Wei and Deng studied the additive result for generalized Drazin inverse under the commutative condition of $PQ = QP$ on a Banach space. In this section, we will give the Drazin inverse of $P + Q$ under the conditions that $P^2Q = 0, PQ + QP = 0$ and $P^DQ = 0, PQ - QP = 0, \mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$, which will be the main tool in our following development.

Theorem 2.1. *Let $P, Q \in C^{n\times n}$ be such that $P^2Q = 0,\ PQ + QP = 0$, then*

$$(P + Q)^D = P^D + (P + Q)(Q^D)^2. \tag{2}$$

Proof. From the conditions of theorem , we can know $P^2Q = -PQP = QP^2 = 0$.

Let $P = R\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}R^{-1}$, where P_1 is nonsingular and P_2 is nilpotent. Write $Q = R\begin{pmatrix} Q_1 & Q_{12} \\ Q_{21} & Q_2 \end{pmatrix}R^{-1}$. Form $P^2Q = 0$ it follows

$$Q_1 = 0,\ Q_{12} = 0,\ P_2^2Q_{21} = 0,\ P_2^2Q_2 = 0 \tag{3}$$

Form $QP^2 = 0$ it follows

$$Q_{21} = 0,\ Q_2P_2^2 = 0. \tag{4}$$

Form $PQ + QP = 0$ it follows $P_2Q_2 + Q_2P_2 = 0$.

Now, using Lemma 1.1 we obtain

$$(P + Q)^D = R\begin{pmatrix} P_1 & 0 \\ 0 & P_2 + Q_2 \end{pmatrix}^D R^{-1} = R\begin{pmatrix} P_1^{-1} & 0 \\ 0 & (P_2 + Q_2)^D \end{pmatrix}^D R^{-1}. \tag{5}$$

Now, we need compute $(P_2 + Q_2)^D$.

$$(P_2 + Q_2)^2 = P_2^2 + Q_2^2 + P_2Q_2 + Q_2P_2 = P_2^2 + Q_2^2,\quad (P_2^2)^D = (P_2^D)^2 = 0.$$

Applying (3) and (4) and Lemma 1.4, we get

$$\left((P_2 + Q_2)^2\right)^D = (P_2^2 + Q_2^2)^D = (Q_2^2)^D.$$

Further,

$$(P_2 + Q_2)^D = (P_2 + Q_2)\left((P_2 + Q_2)^2\right)^D = (P_2 + Q_2)(Q_2^2)^D. \tag{6}$$

By substituting (6) in (5), we get

$$(P + Q)^D = R\begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}R^{-1} + R\begin{pmatrix} 0 & 0 \\ 0 & (P_2 + Q_2)^D \end{pmatrix}R^{-1} = P^D + (P + Q)(Q^D)^2.$$

Using the similar method as in the proof of Theorem 2.1, We get the following two results.

Theorem 2.2. *Let $P, Q \in C^{n\times n}$ be such that $P^2Q = 0,\ QP^\pi = 0$, then*

$$(P + Q)^D = P^D + Q(P^D)^2 + PQ(P^D)^3. \tag{7}$$

Theorem 2.3. *Let $P, Q \in C^{n\times n}$ be such that $P^DQ = 0,\ QP^\pi = 0,\ s = indP$, then*

$$(P + Q)^D = P^D + \sum_{i=0}^{s-1} P^iQ(P^D)^{i+2}. \tag{8}$$

Theorem 2.4. *Let $P, Q \in C^{n\times n}$ be such that $PQ = QP$, $P^D Q = 0$, $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$, then*

$$(P+Q)^D = P^D + \sum_{n=0}^{k-1} (Q^D)^{n+1}(-P)^n. \tag{9}$$

where $k = ind(P)$.

Proof. Let $P = R\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} R^{-1}$, where P_1 and R is nonsingular, and P_2 is nilpotent, then exists a positive real number k, satisfy $P_2^k = 0$, and we can easy know $k = ind(P)$. Write $Q = R\begin{pmatrix} Q_1 & Q_{12} \\ Q_{21} & Q_2 \end{pmatrix} R^{-1}$. From $P^D Q = 0$ it follows $Q_1 = 0,\ Q_2 = 0$.

