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A Categorical Approach to Convergence

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Abstract. We define the concept of a convergence class on an object of a given category by using certain generalized nets for expressing the convergence. The resulting topological category, whose objects are the pairs consisting of objects of the original category and convergence classes on them, is then investigated. We study the full subcategories of this category which are obtained by imposing on it some natural convergence axioms. In particular, we find sufficient conditions for the subcategories to be cartesian closed. We also investigate the behavior of the closure operator associated with the convergence in a natural way.

1. Introduction

The study of topological structures on categories was initiated by Dikranjan and Giuli in their paper [3] on categorical closure operators and now it represents an important branch of categorical topology. Originally, only the topological structures on categories given by closure operators were considered and investigated. Later on, also other types of topological structures on categories were introduced and studied, e.g., convergence structures in [13-14] and neighborhood structures in [7-8]. Different types of topological structures on categories are studied to provide convenient tools for investigating topological features of (objects of) the categories. But only those categories may be investigated which possess such a structure. The approach of the present paper is different - we define new topological structures, convergence classes, on each object of a category possessing no topological structure in general and study the obtained category whose objects are pairs consisting of objects of the original category and convergence classes on them. As a tool for introducing convergence (classes) on objects of a category we use a generalized concept of nets given by a functor from a given category to the category under investigation. We consider certain natural convergence axioms and investigate behavior of the categories of objects with a convergence class satisfying these axioms. In particular, we find a sufficient conditions under which these categories are cartesian closed. Recall that cartesian closedness is a very useful categorical property having many applications. For example, in computer science, cartesian closed categories are used as models of the so-called typed lambda-calculus, which is an important formal programming language. We also show that the introduced categories of objects with a convergence class possess a closure operator in the sense of [3]. This closure operator is studied and, among others, sufficient conditions are given for the operator to be additive and idempotent, respectively.

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2. Generalized-Net Convergence

For the categorical terminology used see [1]. Throughout the paper, $\mathcal{S}$ and $\mathcal{K}$ will be non-empty categories and $\mathcal{F} : \mathcal{S} \to \mathcal{K}$ a functor. For each $\mathcal{K}$-object K, we denote by $(\mathcal{F} \downarrow K)$ the comma category of objects $\mathcal{F}$-over K, i.e., the category with objects all pairs $\langle S, f\rangle$, where S is an $\mathcal{S}$-object and $f : \mathcal{F}S \to K$ is a $\mathcal{K}$-morphism, and with morphisms $h : \langle S, f\rangle \to \langle T, g\rangle$ those $\mathcal{S}$-morphisms $h : S \to T$ satisfying $g \circ \mathcal{F}h = f$.

Definition 2.1. Objects of the category $(\mathcal{F} \downarrow K)$ will be called *$\mathcal{F}$-nets* in K. Given a pair $\langle S, f\rangle, \langle T, g\rangle$ of $\mathcal{F}$-nets in K, $\langle S, f\rangle$ is said to be a *subset* of $\langle T, g\rangle$ if there is an $(\mathcal{F} \downarrow K)$-morphism $h : \langle S, f\rangle \to \langle T, g\rangle$.

Example 2.2. (1) Let $\alpha > 0$ be an ordinal, let $\underline{\alpha}$ be the construct whose only object is α and whose morphisms are isotone injections of α into itself, and let $\mathcal{F} : \underline{\alpha} \to Set$ be the forgetful functor. Then $\mathcal{F}$-nets in a set X and their subnets are precisely the sequences of type α in X and their subsequences. For $\alpha = \omega$ we get the usual sequences and subsequences.

(2) Let Dir be the construct of directed sets and cofinal maps and let $\mathcal{F} : Dir \to Set$ be the forgetful functor. Then $\mathcal{F}$-nets in a set X and their subnets are precisely the usual nets in X and their subnets - see [9].

(3) Let Set^+ be the construct of nonempty sets and let $\mathcal{F} : Set^+ \to Set^+$ be the identity functor. If $\langle S, f\rangle$ and $\langle T, g\rangle$ is a pair of $\mathcal{F}$-nets in a set X, then $\langle S, f\rangle$ is a subnet of $\langle T, g\rangle$ if and only if $f(S) \subseteq g(T)$.

(4) Let $\mathcal{S}$ be a non-empty construct and $\mathcal{F} : \mathcal{S} \to Set$ the forgetful functor. The $\mathcal{F}$-nets in a set X and their subnets coincide with $\mathcal{S}$-nets in X and their subnets introduced in [11] and studied also in [12].

(5) Let $HComp$ be the construct of compact Hausdorff topological spaces and let $\mathcal{F} : HComp \to Set$ be the forgetful functor. A quasi-topology [15] on a set X is nothing but a collection $(Q(S, X))_{S\in HComp}$ where, for each $\mathcal{H}Comp$-object S, $Q(S, X)$ is a set of $\mathcal{F}$-nets $\langle S, f\rangle$ in X satisfying some given axioms.

From now on, we assume that $\mathcal{K}$ has terminal objects and for each $\mathcal{K}$-object K we denote by $\tilde{K}$ the class of all points of K, i.e., $\mathcal{K}$-morphisms $I_{\mathcal{K}} \to K$ where $I_{\mathcal{K}}$ is a terminal object of $\mathcal{K}$. (If $x : I_{\mathcal{K}} \to K$ and $x' : I'_{\mathcal{K}} \to K$ are points with $x' \circ t = x$ where $t : I_{\mathcal{K}} \to I'_{\mathcal{K}}$ is the unique isomorphism, then we write $x \cong x'$ and regard x and x' as identical.)

Now, let K be a $\mathcal{K}$-object and $\pi \subseteq Obj(\mathcal{F} \downarrow K) \times \tilde{K}$ a subclass. Instead of $(\langle S, f\rangle, x) \in \pi$, we will write $\langle S, f\rangle \xrightarrow{\pi} x$ and say that $\langle S, f\rangle$ *converges* to x with respect to π. Analogously, for any $\langle S, f\rangle \in Obj(\mathcal{F} \downarrow K)$ and any $x \in \tilde{K}$, instead of $(\langle S, f\rangle, x) \notin \pi$, we will write $\langle S, f\rangle \overset{\pi}{\not\to} x$ (and say that $\langle S, f\rangle$ does not converge to x with respect to π). The class π is called a *convergence class* on K.

Let K, L be $\mathcal{K}$-objects and let π and ρ be convergence classes on K and L, respectively. A $\mathcal{K}$-morphism $\varphi : K \to L$ is said to be *continuous* (w.r.t. π and ρ) if $\langle S, f\rangle \xrightarrow{\pi} x$ implies $\langle S, \varphi \circ f\rangle \xrightarrow{\rho} \varphi \circ x$. We denote by $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ the category with objects the pairs (K, π), where K is a $\mathcal{K}$-object and π is a convergence class on K, and with morphisms $\varphi : (K, \pi) \to (L, \rho)$ the continuous (w.r.t. π and ρ) $\mathcal{K}$-morphisms $\varphi : K \to L$. Note that the objects of $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ may not form a class so that, according to the terminology introduced in [1], $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ is a so-called quasicategory rather then a category. Since all categorical concepts may naturally be extended to quasicategories, we will avoid using the concept of a quasicategory here, i.e., we will call the quasicategory $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ simply a category. Similarly, (full) subquasicategories of $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ will be called (full) subcategories of $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ or briefly categories.

