Filomat 30:12 (2016), 3371–3376
DOI 10.2298/FIL1612371I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Certain Complex Equations and Some of Their Implications in Relation with Normalized Analytic Functions

Hüseyin Irmak[a]

[a] *Department of Mathematics, Faculty of Science, Çankırı Karatekin University, Uluyazı Campus, Tr-18100, Çankırı, Turkey*

Abstract. The aim of the present investigation is first to reveal some nonlinear relations between certain inequalities (constituted by normalized analytic functions) and equations in the complex plane and then to indicate some useful implications of them.

1. Introduction, Notations and Definitions

As it is known from literature, fractional calculus (FC) is a generalization of ordinary differentiation and integration to arbitrary non-integer order. This subject is as old as the differential calculus, and also goes back to time when Leibniz and Newton invented differential calculus. The efficient usage of FC has been a subject of interest not only among mathematicians but also among physicists and engineers, appearing in rheology, viscoelasticity, electrochemistry, electromagnetism, etc. Fractional differential equations (FDE), i.e., differential equations determined by FC, have also many applications in modeling of physical and chemical processes and also in engineering. In its turn, mathematical aspects of studies on FDE were discussed by several authors. After some ordinary researches, it can be easily arrived at some of them. One may look over their details in the works in [8-10, 13, 14]

The main purpose of this work is both to present a novel investigation dealing with analytic and geometric function theory (AGFT) and FDE, and to reveal some nonlinear relations between certain complex valued functions and complex (differential) equations established by using FC. (See [3, 4] for AGFT, [8-10, 13, 14] for FDE, and also see [5, 6] for certain results between AGFT and FDE.) Principally, a number of special consequences of the main results are also pointed out in the concluding section of this paper.

Now we need to introduce (or remember) some well-known notations and also definitions which will be used in this work.

First of all, let $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ be the set of real numbers, the set of complex numbers and the set of positive integers, respectively. Also let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{C}^* := \mathbb{C} - \{0\}$ and $\mathbb{R}^* := \mathbb{R} - \{0\}$.

For $0 \leq \mu < 1$ and a complex-valued function $\kappa := \kappa(z)$, the symbol $\mathcal{D}_z^{\mu}[\kappa]$ denotes an operator of FC, which is defined as follows (cf., e.g., [1, 2, 7-10, 13, 14]):

2010 *Mathematics Subject Classification*. 26A33, 34A08, 35G10, 35F05, 30C55, 30C45, 30A10
Keywords. Complex plane; Equations and inequalities in the complex plane; Analytic-Normalized-Univalent function; Analytic and geometric properties; Fractional calculus; Linear operators in the complex plane.
Received: 20 September 2014; Accepted: 20 April 2014
Communicated by Dragan S. Djordjević
Research supported by TÜBİTAK (The Scientific and Technological Research Council of TURKEY)
Email address: hisimay@yahoo.com, hirmak@karatekin.edu.tr (Hüseyin Irmak)

Let $\kappa(z)$ be an analytic function in a simply-connected region of the $z-$plane containing the origin. Then, the fractional derivative of order μ is defined by

$$\mathcal{D}_z^{\mu}[\kappa] = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{\kappa(\xi)}{(z-\xi)^{\mu}}\, d\xi \quad \Big(0 \le \mu < 1\Big), \tag{1}$$

where the multiplicity of $(z-\xi)^{-\mu}$ above is removed by requiring $log(z-\xi)$ to be real when $z-\xi > 0$. Here and throughout this investigation, of course, the function Γ is the well-known gamma function that we know.

Under the hypotheses of the definition above, for an analytic function $\kappa(z)$, the fractional derivative of order $m+\mu$ is also defined by

$$\mathcal{D}_z^{m+\mu}[\kappa] = \frac{d^m}{dz^m}\Big(\mathcal{D}_z^{\mu}[\kappa]\Big) \quad \Big(0 \le \mu < 1; m \in \mathbb{N}_0\Big). \tag{2}$$

As an application of the above operators, if we apply the definitions in (1) and (2) to the function $\kappa(z) = z^{\omega}$, it can be easily determined that

$$\mathcal{D}_z^{m+\mu}[z^{\omega}] = \frac{\Gamma(\omega+1)}{\Gamma(\omega-m-\mu+1)} z^{\omega-m-\mu} \quad (\omega > m+\mu-1) \tag{3}$$

for some $0 \le \mu < 1$ and for all $m \in \mathbb{N}_0$.