Now, from $PQ = QP$ it follows $P_2 Q_3 = Q_3 P_1,\ P_2 Q_4 = Q_4 P_2$.

Then $P_2^k Q_3 = Q_3 P_1^k = 0$. Thus $Q_3 = 0$ since P_1^k is invertible. Next we will show that Q_4 is invertible.

If $P_2 = 0$, the assumption $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$ implies $\mathcal{N}(Q_4) = \{0\}$, and we are done. If $P_2 \neq 0$, suppose there exists a $v \neq 0$ such that $v \in \mathcal{N}(Q_4)$. Then

$$P_2^q v \in \mathcal{N}(Q_4) \ \textit{for all integers } q \geq 0, \ \textit{since } P_2 Q_4 = Q_4 P_2.$$

Since P_2 is nilpotent, there exists a nonnegative integer m such that $P_2^m v \neq \{0\}$, which is a contradiction. So we can know Q_4 is invertible.

Since P_2 is nilpotent, the eigenvalue of P_2 is 0, so the eigenvalue of $P_2 Q_4^{-1}$ is 0, then $P_2 Q_4^{-1}$ is nilpotent, then $I + P_2 Q_4^{-1}$ is invertible. From $P_2 Q_4 = Q_4 P_2$ it follows $P_2 + Q_4 = (I + P_2 Q_4^{-1}) Q_4 = Q_4 (I + P_2 Q_4^{-1})$.

By lemma1.6 we obtain

$$(P_2 + Q_4)^D = (I + P_2 Q_4^{-1})^D Q_4^D = \sum_{n=0}^{\infty} Q_4^{-n}(-P_2)^n Q_4^{-1} = \sum_{n=0}^{k-1} Q_4^{-(n+1)}(-P_2)^n.$$

Then we compute $(P+Q)^D$.

$$\begin{aligned}(P+Q)^D &= R\begin{pmatrix} P_1^{-1} & 0 \\ 0 & (P_2+Q_4)^D \end{pmatrix} R^{-1} = R\begin{pmatrix} P_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} R^{-1} + R\begin{pmatrix} 0 & 0 \\ 0 & (P_2+Q_4)^D \end{pmatrix} R^{-1} \\ &= P^D + \sum_{n=0}^{k-1} (Q^D)^{n+1}(-P)^n.\end{aligned} \tag{10}$$

From the conclusion of theorem 2.4, we can know the representation of $(P+Q)^D$ are similarity, when $P,\ Q \in C^{n\times n}$ be such that $PQ = QP$ and $PQ = -QP$. We can choose the correspondingly conclusion to solve questions in a different case. Choose the different conclusion which could simplify the process of proof.

According to lemma 1.7, we can change theorem 2.4 to the following theorem:

Theorem 2.5. *Let $P, Q \in C^{n\times n}$ be such that $PQ = QP$, $P^D Q = 0$, suppose c is such that $(cP + Q)$ is invertible, $k = ind(P)$, then*

$$(P+Q)^D = P^D + \sum_{n=0}^{k-1} (Q^D)^{n+1}(-P)^n. \tag{11}$$

To find a c such that $(cP + Q)$ is invertible, such that $|cP + Q| \neq 0$, one must find a number which is not the root of a certain polynomial. That is a problem which most will agree is considerably simpler than finding a root. We find the conclusion in [33], so we have no introduced here.

3. Applications to the Drazin Inverse of Block Matrix

In this section ,we consider the $n \times n$ block matrices of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{12}$$

where A and D are square, B is $p \times (n-p)$, C is $(n-p) \times p$.
Some results have been provided for the Drazin inverse of M under certain conditions. Djordjevic and Stanimirovic [28] gave explicit representation for M^D under conditions $BC = 0$, $BD = 0$ and $DC = 0$. This result was extended to a case $BC = 0$, $BD = 0$(see[29]). The case $BCA = 0$, $BCB = 0$, $DCA = 0$, $DCB = 0$ has been studied in [12], the case $BCA = 0$, $BCB = 0$, $ABD = 0$, $CBD = 0$ in [30], the case $ABC = 0$, $DC = 0$ or $ABC = 0$, $BD = 0$in [31], and so on.