Proposition 2.3. *$[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ is a topological category over $\mathcal{K}$.*

Proof. It is evident that $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ is a concrete category over $\mathcal{K}$. Clearly, for any family (K_j, π_j), $j \in J$, of $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$-objects and any source $\varphi_j : K \to K_j$, $j \in J$, in $\mathcal{K}$, the convergence class π on K given by $\langle S, f\rangle \xrightarrow{\pi} x$ if and only if $\langle S, \varphi_j \circ f\rangle \xrightarrow{\pi_j} \varphi_j \circ x$ for all $j \in J$ is the initial convergence class on K. □

Though $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ is a topological category, it is too general from the convergence point of view - the convergence classes do not have the proper convergence nature. Therefore, we will introduce and study some full subcategories of $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ obtained by imposing the following natural convergence axioms:

Definition 2.4. Let $(K, \pi) \in Obj[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ and consider the following three axioms:

(i) If $\langle S, f\rangle$ is and $\mathcal{F}$-net in K such that f factors through a point $x \in \tilde{K}$ (i.e., f is a constant), then $\langle S, f\rangle \xrightarrow{\pi} x$ (*constant-net axiom*).

(ii) If $\langle S, f\rangle \xrightarrow{\pi} x$, then $\langle T, g\rangle \xrightarrow{\pi} x$ for each subnet $\langle T, g\rangle$ of $\langle S, f\rangle$ (*subnet axiom*).

(iii) If $\langle S, f\rangle \overset{\pi}{\not\to} x$, then there is a subnet of $\langle S, f\rangle$ whose every subnet $\langle T, g\rangle$ fulfils $\langle T, g\rangle \overset{\pi}{\not\to} x$ (*Urysohn axiom*).

The object (K, π) is called an *$\mathcal{F}$-net space*, an *$\mathcal{F}$-convergence space*, or an *$\mathcal{F}$-limit space* if the axiom (i), the axioms (i) and (ii), or the axioms (i), (ii) and (iii) are fulfilled, respectively.

We denote by $Net_{\mathcal{F}}$, $Conv_{\mathcal{F}}$ and $Lim_{\mathcal{F}}$ the full subcategories of $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ with the objects all $\mathcal{F}$-net spaces, all $\mathcal{F}$-convergence spaces and all $\mathcal{F}$-limit spaces, respectively.

Remark 2.5. As usual, if $\mathcal{A}$ is a concrete category over a category $\mathcal{B}$, then we do not distinguish notationally between $\mathcal{A}$-morphisms and their underlying $\mathcal{B}$-morphisms. Moreover, if $\mathcal{A}$ is a construct with terminal objects and A is and $\mathcal{A}$-object, then we do not distinguish between an element $x \in A$ such that $\{x\}$ is the underlying set of an initial subobject of A with respect to the inclusion map, and the point $I_{\mathcal{A}} \to A$ whose range is $\{x\}$.

Example 2.6. (1) Let $\mathcal{F} : \underline{\alpha} \to Set$ be the forgetful functor (see Example 2.2(1)) and let (X, π) be an $\mathcal{F}$-convergence space. Then π is nothing but the multivalued convergence on X from [10]. The $\mathcal{F}$-convergence or $\mathcal{F}$-limit spaces (X, π) for which $\langle \omega, f\rangle \xrightarrow{\pi} x$ and $\langle \omega, f\rangle \xrightarrow{\pi} y$ imply $x = y$ for each $\mathcal{F}$-net (i.e., sequence) $\langle \omega, f\rangle$ in X are known as the (Fréchet) $\mathcal{L}$-spaces or $\mathcal{L}^*$-spaces, respectively - cf. [5].

(2) The $\mathcal{B}$-convergence structures studied in [16] for special subcategories $\mathcal{B}$ of Dir (see Example 2.2(2)) are nothing but the $\mathcal{F}$-convergence spaces where $\mathcal{F} : \mathcal{B} \to Set$ is the forgetful functor.

(3) Let $\mathcal{F} : Dir \to Set$ or $\mathcal{F} : \underline{\alpha} \to Set$ be the forgetful functor (see Examples 2.2 (1) and (2)). Let $(X, \mathcal{O})$ be a topological space (given by the set $\mathcal{O}$ of open subsets). For an $\mathcal{F}$-net $\langle S, f\rangle$ in a set X, put $\langle S, f\rangle \xrightarrow{\pi} x$ if and only if, for each $A \in \mathcal{O}$ with $x \in A$, there exists $s_A \in \mathcal{F}S$ such that $f(s) \in A$ for every $s \in \mathcal{F}S$ with $s \geq s_A$. Then (X, π) is an $\mathcal{F}$-limit space.

(4) If $\mathcal{F} : Dir \to Set$ is the forgetful functor (see Example 2.2 (2)), then $Lim_{\mathcal{F}}$ coincides with the category of L^*-spaces studied in [6].

(5) If $\mathcal{S}$ is a non-empty construct and $\mathcal{F} : \mathcal{S} \to Set$ the forgetful functor, then $Net_{\mathcal{F}}$, $Conv_{\mathcal{F}}$ and $Lim_{\mathcal{F}}$ coincide with the categories $Net_{\mathcal{S}}$, $Conv_{\mathcal{S}}$ and $Lim_{\mathcal{S}}$ studied in [11] and [12].

Theorem 2.7. *Let $\mathcal{K}$ be a construct with finite concrete products and discrete terminal objects, let $\mathcal{K}$ have initial subobjects with respect to inclusion maps, and let $\mathcal{F} : \mathcal{K} \to Set$ be the forgetful functor. If $\mathcal{K}$ is cartesian closed, then so is $Net_{\mathcal{F}}$.*

Proof. Under the assumptions of the statement, the cartesian closedness of $\mathcal{K}$ implies that $\mathcal{K}$ has function spaces. Let (K, π), (L, ρ) be $\mathcal{F}$-net spaces and L^K be the function space of K and L in $\mathcal{K}$. Let M be the initial subobject (with respect to the inclusion map) of L^K whose underlying set is the set of all $\mathcal{K}$-morphisms $\psi : K \to L$ that are continuous w.r.t. π and ρ. Let σ be the convergence class on M given by $\langle S, g\rangle \xrightarrow{\sigma} z$ if and only if $\langle S, f\rangle \xrightarrow{\pi} x$ implies that $\langle S, g^f\rangle \xrightarrow{\rho} z(x)$ where $g^f : \mathcal{F}S \to L$ is the $\mathcal{K}$-morphism defined by $g^f(s) = g(s)(f(s))$. Indeed, g^f is a $\mathcal{K}$-morphism because $g^f = e \circ (f \times g) \circ d$ where $d : \mathcal{F}S \to \mathcal{F}S \times \mathcal{F}S$ is the diagonal, i.e., the map given by $d(s) = (s, s)$ (which is obviously a $\mathcal{K}$-morphism), and $e : K \times M \to L$ is the evaluation map, i.e., the map given by $e(x, z) = z(x)$. As $\mathcal{K}$ has function spaces, the evaluation map $ev : K \times L^K \to L$ is a $\mathcal{K}$-morphism. But $K \times M$ is an initial subobject of $K \times L^K$ w.r.t. the inclusion map, so that $e : K \times M \to L$ is a $\mathcal{K}$-morphism, too. Let $\langle S, g\rangle$ be a constant $\mathcal{F}$-net in M with $g(s) = z$ for every $s \in \mathcal{F}S$ and let $\langle S, f\rangle$ be an $\mathcal{F}$-net in K with $\langle S, f\rangle \xrightarrow{\pi} x$. Then $g^f = z \circ f$, hence $\langle S, g^f\rangle \xrightarrow{\rho} z(x)$. Therefore, $\langle S, g\rangle \xrightarrow{\sigma} z$ and we have shown that (M, σ) is an $\mathcal{F}$-net space.