Next, let $\mathcal{A}$ denote the family of the functions $f(z)$ normalized by the following Taylor-Maclaurin series:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots \quad \Big(a_{n+1} \in \mathbb{C};\ n \in \mathbb{N}\Big), \tag{4}$$

which are analytic in the unit open disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. This functions family $\mathcal{A}$ has an important roles for AGFT. For certain examples concerning functions in the related family, see the results given by the references in [1, 2, 5-7].

By applying the operator $\mathcal{D}_z^{\mu}[\cdot]$ to a complex function $f(z)$ belonging to the class $\mathcal{A}$, given by (1), we can then define a linear operator $J_z^{\mu}[f]$ in the following form:

$$J_z^{\mu}[f] = \Gamma(2-\mu) z^{\mu} \mathcal{D}_z^{\mu}[f] = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k-\mu+1)} a_k z^k, \tag{5}$$

where $\mu \in \mathbf{R} := \mathbb{R} - \{2, 3, 4, \cdots\}$.

In order to prove main results in the next section, we need to recall the following assertion given by [11]:

Lemma 1. *Let $p(z)$ be an analytic function in the disk $\mathbb{U}$ with $p(0) = 1$. If there exists a point z_0 in $\mathbb{U}$ such that*

$$\Re e\Big(p(z)\Big) > 0 \ \Big(|z| < |z_0|\Big) \ , \ \Re e\Big(p(z_0)\Big) = 0 \ \text{and} \ p(z_0) \neq 0, \tag{6}$$

then

$$p(z_0) = ia \quad \text{and} \quad \frac{zp'(z)}{p(z)}\bigg|_{z=z_0} = ik\Big(a + \frac{1}{a}\Big) \ \Big(k \ge \frac{1}{2}; a \in \mathbb{R}^*\Big). \tag{7}$$

2. The Main Results and Some Implications

In this section, for stating of the main results and also their implications, it needs to introduce three important and comprehensive definitions which are in the following forms:

$$\mathcal{F}(z) = \Big(1-\lambda\Big)f(z) + \lambda z f'(z) \quad \Big(\lambda \in \mathbb{R}; f(z) \in \mathcal{A}\Big), \tag{8}$$

$$(1-\mu)J_z^{1+\mu}[\mathcal{F}] \ + \ (1+\mu)J_z^{\mu}[\mathcal{F}] \ = \ z\Phi(z) \quad \Big(\mu \in \mathbf{R}\Big) \tag{9}$$

and

$$\left(\mu + (1-\mu)\frac{J_z^{1+\mu}[\mathcal{F}]}{J_z^{\mu}[\mathcal{F}]}\right)\left(\frac{J_z^{\mu}[\mathcal{F}]}{z}\right)^{\delta} \ = \ \Psi(z) \quad \Big(\delta \in \mathbb{R}^*; \mu \in \mathbf{R}\Big). \tag{10}$$

In relation to the definitions in (8)-(10), of course, after simple calculations, when focusing on the related definitions (8)-(10), it is easily seen that the function $\mathcal{F}(z)$ is a member of the class $\mathcal{A}$, and the functions $\Phi(z)$ and $\Psi(z)$ are analytic in $\mathbb{U}$, and also both of the complex equations include several certain types of complex differential equations (CDE) by choosing suitable values of the parameters λ, μ and/or δ there. Here and throughout this work, note that the value of the complex power in (10) is considered to be its principle value.

We now begin by setting and then proving the theorems consisting of several useful results between certain complex - differential - equations and normalized analytic functions. The first of the main results is contained in the following form.