In the following, we illustrate an application of our result obtained in the previous section to establish representations for M^D under some conditions.

Lemma 3.1. ([15]) *Let*$T \in C^{n\times n}$ *be such that* $T = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$, $B \in C^{p\times(n-p)}$, $C \in C^{(n-p)\times p}$. *Then*

$$T^D = \begin{pmatrix} 0 & B(CB)^D \\ (CB)^D C & 0 \end{pmatrix}.$$

Theorem 3.2. *Let M be as in* (12) *such that* $A^2B = 0$, $D^2C = 0$, $BD^\pi = 0$, $CA^\pi = 0$. *Then*

$$M^D = \begin{pmatrix} A^D & B(D^D)^2 + AB(D^D)^3 \\ C(A^D)^2 + DC(A^D)^3 & D^D \end{pmatrix}. \tag{13}$$

Proof. Consider the splitting of matrix M

$$M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \triangleq P + Q,$$

Since the conditions we can obtain $P^2Q = 0$, $QP^\pi = 0$. Hence matrices P and Q satisfy the conditions of Theorem2.3 and

$$M^D = (P+Q)^D = P^D + Q(P^D)^2 + PQ(P^D)^3. \tag{14}$$

So we can compute M^D.

Theorem 3.3. *Let M be as in* (12) *such that* $BCA = 0$, $CBD = 0$, $A(BC)^\pi = 0$, $D(CB)^\pi = 0$, *then*

$$M^D = \begin{pmatrix} A(BC)^D + BD\big((CB)^D\big)^2 C & B(CB)^D \\ (CB)^D C & D(CB)^D + CA(BC)^D BC(CB)^D \end{pmatrix}. \tag{15}$$

Proof. Consider the splitting $M = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \triangleq P + Q$. By applying lemma 3.1, we get $P^D = \begin{pmatrix} 0 & B(CB)^D \\ (CB)^D C & 0 \end{pmatrix}$, $P^\pi = \begin{pmatrix} (BC)^\pi & 0 \\ 0 & (CB)^\pi \end{pmatrix}$.

The remaining proof is similar to that of Theorem 3.2. Hence, we omit the details.

As we known, M is nonsingular such that A and the generalized Schur complement $S = D - CA^{-1}B$ are nonsingular, and

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}.$$

The generalized Schur complement of A in M denoted by $S = D - CA^DB$ piays an important role in the representations for M^D. When S is nonsingular, Wei[21] gave the representation of M^D. Our purpose is to explore the case in which the generalized Schur complement S is nonsingular under new conditions.

Lemma 3.4. ([21]) *Let M be as in* (12) *such that S is nonsingular. If $A^\pi B = 0$ and $CA^\pi = 0$, then*

$$M^D = \begin{pmatrix} A^D + A^D B S^{-1} C A^D & -A^D B S^{-1} \\ -S^{-1} C A^D & S^{-1} \end{pmatrix}.$$

Theorem 3.5. *Let M be as in* (12) *such that S is nonsingular. If $A^\pi BC = 0,\ CA^\pi B = 0,\ BD + AB = 0$, then*

$$M^D = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \left[\sum_{i=1}^{k} (Q_2^D)^{i+2} \begin{pmatrix} A^i A^\pi & 0 \\ CA^{i-1} A^\pi & 0 \end{pmatrix} + (Q_2^D)^2 \right]. \tag{16}$$

where $Q_2^D = \begin{pmatrix} A^D + A^D B S^{-1} C A^D & -A^D B S^{-1} \\ -S^{-1} C A^D & S^{-1} \end{pmatrix}$, $k = ind(A)$.

Proof. We rewrite M as

$$M = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A & AA^D B \\ C & D \end{pmatrix} \triangleq P + Q,$$

From the conditions, we have $PQ + QP = 0$, moreover $P^2 = 0$, $P^D = 0$. By Theorem 2.1, we get $M^D = (P + Q)(Q^D)^2$, now just need calculate Q^D.