We will show that $e : (K, \pi) \times (M, \sigma) \to (L, \rho)$ is continuous. To this end, put $(N, \tau) = (K, \pi) \times (M, \sigma)$ and let $\langle U, h\rangle \xrightarrow{\tau} (x, z)$. Then $\langle U, pr_K \circ h\rangle \xrightarrow{\pi} x$ and $\langle U, pr_M \circ h\rangle \xrightarrow{\sigma} z$. Hence, $\langle U, q^p\rangle \xrightarrow{\rho} z(x)$ where $q = pr_M \circ h$ and

$p = pr_K \circ h$. We have $q^p(u) = q(u)(p(u)) = e(p(u), q(u)) = e(pr_K(h(u)), pr_M(h(u)) = e(h(u))$ for every $u \in \mathcal{F}U$. Therefore, $q^p = e \circ h$, which yields $\langle U, q^p\rangle \xrightarrow{\rho} w(x,z)$. Consequently, $e : (K,\pi) \times (M,\sigma) \to (L,\rho)$ is continuous.

Now, let (N,μ) be an $\mathcal{F}$-net space, let $\varphi : (K,\pi) \times (N,\mu) \to (L,\rho)$ be a continuous $\mathcal{K}$-morphism and let $\varphi^* : N \to L^K$ be the map given by $\varphi^*(y)(x) = \varphi(x,y)$. Put $(K \times N, \nu) = (K,\pi) \times (N,\mu)$ and let $y \in N$. Let $\langle S, f\rangle \xrightarrow{\pi} x$ and let $c : \mathcal{F}S \to N$ be the constant map given by $c(s) = y$ for each $s \in \mathcal{F}S$. Then c is a morphism in $\mathcal{K}$ (because $\mathcal{K}$ has discrete terminal objects). Therefore, $\langle S, c\rangle \xrightarrow{\mu} y$, and thus $\langle S, (f,c)\rangle \xrightarrow{\nu} (x,y)$. Consequently, $\langle S, \varphi \circ (f,c)\rangle \xrightarrow{\rho} \varphi(x,y)$. For each $s \in \mathcal{F}S$ we have $\varphi((f,c)(s)) = \varphi(f(s), y) = \varphi^*(y)(f(s))$, so that $\varphi \circ (f,c) = \varphi^*(y) \circ f$. We have $\langle S, \varphi^*(y) \circ f\rangle \xrightarrow{\rho} \varphi^*(y)(x)$. Therefore, $\varphi^*(y) \in M$ for each $y \in N$, so that we may consider φ^* to be a map $\varphi^* : N \to M$. As $\mathcal{K}$ has function spaces, $\varphi^* : N \to L^K$ is a $\mathcal{K}$-morphism. Consequently, since $\mathcal{K}$ has initial subobjects with respect to inclusion maps, $\varphi^* : N \to M$ is a $\mathcal{K}$-morphism, too. Let $\langle S, r\rangle \xrightarrow{\mu} y$ and $\langle S, f\rangle \xrightarrow{\pi} x$. Then $\langle S, (f,r)\rangle \xrightarrow{\nu} (x,y)$ and hence $\langle S, \varphi \circ (f,r)\rangle \xrightarrow{\rho} \varphi(x,y)$. This results in $\langle S, \varphi^* \circ r\rangle \xrightarrow{\sigma} \varphi^*(y)$ because $\varphi((f,r)(s)) = \varphi^*(r(s))(f(s)) = (\varphi^* \circ r)^f(s)$ whenever $s \in \mathcal{F}S$. Consequently, $\varphi^* : (N,\mu) \to (M,\sigma)$ is continuous. Since clearly $e \circ (id_K \times \varphi^*) = \varphi$, we have shown that (M,σ) is a function space of (K,π) and (L,ρ) in $Net_{\mathcal{F}}$. Therefore, $Net_{\mathcal{F}}$ is cartesian closed. □

It can easily be seen that $Conv_{\mathcal{F}}$ and $Lim_{\mathcal{F}}$ are initially closed in $Net_{\mathcal{F}}$ so that they are topological. Consequently, $Conv_{\mathcal{F}}$ is a concretely reflective subcategory of $Net_{\mathcal{F}}$ and $Lim_{\mathcal{F}}$ is a concretely reflective subcategory of $Conv_{\mathcal{F}}$. The concrete reflection of an $\mathcal{F}$-net space (K,π) in $Conv_{\mathcal{F}}$ is given by the identity $\mathcal{K}$-morphism $id_K : (K,\pi) \to (K,\hat{\pi})$ where $\hat{\pi}$ is the convergence class on K defined by $\langle S, f\rangle \xrightarrow{\hat{\pi}} x$ if and only if there is an $\mathcal{F}$-net $\langle T, g\rangle$ in K with $\langle T, g\rangle \xrightarrow{\pi} x$ such that $\langle S, f\rangle$ is a subnet of $\langle T, g\rangle$. The concrete reflection of an $\mathcal{F}$-convergence space (K,π) in $Lim_{\mathcal{F}}$ is given by the identity $\mathcal{L}$-morphism $id_K : (K,\pi) \to (K,\hat{\pi})$ where $\hat{\pi}$ is the convergence class on K defined by $\langle S, f\rangle \xrightarrow{\hat{\pi}} x$ if and only if each subnet of $\langle S, f\rangle$ has a subnet $\langle T, g\rangle$ with $\langle T, g\rangle \xrightarrow{\pi} x$.

Theorem 2.8. *$Conv_{\mathcal{F}}$ is a concretely coreflective subcategory of $Net_{\mathcal{F}}$.*

Proof. Let (K,π) be an $\mathcal{F}$-net space and let π^* be the convergence class on K given by $\langle T, g\rangle \xrightarrow{\pi^*} x$ if and only if $\langle S, f\rangle \xrightarrow{\pi} x$ for every subnet $\langle S, f\rangle$ of $\langle T, g\rangle$. Clearly, (K,π^*) is an $\mathcal{F}$-convergence space and $id_K : (K,\pi^*) \to (K,\pi)$ is continuous. Let (L,ρ) be an $\mathcal{F}$-convergence space and let $\varphi : (L,\rho) \to (K,\pi)$ be a continuous $\mathcal{K}$-morphism. Let $\langle U, h\rangle \xrightarrow{\rho} y$. Then each subnet $\langle V, p\rangle$ of $\langle U, h\rangle$ satisfies $\langle V, p\rangle \xrightarrow{\rho} y$. Let $\langle W, q\rangle$ be a subnet of $\langle U, \varphi \circ h\rangle$. Then there is an $(\mathcal{F} \downarrow K)$-morphism $r : \langle W, q\rangle \to \langle U, \varphi \circ h\rangle$, i.e., an $\mathcal{S}$-morphism $r : W \to U$ with $q = \varphi \circ h \circ \mathcal{F}r$. Since $\langle W, h \circ \mathcal{F}r\rangle$ is a subnet of $\langle U, h\rangle$, we have $\langle W, h \circ \mathcal{F}r\rangle \xrightarrow{\rho} y$. Consequently, $\langle W, q\rangle = \langle W, \varphi \circ h \circ \mathcal{F}r\rangle \xrightarrow{\pi} \varphi(y)$. Hence, $\langle U, \varphi \circ h\rangle \xrightarrow{\pi^*} \varphi(y)$, i.e., $\varphi : (L,\rho) \to (K,\pi^*)$ is continuous. Thus, $id_K : (K,\pi^*) \to (K,\pi)$ is a concrete reflection of (K,π) in $Conv_{\mathcal{F}}$ and the proof is complete. □