Theorem 1. *Let the function* $\mathcal{F}(z)$ be defined by (8) *and let the function* $\Phi(z)$ *be an analytic in* $\mathbb{U}$ *also satisfy any one of the following conditions:*

$$\Im m\Big(\Phi(z)\Big) = 0 \quad \textit{and} \quad \Big|\Re e\Big(\Phi(z)\Big)\Big| < 1. \tag{11}$$

If the function $\mathcal{F} := \mathcal{F}(z)$ *is a solution for the complex equation given by* (9), *then*

$$\Re e\left(\frac{J_z^{\mu}\Big[\mathcal{F}\Big]}{z}\right) > 0 \quad \Big(\mu \in \mathbf{R}; z \in \mathbb{U}\Big). \tag{12}$$

Proof. Let $f(z) \in \mathcal{A}$ and also let the function $\mathcal{F}(z)$ be defined by (8). By means of the definition of the operator in (5), it can be easily derived that

$$\frac{J_z^{\mu}\Big[\mathcal{F}\Big]}{z} = 1 + \sum_{n=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k-\mu+1)}\Big(k\lambda - \lambda + 1\Big)a_k z^{k-1}, \tag{13}$$

where $\lambda \in \mathbb{R}$, $\mu \in \mathbf{R}$ and $z \in \mathbb{U}$.

If we define $p(z)$ by

$$\frac{J_z^{\mu}\Big[\mathcal{F}\Big]}{z} = p(z) \quad \Big(\mu \in \mathbf{R}; z \in \mathbb{U}\Big), \tag{14}$$

then, clearly, the function $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. By means of the equality in (14), we easily obtain that

$$\Big(J_z^{\mu}\Big[\mathcal{F}\Big]\Big)' + \frac{J_z^{\mu}\Big[\mathcal{F}\Big]}{z} = p(z)\Big(2 + \frac{zp'(z)}{p(z)}\Big). \tag{15}$$

By using (2.8) and also the well known identy:

$$z \cdot \Big(J_z^{\mu}\Big[\mathcal{F}\Big]\Big)' \ = \ (1-\mu)\ J_z^{1+\mu}\Big[\mathcal{F}\Big] \ + \ \mu\ J_z^{\mu}\Big[\mathcal{F}\Big], \tag{16}$$

we easily get that

$$(1-\mu)J_z^{1+\mu}\Big[\mathcal{F}\Big] \ + \ (1+\mu)J_z^{\mu}\Big[\mathcal{F}\Big] \ \left(= zp(z)\Big(2 + \frac{zp'(z)}{p(z)}\Big)\right) =: z\Phi(z). \quad (say) \tag{17}$$

Then, it is clear that $\Phi(z)$ is both an analytic function in $\mathbb{U}$ and satisfies the complex equation in (9).

Now assume that there exists a point z_0 in $\mathbb{U}$ such that

$$\Re e\big(p(z)\big) > 0 \ \big(|z| < |z_0|\big) \ , \ \Re e\big(p(z_0)\big) = 0 \ \ and \ \ p(z_0) \neq 0.$$

From (6) of the Lemma 1, we then obtain that

$$p(z_0) = ia \quad \text{and} \quad \frac{zp'(z)}{p(z)}\bigg|_{z=z_0} = ik\Big(a + \frac{1}{a}\Big) \quad \Big(k \geq \frac{1}{2}; a \in \mathbb{R}^*\Big).$$

By using the assertions just above, (17) follows that

$$\Im m\big(\Phi(z_0)\big) = \Im m\left[\ p(z)\Big(2 + \frac{zp'(z)}{p(z)}\Big)\bigg|_{z=z_0}\right] = 2a \neq 0$$

and

$$\Big|\Re e\big(\Phi(z_0)\big)\Big| = \left|\Re e\left[\ p(z)\Big(2 + \frac{zp'(z)}{p(z)}\Big)\bigg|_{z=z_0}\right]\right| = \left|-k\Big(a + \frac{1}{a}\Big)\right| \geq 2k \geq 1.$$

But, these results overtly contradict our assumptions in (11), respectively. Hence, the equality in (13) immediately yields that $\Re e\big(p(z)\big) > 0$ for all $z \in \mathbb{U}$. Therefore, we evidently receive the inequality in (12). This completes the related proof.

The second of the main results is also contained in following form.