We consider the splitting $Q = Q_1 + Q_2$, where $Q_1 = \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}$, $Q_2 = \begin{pmatrix} A^2 A^D & AA^D B \\ CAA^D & D \end{pmatrix}$. We notice that $Q_1 Q_2 = 0$, moreover Q_1 satisfy the conditions of Lemma 1.2 and Q_1 is $k+1$-nilpotent. By Lemma 1.2,

$$Q^D = \sum_{i=0}^{k} (Q_2^D)^{i+1} Q_1^i = Q_2^D + \sum_{i=1}^{k} (Q_2^D)^{i+1} Q_1^i.$$

By induction, we get $(Q^D)^j = \sum_{i=0}^{k} (Q_2^D)^{i+j} Q_1^i,\ \forall j \geq 1$.

For Q_2, the generalized Schur complement of $A^2 A^D$ is nonsingular, and Q_2 satisfy the conditions of Lemma 3.2, so we know Q_2^D. Hence we could compute M^D.

Theorem 3.6. *Let M be as in* (12) *such that S is nonsingular. If $BCA^\pi = 0,\ CA^\pi B = 0,\ CA + DC = 0$, then*

$$M^D = \sum_{i=1}^{k} \begin{pmatrix} A^{i+1} A^\pi & A^i A^\pi B \\ 0 & CA^{i-1} A^\pi B \end{pmatrix} (Q_2^D)^{i+2} + \begin{pmatrix} A & B \\ C & D \end{pmatrix} (Q_2^D)^2. \tag{17}$$

where $Q_2^D = \begin{pmatrix} A^D + A^D B S^{-1} C A^D & -A^D B S^{-1} \\ -S^{-1} C A^D & S^{-1} \end{pmatrix}$, $k = ind(A)$.

Proof. Consider the splitting of M

$$M = \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix} + \begin{pmatrix} A & B \\ CAA^D & D \end{pmatrix} \triangleq P + Q.$$

The remaining proof is similar to that of Theorem 3.5. Hence, we omit the details.

4. Numerical Example

We give the following example to illustrate the application of the representation given in Theorem 2.1.

Example 4.1. *Consider the block matrix $M \in C^{8\times 8}$,*

$$M = \begin{pmatrix} 0.6024 & 0.5793 & 0.7203 & -0.1819 & -0.5055 & -0.5310 & 0.0448 & 0.5580 \\ 0.0382 & 0.8535 & 1.0953 & -0.2901 & -0.1723 & -0.1960 & -0.2015 & 0.2748 \\ 0.0015 & 0.8127 & 0.9551 & 0.1958 & -0.5880 & -0.8991 & 0.0871 & 0.1239 \\ -0.6484 & 1.1865 & 3.7065 & 0.6480 & -0.5028 & -2.2071 & -0.1799 & 0.2024 \\ 0.2111 & 0.1855 & 1.3421 & -0.1598 & 0.6033 & -0.8221 & -0.2392 & 0.1312 \\ 0.6602 & 0.4984 & 1.1177 & 0.2857 & -0.3202 & -0.5843 & -0.2467 & -0.6442 \\ 1.1049 & -0.9367 & 0.5185 & -0.0839 & 0.0210 & 0.0387 & 0.5035 & -0.3400 \\ 0.7542 & -0.0112 & 0.6711 & 0.1958 & -0.1133 & -0.8857 & -0.2795 & 0.4185 \end{pmatrix},$$