As $Conv_{\mathcal{F}}$ is a full isomorphism closed subcategory of $Net_{\mathcal{F}}$ which is closed under formation of products in $Net_{\mathcal{F}}$, Theorems 2.7 and 2.8 result in

Corollary 2.9. *Let $\mathcal{K}$ be a construct with finite concrete products and discrete terminal objects, let $\mathcal{K}$ have initial subobjects with respect to inclusion maps and let $\mathcal{F} : \mathcal{K} \to Set$ be the forgetful functor. If $\mathcal{K}$ is cartesian closed, then so is $Conv_{\mathcal{F}}$.*

Theorem 2.10. *Let $\mathcal{K}$ be a construct with finite concrete products and discrete terminal objects, let $\mathcal{K}$ have initial subobjects with respect to inclusion maps, and let $\mathcal{F} : \mathcal{K} \to Set$ be the forgetful functor. Let $\mathcal{S}$ have finite products and let these products be preserved by $\mathcal{F}$. If $\mathcal{K}$ is cartesian closed, the so is $Lim_{\mathcal{F}}$.*

Proof. Let (K,π), (L,ρ) be $\mathcal{F}$-limit spaces and L^K be the function space of K and L in $\mathcal{K}$. Let M be the initial subobject (with respect to inclusion map) of L^K whose underlying set is the set of all $\mathcal{K}$-morphisms $f : K \to L$ that are continuous w.r.t. π and ρ. Let σ be the convergence class on M defined by $\langle T, g\rangle \xrightarrow{\sigma} z$

if and only if $\langle S, f\rangle \xrightarrow{\pi} x$ implies $\langle S\times T, g^f\rangle \xrightarrow{\rho} z(x)$ where $g^f : \mathcal{F}S\times\mathcal{F}T \to L$ is the $\mathcal{K}$-morphism given by $g^f(s,t) = g(t)(f(s))$. Indeed, g^f is a $\mathcal{K}$-morphism because $g^f = e\circ(f\times g)$ where $e : K\times L^K \to L$ is the evaluation map. And for the same reasons as in the proof of Theorem 2.7, e is a $\mathcal{K}$-morphism.

We will show that (M,σ) is an $\mathcal{F}$-limit space. To this end, let $\langle T, g\rangle$ be an $\mathcal{F}$-net in M and let $z\in M$ be an element such that $g(t) = z$ for all $t\in\mathcal{F}T$. Let $\langle S, f\rangle \xrightarrow{\pi} x$. Then $g^f(s,t) = g(t)(f(s)) = z(f(s))$ for each $(s,t)\in\mathcal{F}S\times\mathcal{F}T$. Hence, $g^f = z\circ f\circ pr_{\mathcal{F}S}$, so that $\langle S\times T, g^f\rangle$ is a subnet of $\langle S, z\circ f\rangle$. Since z is continuous, we have $\langle S, z\circ f\rangle \xrightarrow{\pi} z(x)$. Therefore, $\langle S\times T, g^f\rangle \xrightarrow{\rho} z(x)$, i.e., $\langle T, g\rangle \xrightarrow{\sigma} z$. Thus, the constant net axiom is satisfied.

Let $\langle T, g\rangle \xrightarrow{\sigma} z$, $\langle S, f\rangle \xrightarrow{\pi} x$, and let $\langle U, h\rangle$ be a subnet of $\langle T, g\rangle$. Then there is an $\mathcal{S}$-morphism $p : U \to T$ with $h = g\circ\mathcal{F}p$. We have $\langle S\times T, g^f\rangle \xrightarrow{\rho} z(x)$. As the map $q : S\times U \to S\times T$ given by $q = id_s\times p$ is an $\mathcal{S}$-morphism, $\langle S\times U, g^f\circ\mathcal{F}q\rangle$ is a subnet of $\langle S\times T, g^f\rangle$. Thus, there holds $\langle S\times U, g^f\circ\mathcal{F}q\rangle \xrightarrow{\sigma} z(x)$. Since $\mathcal{F}q = \mathcal{F}id_s\times\mathcal{F}p = id_{\mathcal{F}s}\times\mathcal{F}p$, we have $g^f\circ\mathcal{F}q(s,u) = g^f(s,\mathcal{F}p(u)) = g(\mathcal{F}p(u))(f(s)) = h(u)(f(s)) = h^f(s,u)$ for every $(s,u)\in\mathcal{F}S\times\mathcal{F}U$. Consequently, $\langle U, h\rangle \xrightarrow{\sigma} z$ and the validity of the subnet axiom is shown.

Let $\langle T, g\rangle$ be an $\mathcal{F}$-net in M, $\langle T, g\rangle \overset{\sigma}{\nrightarrow} z$. Then there exists an $\mathcal{F}$-net $\langle S, f\rangle$ in K with $\langle S, f\rangle \xrightarrow{\pi} x$ such that $\langle S\times T, g^f\rangle \overset{\rho}{\nrightarrow} z(x)$. Thus, there exists a subnet $\langle U, h\rangle$ of $\langle S\times T, g^f\rangle$ such that $\langle V, s\rangle \overset{\rho}{\nrightarrow} z(x)$ for any subnet $\langle V, s\rangle$ of $\langle U, h\rangle$. Let $p : U \to S\times T$ be the $\mathcal{S}$-morphism with $h = g^f\circ\mathcal{F}p$. Then $pr_T\circ p : U \to T$ is an $\mathcal{S}$-morphism, so that $\langle U, g\circ\mathcal{F}pr_T\circ\mathcal{F}p\rangle$ is a subnet of $\langle T, g\rangle$. Of course, $\mathcal{F}pr_T = pr_{\mathcal{F}T}$ where $pr_{\mathcal{F}T} : \mathcal{F}S\times\mathcal{F}T \to \mathcal{F}T$ is the projection. Let $\langle W, t\rangle$ be a subnet of $\langle U, g\circ pr_{\mathcal{F}T}\circ\mathcal{F}p\rangle$. Then there is an $\mathcal{S}$-morphism $q : W \to U$ such that $t = g\circ pr_{\mathcal{F}T}\circ\mathcal{F}p\circ\mathcal{F}q$. Let $r : W \to S\times W$ be the map given by $r = (pr_S\circ p\circ q\circ id_W)\circ d_W$ where $d_W : W \to W\times W$ is the diagonal. Then r is an $\mathcal{S}$-morphism and we have $g^f\circ\mathcal{F}p\circ\mathcal{F}q(w) = g^f(pr_{\mathcal{F}S}(\mathcal{F}p(\mathcal{F}q(w))), pr_{\mathcal{F}T}(\mathcal{F}p(\mathcal{F}q(w)))) = g(pr_{\mathcal{F}T}(\mathcal{F}p(\mathcal{F}q(w))))(f(pr_{\mathcal{F}S}(\mathcal{F}p(\mathcal{F}q(w))))) = t(w)(f(pr_{\mathcal{F}S}(\mathcal{F}r(w)))) = t^f(pr_{\mathcal{F}S}(\mathcal{F}r(w)), w) = t^f(\mathcal{F}r(w))$ for each $w\in\mathcal{F}W$. Hence, $\langle W, g^f\circ\mathcal{F}p\circ\mathcal{F}q\rangle$ is a subnet of $\langle S\times W, t^f\rangle$. As $\langle W, g^f\circ\mathcal{F}p\circ\mathcal{F}q\rangle$ is a subnet of $\langle U, h\rangle$, we have $\langle W, g^f\circ\mathcal{F}p\circ\mathcal{F}q\rangle \overset{\rho}{\nrightarrow} z(x)$. Consequently, $\langle S\times W, r^f\rangle \overset{\rho}{\nrightarrow} z(x)$. Hence, $\langle W, r\rangle \overset{\sigma}{\nrightarrow} z$, which means that no subnet of $\langle U, g\circ pr_{\mathcal{F}T}\circ\mathcal{F}p\rangle$ converges to z (w.r.t. σ). Thus, the Urysohn axiom is satisfied and we have shown that (M,σ) is an $\mathcal{F}$-limit space.