Theorem 2. *Let the function $\mathcal{F}(z)$ be defined by* (8) *and let the function $\Psi(z)$ be an analytic in $\mathbb{U}$ and also satisfy any one of the following conditions:*

$$\Im m\big(\Psi(z)\big) = 0 \quad and \quad \Big|\Re e\big(\Psi(z)\big)\Big| < \frac{1}{|\delta|} \quad \big(\delta \in \mathbb{R}^*\big). \tag{18}$$

If the function $\mathcal{F} := \mathcal{F}(z)$ is a solution for the complex equation given by (10), *then*

$$\Re e\left[\left(\frac{J_z^{\mu}\big[\mathcal{F}\big]}{z}\right)^{\delta}\right] > 0 \quad \big(\delta \in \mathbb{R}^*; \mu \in \mathbf{R}; z \in \mathbb{U}\big). \tag{19}$$

Proof. Let $f(z) \in \mathcal{A}$ and also let $\mathcal{F}(z)$ be defined by (8). Then, in view of (13), if we first consider a function $p(z)$ (which is analytic in $\mathbb{U}$ and $p(0) = 1$) as in the form:

$$\left(\frac{J_z^{\mu}\big[\mathcal{F}\big]}{z}\right)^{\delta} = p(z) \quad (\delta \in \mathbb{R}^*; \mu \in \mathbf{R}; z \in \mathbb{U}) \tag{20}$$

and also follow the smilar ways pursued in (15)-(17), we then derive that

$$\left(\frac{J_z^{\mu}\big[\mathcal{F}\big]}{z}\right)^{\delta} + z\left(\frac{J_z^{\mu}\big[\mathcal{F}\big]}{z}\right)'\left(\frac{J_z^{\mu}\big[\mathcal{F}\big]}{z}\right)^{\delta-1} = p(z)\left(1 + \frac{1}{\delta}\frac{zp'(z)}{p(z)}\right),$$

or, equivalently,

$$\left(\mu + (1-\mu)\frac{J_z^{1+\mu}[\mathcal{F}]}{J_z^{\mu}[\mathcal{F}]}\right)\left(\frac{J_z^{\mu}[\mathcal{F}]}{z}\right)^{\delta}\left(= p(z)\left(1 + \frac{1}{\delta}\frac{zp'(z)}{p(z)}\right)\right) = \Psi(z). \quad (say) \tag{21}$$

Obviously, $\Psi(z)$ is an analytic function in $\mathbb{U}$ and also satisfies the equation in (10). As we did in the proof of the Theorem 1, if we again the assumptions of the Lemma 1 to the equality in (21), we easily determine that

$$\Im m\Big(\Psi(z_0)\Big) = \Im m\left[\ p(z)\Big(1+\frac{1}{\delta}\frac{zp'(z)}{p(z)}\Big)\Big|_{z=z_0}\right] = a \neq 0$$

and

$$\Big|\Re e\Big(\Psi(z_0)\Big)\Big| = \left|\Re e\left[\ p(z)\Big(1+\frac{1}{\delta}\frac{zp'(z)}{p(z)}\Big)\Big|_{z=z_0}\right]\right| = \left|-\frac{k}{\delta}\Big(a+\frac{1}{a}\Big)\right| \geq \frac{1}{|\delta|},$$

which are contradictions with the assertions of Theorem 2 in (18), respectively. Hence, the equality in (15) yields that $\Re e\big(p(z)\big) > 0$ for all $z \in \mathbb{U}$. Therefore, we receive the inequalty in (15). Thus, the proof of the Theorem 2 is completed.

As we emphasized in the section 1, the theorems above include several useful and comprehensive results between certain complex differential equations, which were given by (9) and (10), and normalized analytic functions, which were given by (1). It is impossible to list all of them. But, we want to center on only four of them, which are directly related to the results between CDE and AGFT. Both the consequences of these (i.e., Corollaries 1-4 below) and the other possible results of the related theorems (which are here omitted) are presented to the attention of the researchers who have been working on the theory relating to differential equation or univalent function.

By putting $\mu := 0$ and $\lambda := 0$ in (8), (9) and also in the Theorem 1, the following result (which is Corollary 1 below) dealing with close-to-starlikeness (w.r.t. origin) can be first revealed.

Corollary 1. *Let $w := f(z) \in \mathcal{A}$ and also let $\Phi(z)$ be an analytic function and satisfy any one of the conditions given by* (11). *Then,*

$$z\frac{dw}{dz} + w = z\Phi(z) \ \Rightarrow\ \Re e\left(\frac{w}{z}\right) > 0 \quad \big(z \in \mathbb{U}\big).$$

By setting $\mu := 0$ and $\lambda := 1$ in (8), (9) and also in the Theorem 1, the following result (which is Corollary 2 below) relating to close-to-convexity (w.r.t. origin) can be second obtained.