we can easy know $ind(M) = 4$. Consider the splitting $M = P + Q$, where

$$P = \begin{pmatrix} 0.5934 & 0.4427 & 0.4715 & -0.2610 & -0.5740 & -0.2310 & 0.0465 & 0.7766 \\ 0.6936 & 0.1383 & 0.1764 & -0.2822 & -0.1957 & 0.4997 & -0.3672 & 0.2907 \\ -0.6612 & 1.2641 & 1.3993 & 0.0414 & -0.6918 & -1.0287 & 0.2536 & 0.5133 \\ 0.3416 & -0.0345 & 2.0672 & 0.5841 & -0.6043 & -0.8588 & -0.4308 & 0.4368 \\ 0.6830 & -0.4529 & 0.4602 & -0.2206 & 0.5286 & -0.0606 & -0.3590 & 0.3269 \\ 0.6523 & 0.2908 & 0.7429 & 0.1691 & 0.4214 & -0.1360 & -0.2456 & 0.3216 \\ 1.8787 & -1.7617 & -0.5318 & -0.0641 & 0.0024 & 0.8191 & 0.3079 & -0.3502 \\ 0.9200 & -0.3006 & 0.2451 & 0.1393 & -0.1701 & -0.4824 & -0.3218 & 0.5844 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0.0091 & 0.1367 & 0.2488 & 0.0791 & 0.0685 & -0.2999 & -0.0017 & -0.2186 \\ -0.6554 & 0.7151 & 0.9189 & -0.0079 & 0.0234 & -0.6957 & 0.1657 & -0.0160 \\ 0.6627 & -0.4514 & -0.4443 & 0.1544 & 0.1038 & 0.1296 & -0.1665 & -0.3894 \\ -0.9900 & 1.2210 & 1.6393 & 0.0640 & 0.1015 & -1.3483 & 0.2509 & -0.2344 \\ -0.4719 & 0.6383 & 0.8818 & 0.0608 & 0.0748 & -0.7615 & 0.1198 & -0.1957 \\ 0.0079 & 0.2076 & 1.3748 & 0.1166 & 0.1012 & -0.4483 & -0.0011 & -0.3226 \\ -0.7738 & 0.8250 & 1.0503 & -0.0198 & 0.0185 & -0.7804 & 0.1956 & 0.0101 \\ -0.1658 & 0.2893 & 0.4260 & 0.0564 & 0.0568 & -0.4050 & 0.0424 & -0.1659 \end{pmatrix},$$

we get $ind(P) = 4$, Q is 42−nilpotent matrix, and $PQ + QP = 0$, $P^2Q = 0$. From Theorem 2.1 we obtain $M^D = P^D + (P + Q)(Q^D)^2$. Now, we just compute P^D and Q^D.

$$P^D = \begin{pmatrix} 1.3063 & 0.0867 & -0.6461 & -0.1342 & -0.8956 & 0.0919 & -0.0748 & 0.8500 \\ 0.6958 & 0.6547 & -0.3382 & -0.1828 & -0.7677 & 0.1611 & -0.2711 & 0.6482 \\ 0.2189 & 0.4163 & 0.1679 & 0.0878 & -0.7830 & -0.2281 & 0.0445 & 0.5230 \\ -0.6483 & 1.6361 & 1.4816 & 0.5109 & -2.0090 & -1.5672 & 0.0211 & 1.9148 \\ 0.7858 & -0.0102 & -0.4789 & -0.1442 & -0.2273 & -0.1860 & -0.2653 & 0.9068 \\ 0.1909 & 0.8752 & 0.2373 & 0.0446 & -1.1027 & -0.1158 & -0.0432 & 0.5676 \\ 1.6551 & -1.00957 & -0.8905 & 0.0090 & -0.6469 & 0.3377 & 0.4704 & 0.1293 \\ 0.4753 & 0.2825 & 0.0203 & 0.0585 & -0.6885 & -0.6270 & -0.1417 & 1.2330 \end{pmatrix},$$

Hence, we can compute M^D.

From the above calculate process, if we compute M^D directly, it needs 0.0160s. But by applying Theorem 2.1, we first solve P^D and Q^D, then use them to calculate M^D, it will shorten 0.0010s on the time, and equivalent reduction the calculate process virtually.

If a square matrix with a large order, we can also use the method to calculate the Drazin inverse of a square matrix, it needs find a suitable nonsingular matrix R, and applying the core-nilpotent method to solve the Drazin inverse.

Remark 4.2. *The above example is generated randomly,so there exist some errors,but these errors do not affect the results.*

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