We will show that the evaluation map $e : (K,\pi)\times(M,\sigma) \to (L,\rho)$ is continuous. Put $(N,\tau) = (K,\pi)\times(M,\sigma)$ and let $\langle S, f\rangle \xrightarrow{\tau} (x,z)$. Then $\langle S, pr_K\circ f\rangle \xrightarrow{\pi} x$ and $\langle S, pr_M\circ f\rangle \xrightarrow{\sigma} z$. Hence, $\langle S\times S, h^g\rangle \xrightarrow{\rho} z(x)$ where $g = pr_K\circ f$ and $h = pr_M\circ f$. As the diagonal $d_S : S \to S\times S$ is an $\mathcal{S}$-morphism, $\langle S, h^g\circ d_{\mathcal{F}S}\rangle$ is a subnet of $\langle S\times S, h^g\rangle$. Thus, $\langle S, h^g\circ d_{\mathcal{F}S}\rangle \xrightarrow{\rho} z(x)$. Whenever $s\in\mathcal{F}S$, we have $h^g(d_{\mathcal{F}S}(s)) = h^g(s,s) = pr_M(f(s))(pr_K(f(s)) = e(pr_K(f(s)), pr_M(f(s)) = e(f(s))$. Therefore, $\langle S, e\circ f\rangle$ is a subnet of $\langle S\times S, h^g\rangle$ and, consequently, $\langle S, e\circ f\rangle \xrightarrow{\rho} z(x) = e(x,z)$. Hence, e is continuous.

Let (N,τ) be an $\mathcal{F}$-limit space and let $\varphi : (K,\pi)\times(N,\tau) \to (L,\rho)$ be a continuous $\mathcal{K}$-morphism. Put $\varphi^*(w)(x) = \varphi(x,w)$ for any $w\in N$ and any $x\in L$. Then, for the same reasons as in the proof of Theorem 2.7, $\varphi^* : N \to M$ is a $\mathcal{K}$-morphism. Let $\langle S, f\rangle \xrightarrow{\pi} x$ and $\langle T, g\rangle \xrightarrow{\tau} y$ and put $(P,\mu) = (K,\pi)\times(N,\tau)$. Then $\langle S\times T, f\times g\rangle \xrightarrow{\mu} (x,y)$ and thus $\langle S\times T, \varphi\circ(f\times g)\rangle \xrightarrow{\rho} \varphi(x,y)$. For every $(s,t)\in\mathcal{F}S\times\mathcal{F}T$, we have $\varphi((f\times g)(s,t)) = \varphi(f(s), g(t)) = \varphi^*(g(t))(f(s)) = (\varphi^*\circ g)^f(s,t)$. Hence, $\langle S\times T, (\varphi^*\circ g)^f\rangle \xrightarrow{\rho} \varphi(x,y) = \varphi^*(y)(x)$, which yields $\langle S, \varphi^*\circ g\rangle \xrightarrow{\sigma} \varphi^*(y)$. Thus, $\varphi^* : (N,\tau) \to (M,\sigma)$ is continuous. As the equality $e\circ(id_K\times\varphi^*) = \varphi$ is clearly valid, we have shown that (M,σ) is a function space of (K,π) and (L,ρ) in $Lim_{\mathcal{F}}$. The proof is complete. □

3. Categorical Structures of Convergence and Closure

Throughout this section, we assume that there is given a class $\mathcal{M}$ of monomorphisms in $\mathcal{K}$ which contains all isomorphisms and is closed under compositions. Further, we assume that $\mathcal{K}$ is $\mathcal{M}$-complete, i.e., that all inverse images and intersections of $\mathcal{M}$-subobjects exist and are $\mathcal{M}$-subobjects. Consequently, there is a class $\mathcal{E}$ of $\mathcal{K}$-morphisms such that the pair $(\mathcal{E},\mathcal{M})$ is a factorization system for morphisms in $\mathcal{K}$. Given a $\mathcal{K}$-object K, $\mathcal{M}|K$ will denote the class of all (equivalence classes of) $\mathcal{M}$-subobjects of K. Of course, $\mathcal{M}|K$ is a large-complete lattice for each $\mathcal{K}$-object K - cf. [4]. Its joins will be denoted by $\vee$ and $\bigvee$ and its least element by o_K. A *closure operator* [3] on $\mathcal{K}$ (with respect to $\mathcal{M}$) is given by assigning to every $\mathcal{K}$-object K a map $c_K : \mathcal{M}|K \to \mathcal{M}|K$ satisfying the following three axioms:

(1) $m \leq c_K(m)$ for all $m \in \mathcal{M}|K$,

(2) if $m \leq n$ in $\mathcal{M}|K$, then $c_K(m) \leq c_K(n)$,

(3) $\varphi(c_K(m)) \leq c_L(\varphi(m))$ for each $\mathcal{K}$-morphism $\varphi : K \to L$ and each $m \in \mathcal{M}|K$; here, $\varphi(m)$ denotes the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of $\varphi \circ m$.

A closure operator $c = (c_K)_{K \in Obj\mathcal{K}}$ on $\mathcal{K}$ is called:
grounded if $c_K(o_K) \cong o_K$ for all $K \in Obj\mathcal{K}$,
additive if $c_K(m \vee n) \cong c_K(m) \vee c_K(n)$ whenever $K \in Obj\mathcal{K}$ and $m, n \in \mathcal{M}|K$,
idempotent if $c_K(c_K(m)) \cong c_K(m)$ whenever $K \in Obj\mathcal{K}$ and $m \in \mathcal{M}|K$.

Since $\mathcal{K}$ is $\mathcal{M}$-complete, $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ is $Emb_{\mathcal{M}}$-complete where $Emb_{\mathcal{M}}$ denotes the class of all $\mathcal{M}$-embeddings in $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$. In the sequel, we assume that $\mathcal{M}$ contains all points (of $\mathcal{K}$-objects). For each $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$-object $\mathbf{K} = (K, \pi)$ and each $m \in \mathcal{M}|K$, we put $c_{\mathbf{K}}(m) = m \vee \bigvee\{x \in \tilde{K}$; there is an $\mathcal{F}$-net $\langle S, f\rangle$ in K such that f factors through m and $\langle S, f\rangle \xrightarrow{\pi} x\}$. We get a map $c_{\mathbf{K}} : \mathcal{M}|K \to \mathcal{M}|K$.

As there is an isomorphism between the (large) complete lattices $\mathcal{M}|K$ and $Emb_{\mathcal{M}}|\mathbf{K}$ (= the class of all initial subobjects of $\mathbf{K}$ with respect to $\mathcal{M}$-morphisms), $c_{\mathbf{K}}$ determines a unique map $Emb_{\mathcal{M}}|\mathbf{K} \to Emb_{\mathcal{M}}|\mathbf{K}$ which will also be denoted by $c_{\mathbf{K}}$.