Corollary 2. *Let $w := f(z) \in \mathcal{A}$ and also let $\Phi(z)$ be an analytic function and satisfy any one of the conditions given by* (11). *Then,*

$$z\frac{d^2w}{dz^2} + 2\frac{dw}{dz} = \Phi(z) \ \Rightarrow\ \Re e\left(\frac{dw}{dz}\right) > 0 \quad \big(z \in \mathbb{U}\big).$$

By letting $\mu := 0$ and $\lambda := 0$ in (8), (9) and also in the Theorem 2, the following result (which is Corollary 3 below) can be then received.

Corollary 3. *Let $w := f(z) \in \mathcal{A}$ and also let $\Psi(z)$ be an analytic function and satisfy any one of the conditions given by* (18). *Then,*

$$z\left(\frac{w}{z}\right)^{\delta}\frac{dw}{dz} = w\,\Psi(z) \ \Rightarrow\ \Re e\left[\ \Big(\frac{w}{z}\Big)^{\delta}\ \right] > 0 \quad \big(\delta \in \mathbb{R}^*; z \in \mathbb{U}\big).$$

By choosing $\mu := 0$ and $\lambda := 1$ in (8), (10) and also in the Theorem 3, the following result (which is Corollary 4 below) can be also obtained.

Corollary 4. *Let $w := f(z) \in \mathcal{A}$ and also let $\Psi(z)$ be an analytic function and satisfy any one of the conditions given by* (18). *Then,*

$$\left(\frac{dw}{dz}\right)^{\delta}\left(z\frac{d^2w}{dz^2} + \frac{dw}{dz}\right) - \Psi(z)\frac{dw}{dz} = 0 \ \Rightarrow\ \Re e\left[\ \Big(\frac{dw}{dz}\Big)^{\delta}\ \right] > 0 \quad \big(\delta \in \mathbb{R}^*; z \in \mathbb{U}\big).$$

Acknowledgements

The work in this investigation was supported by TÜBİTAK (The Scientific and Technological Research Council of TURKEY) with the Project Number 105T056.

References

[1] M.P. Chen, H. Irmak, H. M. Srivastava, Some families of multivalently analytic functions with negative coefficients, J. Math. Anal. Appl. 214 (2) (1997) 674-690.

[2] M.P. Chen, H. Irmak, H.M. Srivastava, A certain subclass of analytic functions involving operators of fractional calculus, Comput. Math. Appl. 35 (2) (1998) 83-91.

[3] P.L. Duren, Grundlehren der Mathematischen Wissenchaffen, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[4] A.W. Goodman, Univalent functions. Vols. I and II., Polygonal Publishing House, Washington-New Jersey, 1983.

[5] H. Irmak, B. A. Frasin, A few complex equations constituted by an operator consisting of fractional calculus and their consequences, Chinese J. Math. vol. 2014, Article ID 718389, 4 pages, 2014. doi:10.1155/2014/718389.

[6] H. Irmak, Some computational results for functions belonging to a family consisting of Cauchy-Euler type differential equation, Fract. Differ. Calc. 2 (1) (2012), 109-117.

[7] H. Irmak, N. Tuneski, Fractional calculus operator and certain applications in geometric function theory, Sarajevo J. Math. 6 (1) (18) (2010) 51-57.

[8] A. A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V., Amsterdam, 2006.

[9] A. A. Kilbas, O. I. Marichev, S. G. Samko, Fractional Integral and Derivatives (Theory and Applications), Gordon and Breach, Switzerland, 1993.

[10] V. Kiryakova, Generalized Fractional Calculus and Applications, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1994.

[11] M. Nunokawa, On properties of non-Caratheodory functions, Proc. Japan acad. Ser. A Math. Sci. 68 (1992), 152-153.

[12] S. Owa, On the distortion theorems, Kyungpook Math. J. 18 (1978) 53-59.

[13] H.M. Srivastava, S. Owa (Editors), Univalent Functions, Fractional Calculus and Their Applications, Halsted Press, John Wiley and Sons. New york, Chieschester, Brisbane, Toronto, 1989.

[14] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integral and Derivative, Theory and Applications, Gordon and Breach, 1993.