Proposition 3.1. *The maps $c_{\mathbf{K}}$, $\mathbf{K} \in Obj[\mathcal{S}, \mathcal{F}, \mathcal{K}]$, constitute a closure operator on $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ with respect to $Emb_{\mathcal{M}}$.*

Proof. Let $\mathbf{K} = (K, r) \in Obj[\mathcal{S}, \mathcal{F}, \mathcal{K}]$. Clearly, if $m \in \mathcal{M}|\mathcal{K}$, then $m \leq c_{\mathbf{K}}(m)$ and, if also $m' \in \mathcal{M}|K$, then $m \leq m' \Rightarrow c_{\mathbf{K}}(m) \leq c_{\mathbf{K}}(m')$. Consequently, the same is also valid when $m, m' \in Emb_{\mathcal{M}}|\mathcal{K}$. Let $\varphi : \mathbf{K} \to \mathbf{L}$ be a morphism in $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$, $\mathbf{K} = (K, \pi)$, $\mathbf{L} = (L, \rho)$. Then $\varphi : K \to L$ is a $\mathcal{K}$-morphism and for every $m \in \mathcal{M}|K$ we have $\varphi(c_{\mathbf{K}}(m)) = \varphi(m) \vee \bigvee\{\varphi(x); x \in \tilde{K}$ and there is and $\mathcal{F}$-net $\langle S, f\rangle$ in K such that f factors through m and $\langle S, f\rangle \xrightarrow{\pi} x\} \leq \varphi(m) \vee \bigvee\{y \in \tilde{L}$; there is an $\mathcal{F}$-net $\langle S, \varphi \circ f\rangle$ in L such that $\varphi \circ f$ factors through $\varphi(m)$ and $\langle S, \varphi \circ f\rangle \xrightarrow{\rho} y\} \leq \varphi(m) \vee \bigvee\{y \in \tilde{L}$; there is and $\mathcal{F}$-net $\langle T, g\rangle$ in L such that g factors through $\varphi(m)$ and $\langle T, g\rangle \xrightarrow{\rho} y\} = c_{\mathbf{K}}(\varphi(m))$. Hence, $\varphi(c_{\mathbf{K}}(m)) \leq c_{\mathbf{K}}(\varphi(m))$ and the same is also true when $m \in Emb_{\mathcal{M}}|\mathbf{K}$. Thus, $c_{\mathbf{K}}$, $\mathbf{K} \in Obj[\mathcal{S}, \mathcal{F}, \mathcal{K}]$, is a closure operator on $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ w.r.t. $Emb_{\mathcal{M}}$. □

The closure operator $c_{\mathbf{K}}$, $\mathbf{K} \in Obj[\mathcal{S}, \mathcal{F}, \mathcal{K}]$, on $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$ (with respect to $Emb_{\mathcal{M}}$) will be called $\mathcal{M}$-*natural* and so will be called also its restriction to (objects of) any full subcategory of $[\mathcal{S}, \mathcal{F}, \mathcal{K}]$.

We say that a $\mathcal{K}$-object K has enough points [2] if $\bigvee \tilde{K} \cong id_K$. If every $\mathcal{K}$-object has enough points, then we say that $\mathcal{K}$ has enough points.

Lemma 3.2. *Let $(K, \pi) \in Net_{\mathcal{F}}$. If the domain of each $\mathcal{M}$-subobject of K has enough points, then for each $m \in \mathcal{M}|K$ we have $m \leq \bigvee\{x \in \tilde{K}$; there is an $\mathcal{F}$-net $\langle S, f\rangle$ in K such that f factors through m and $\langle S, f\rangle \xrightarrow{\pi} x\}$.*

Proof. Let $m : M \to K$ and let $\tilde{M} = \{x_j; j \in J\}$. Then $\bigvee \tilde{M} = \bigvee\{x_j; j \in J\} = id_M$. Let $\langle S, g\rangle$ be an $\mathcal{F}$-net in $I_{\mathcal{K}}$. Then $\langle S, m \circ x_j \circ g\rangle$ is an $\mathcal{F}$-net in K and $\langle S, m \circ x_j \circ g\rangle \xrightarrow{\pi} m \circ x_j \in \tilde{K}$ for each $j \in J$. We have $m \cong m \circ id_M \cong m \circ \bigvee\{x_j; j \in J\} \cong \bigvee\{m \circ x_j; j \in J\}$. This results in the statement. □

Let $\mathcal{A}$, $\mathcal{B}$ be categories and $\mathcal{G} : \mathcal{A} \to \mathcal{B}$ a functor. Let $\mathcal{W}$ be a $\mathcal{G}$-structured source, i.e., a source $f_j : P \to \mathcal{F}U_j$, $j \in J$, in $\mathcal{B}$ where U_j, $j \in J$, are $\mathcal{A}$-objects. A *lift* $\mathcal{V}$ of the source $\mathcal{W}$ is any source $h_j : U \to U_j$, $j \in J$, in $\mathcal{A}$ such that $\mathcal{F}V = \mathcal{W}$.

Lemma 3.3. *Let every $\mathcal{F}$-structured source consisting of a pair of points has a lift. Suppose that for every pair of $\mathcal{K}$-morphisms $e : M \to L, p : N \to L$ such that $e \in \mathcal{E}$ or $p \in \mathcal{E}$ there are $x \in \tilde{M}$ and $y \in \tilde{N}$ such that $e \circ x = p \circ y$. If $\langle S, f\rangle$, $\langle T, g\rangle$ are $\mathcal{F}$-nets in a $\mathcal{K}$-object K such that g factors through the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of f, then there is an $\mathcal{F}$-net in K which is a subnet of both $\langle S, f\rangle$ and $\langle T, g\rangle$.*

Proof. Let $f = m \circ e$ be the $(\mathcal{E}, \mathcal{M})$-factorization of f where $e : \mathcal{F}(S) \to L$ and let $g = m \circ p$. Then there are $x \in \widetilde{\mathcal{F}(S)}$ and $y \in \widetilde{\mathcal{F}(T)}$ such that $e \circ x = p \circ y$. Let $\mathcal{W}$ be the $\mathcal{F}$-structured source consisting of x and y and let $\mathcal{V}$ be a lift of $\mathcal{W}$ consisting of a pair of $\mathcal{S}$-morphisms $h : U \to S$ and $h' : U \to T$ where $\mathcal{F}(h) = x$ and $\mathcal{F}(h') = y$. Clearly, the $\mathcal{F}$-net $\langle U, f \circ x\rangle = \langle U, g \circ y\rangle$ is a subnet of both $\langle S, f\rangle$ and $\langle T, g\rangle$. □

Example 3.4. 1. The assumptions of Lemma 3.3 are clearly satisfied if $\mathcal{S}$ is a topological category over $\mathcal{K}$.

2. If $\mathcal{F} : Dir \to Set$ is the forgetful functor (where Dir is the construct of directed sets and cofinal maps), then the first assumption of Lemma 3.3 is not satisfied and the statement of the Lemma is not true in general (note that, in the subnet $\langle U, f \circ x\rangle$ of $\langle S, f\rangle$ and $\langle T, g\rangle$ found in the proof of the Lemma, the morphism $f \circ x$ is a point in $\mathcal{K}$; but points need not be morphisms in Dir in general). The first assumption of Lemma 3.3 is satisfied if, for example, $\mathcal{S}$ has a unique (up to isomorphisms) object U such that $\mathcal{F}U$ is a terminal object of $\mathcal{K}$ and, for every $\mathcal{S}$-object V and every point $x \in \tilde{\mathcal{F}V}$, there is an $\mathcal{S}$-morphism h with $\mathcal{F}h = x$.

Recall that a point $x \in \tilde{K}$, where $K \in Obj\mathcal{K}$, is said to be $\bigvee$-prime (cf. [4]) if, for every subclass $\mathcal{L} \subseteq \tilde{K}$, $x \le \bigvee \mathcal{L}$ implies $x \cong y$ for some $y \in \mathcal{L}$. We say that the category $\mathcal{K}$ is *$\bigvee$-prime* if, whenever $K \in Obj\mathcal{K}$, each point $x \in \tilde{K}$ is $\bigvee$-prime, i.e., $\widetilde{\bigvee \mathcal{L}} = \bigcup\{j \in Mor\mathcal{K};$ there exists $x \in \mathcal{L}$ with $\bigvee \mathcal{L} \circ j \cong x\}$ for each subclass $\mathcal{L} \subseteq \tilde{K}$.

Lemma 3.5. *Let $\mathcal{K}$ have enough points, let $\mathcal{K}$ be $\bigvee$-prime and let the assumptions of Lemma 3.3 be fulfilled. Let $\mathbf{K} = (K, \pi) \in ObjLim_{\mathcal{F}}$, let $\langle S, f\rangle$ be an $\mathcal{F}$-net in K and let $x \in \tilde{K}$ be a point. Then $\langle S, f\rangle \xrightarrow{\pi} x$ if and only if, for each subnet $\langle T, g\rangle$ of $\langle S, f\rangle$, we have $x \le c_{\mathbf{K}}(m)$ where m is the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of g.*

Proof. Suppose that $\langle S, f\rangle \overset{\pi}{\not\to} x$. Then there is a subnet $\langle T, g\rangle$ of $\langle S, f\rangle$ such that $\langle U, h\rangle \overset{\pi}{\not\to} x$ for any subnet $\langle U, h\rangle$ of $\langle T, g\rangle$. Let m be the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of g and let $\langle V, p\rangle$ be an $\mathcal{F}$-net in K such that p factors through m. By Lemma 3.3, there is an $\mathcal{F}$-net $\langle W, q\rangle$ in K that is a subnet of both $\langle V, p\rangle$ and $\langle T, g\rangle$. But then $\langle W, q\rangle \overset{\pi}{\not\to} x$, hence $\langle V, p\rangle \overset{\pi}{\not\to} x$. As x is $\bigvee$-prime, $x \not\le \bigvee\{x \in \tilde{K};$ there is an $\mathcal{F}$-net $\langle S, f\rangle$ in K such that f factors through m and $\langle S, f\rangle \xrightarrow{\pi} x\} = c_{\mathbf{K}}(m)$ (for the last equality see Lemma 3.2). We have proved that from $x \le c_{\mathbf{K}}(m)$ for each subnet $\langle T, g\rangle$ of $\langle S, f\rangle$, where m is the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of g, it follows that $\langle S, f\rangle \xrightarrow{\pi} x$. As the converse implication is obvious, the proof is complete. □

Theorem 3.6. *Let $\mathcal{K}$ have enough points, let $\mathcal{K}$ be $\bigvee$-prime and let the assumptions of Lemma 3.3 be fulfilled. Then for every pair $\mathbf{K}, \mathbf{L} \in ObjLim_{\mathcal{F}}$ we have:*

(1) if $\mathbf{K} \ne \mathbf{L}$, then $c_{\mathbf{K}} \ne c_{\mathbf{L}}$,

(2) if $\mathbf{K} = (\mathbf{K}, \pi)$, $\mathbf{L} = (\mathbf{L}, \varrho)$ and $\varphi : K \to L$ is a $\mathcal{K}$-morphism with $\varphi(c_{\mathbf{K}}(m)) \le c_{\mathbf{L}}(\varphi(m))$ for each $m \in \mathcal{M}|K$, then φ is continuous w.r.t. π and ϱ.

Proof. (1) Let $\mathbf{K} = (K, \pi)$, $\mathbf{L} = (L, \varrho)$ and let $\pi \ne \rho$. Without loss of generality we can suppose that there is an $\mathcal{F}$-net $\langle S, f\rangle$ in K and a point $x \in \tilde{K}$ such that $\langle S, f\rangle \xrightarrow{\pi} x$ but $\langle S, f\rangle \overset{\varrho}{\not\to} x$. Thus, by Lemma 3.5, $x \le c_{\mathbf{K}}(m)$ for each subnet $\langle T, g\rangle$ of $\langle S, f\rangle$ where m is the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of g, and there is a subnet $\langle T_0, g_0\rangle$ of $\langle S, f\rangle$ such that $x \not\le c_{\mathbf{K}}(n)$ where n is the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of g_0. Hence, $c_{\mathbf{K}} \ne c_{\mathbf{L}}$.

(2) Let $\langle S, f\rangle$ be an $\mathcal{F}$-net in K and let $\langle S, f\rangle \xrightarrow{\pi} x$. Then, by Lemma 3.5, for each subnet $\langle T, g\rangle$ of $\langle S, f\rangle$ we have $x \le c_{\mathbf{K}}(m)$ where m is the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of g. Consequently, $\varphi \circ x \le \varphi(c_{\mathbf{K}}(m)) \le c_{\mathbf{K}}(\varphi(m))$. But every subnet of $\langle S, \varphi \circ f\rangle$ has the form $\langle T, \varphi \circ g\rangle$ where $\langle T, g\rangle$ is a subnet of $\langle S, f\rangle$. Since $\varphi(m)$ is clerly the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of $\varphi \circ g$, we have $\langle S, \varphi \circ f\rangle \xrightarrow{\varrho} \varphi \circ x$ by Lemma 3.5. Therefore, φ is continuous w.r.t. π and ϱ. □

Remark 3.7. Theorem 3.6 states that the "large" category of all categories $Lim_{\mathcal{F}}$ such that $\mathcal{K}$ has enough points, $\mathcal{K}$ is $\bigvee$-prime and the assumptions of Lemma 3.3 are satisfied, can be fully concretely embedded into the "large" category of all categories with a closure operator.

Remark 3.8. Clearly, the $\mathcal{M}$-natural closure operator on $[\mathcal{S},\mathcal{F},\mathcal{K}]$ is grounded if and only if, for each $\mathbf{K} = (K,\pi) \in Obj[\mathcal{S},\mathcal{F},\mathcal{K}]$ and each $x \in \tilde{K}, x \cong o_{\mathbf{K}}$ whenever there is an $\mathcal{F}$-net $\langle S, f\rangle$ in K such that f factors through $o_{\mathbf{K}}$ and $\langle S, f\rangle \xrightarrow{\pi} x$.

Example 3.9. Let $\mathcal{S}$ be a construct with the forgetful functor $\mathcal{F} : \mathcal{S} \to Set$ and let $\mathcal{M}$ be the class of inclusion maps. If $\mathcal{F}(X) \neq \emptyset$ for each $X \in Obj\mathcal{S}$, then the $\mathcal{M}$-natural closure operator on $[\mathcal{S},\mathcal{F}, Set]$ is grounded.

We will give sufficient conditions for the $\mathcal{M}$-natural closure operator on $[\mathcal{S},\mathcal{F},\mathcal{K}]$ to be additive and idempotent, respectively.

Definition 3.10. The category $\mathcal{S}$ is said to be *join-hereditary* with respect to $\mathcal{F}$ and $\mathcal{M}$ if, for any $\mathcal{K}$-object K, any pair m, n of $\mathcal{M}$-subobjects of K and any $\mathcal{F}$-net $\langle S, f\rangle$ in K such that f factors through $m \vee n$, there are an $\mathcal{S}$-object T, an $\mathcal{S}$-morphism $h : T \to S$ and a $\mathcal{K}$-morphism g such that $f \circ \mathcal{F}(h) = m \circ g$ or $f \circ \mathcal{F}(h) = n \circ g$.

Theorem 3.11. *The $\mathcal{M}$-natural closure operator on $Conv_{\mathcal{F}}$ is additive if $\mathcal{S}$ is join-hereditary with respect to $\mathcal{F}$ and $\mathcal{M}$.*

Proof. Let $\mathcal{S}$ be join-hereditary with respect to $\mathcal{F}$ and $\mathcal{M}$. Let $\mathbf{K} = (K,\pi) \in ObjConv_{\mathcal{F}}$ and $m, n \in \mathcal{M}|K$. Let $\langle S, f\rangle$ be an $\mathcal{F}$-net in K which factors through $m \vee n$ and let $\langle S, f\rangle \xrightarrow{\pi} x$. Then there is a subnet $\langle T, m \circ g\rangle$ or $\langle T, n \circ g\rangle$ of $\langle S, f\rangle$ which converges to x w.r.t. π (as (K,π) is an $\mathcal{F}$-convergence space). Thus, we have $c_{\mathbf{K}}(m \vee n) = m \vee n \vee \bigvee\{x \in \tilde{K};$ there is an $\mathcal{F}$-net $\langle S, f\rangle$ in K such that f factors through $m \vee n$ and $\langle S, f\rangle \xrightarrow{\pi} x\} \leq (m \vee \bigvee\{x \in \tilde{K};$ there is an $\mathcal{F}$-net $\langle T, p\rangle$ in K such that p factors through m and $\langle T, p\rangle \xrightarrow{\pi} x\}) \vee (n \vee \bigvee\{x \in \tilde{K};$ there is an $\mathcal{F}$-net $\langle T, p\rangle$ in K such that p factors through n and $\langle T, p\rangle \xrightarrow{\pi} x\}) = c_{\mathbf{K}}(m) \vee c_{\mathbf{K}}(n)$. □

Proposition 3.12. *Let $\mathcal{S}$ be a construct, $\mathcal{F} : \mathcal{S} \to Set$ the forgetful functor and $\mathcal{M}$ the class of inclusion maps. Suppose that for any $\mathcal{S}$-object S and any pair U, V of sets with $\mathcal{F}(S) = U \cup V$, there exists an initial subobject T of S with respect to an $\mathcal{M}$-morphism such that $\mathcal{F}(T) = U$ or $\mathcal{F}(T) = V$. Then $\mathcal{S}$ is join-hereditary with respect to $\mathcal{F}$ and $\mathcal{M}$.*

Proof. Let the assumptions of the statement be fulfilled. For any set K, any pair of subsets $M, N \subseteq K$ and any $\mathcal{F}$-net $\langle S, f\rangle$ in K such that $f(\mathcal{F}S) \subseteq M \cup N$, put $U = f^{-1}(M)$ and $V = f^{-1}(N)$. Then $\mathcal{F}\mathcal{S} = U \cup V$ and thus there is an initial subobject T of S with respect to an $\mathcal{M}$-morphism such that $\mathcal{F}(T) = U$ or $\mathcal{F}(T) = V$. Let $h : T \to S$ be an embedding and put $g = f|U$ if $\mathcal{F}(T) = U$ or $g = f|V$ if $\mathcal{F}(T) = V$. Then clearly $f \circ h = g$ and $f(h(t)) \in M$ for each $t \in \mathcal{F}(T)$ or $f(h(t)) \in N$ for each $t \in \mathcal{F}(T)$. Therefore, $\mathcal{S}$ is join-hereditary with respect to $\mathcal{F}$ and $\mathcal{M}$. □

Example 3.13. Let $\alpha \geq \omega$ be an initial ordinal and let $[\alpha]$ be the construct of well-ordered sets isomorphic to α with isotone injections as morphisms. If $\mathcal{S} = [\alpha]$ or $\mathcal{S} = Dir$ (the construct of directed sets and cofinal maps), then $\mathcal{S}$ satisfies the assumptions of Proposition 3.12 and, therefore, $\mathcal{S}$ is join-hereditary with respect to the forgetful functor $\mathcal{F} : \mathcal{S} \to Set$ and the class $\mathcal{M}$ of inclusions in *Set*.

Definition 3.14. We say that an object $\mathbf{K} = (\mathbf{K},\pi) \in [\mathcal{S},\mathcal{F},\mathcal{K}]$ fulfils the *weak condition of iterated limits* if the following condition is satisfied:
If $\langle S, f\rangle \xrightarrow{\pi} x$, $\widetilde{\mathcal{F}S} \neq \emptyset$ and $\langle T_s, g_s\rangle \xrightarrow{\pi} f \circ s$ for each $s \in \widetilde{\mathcal{F}S}$, then there is an $\mathcal{F}$-net $\langle U, h\rangle$ in K such that $\langle U, h\rangle \xrightarrow{\pi} x$ and the $\mathcal{M}$-part n of the $(\mathcal{E},\mathcal{M})$-factorization of h fulfils $n \leq \bigvee\{m_s;\ s \in \widetilde{\mathcal{F}S}\}$ where m_s denotes the $\mathcal{M}$-part of the $(\mathcal{E},\mathcal{M})$-factorization of g_s for each $s \in \widetilde{\mathcal{F}S}$.

Theorem 3.15. *Let $\mathcal{K}$ have enough points and be $\bigvee$-prime and let $\widetilde{\mathcal{F}S} \neq \emptyset$ for each $S \in Obj\mathcal{S}$. If each object of $Net_{\mathcal{F}}$ fulfils the weak condition of iterated limits, then the $\mathcal{M}$-natural closure operator on $Net_{\mathcal{F}}$ is idempotent.*

Proof. Let $\mathbf{K} = (K,\pi) \in ObjNet_{\mathcal{F}}$ and $m \in \mathcal{M}|\mathbf{K}$. By Lemma 3.2, $c_{\mathbf{K}}c_{\mathbf{K}}(m) = \bigvee\{x \in \tilde{K};$ there is an $\mathcal{F}$-net $\langle S, f\rangle$ in K such that f factors through $c_{\mathbf{K}}(m)$ and $\langle S, f\rangle \xrightarrow{\pi} x\}$. If f factors through $c_{\mathbf{K}}(m)$, then $f \circ s \leq c_{\mathbf{K}}(m)$ for each

$s \in \widetilde{\mathcal{F}S}$ and, by Lemma 3.2, $c_{\mathbf{K}}(m) = \bigvee\{x \in \tilde{K};$ there is an $\mathcal{F}$-net $\langle T, g\rangle$ in K such that f factors through m and $\langle T, g\rangle \xrightarrow{\pi} x\}$. As $\mathcal{K}$ is $\bigvee$-prime, for each $s \in \widetilde{\mathcal{F}S}$ there is an $\mathcal{F}$-net $\langle T_s, g_s\rangle$ in K such that g_s factors through m and $\langle T_s, g_s\rangle \xrightarrow{\pi} f \circ s$. As $\mathbf{K}$ fulfills the weak condition of iterated limits, there is an $\mathcal{F}$-net $\langle U, h\rangle$ in K such that $\langle U, h\rangle \xrightarrow{\pi} x$ and the $\mathcal{M}$-part n of the $(\mathcal{E}, \mathcal{M})$-factorization of h fulfills $n \leq \bigvee\{m_s;\ s \in \widetilde{\mathcal{F}S}\}$ where m_s denotes the $\mathcal{M}$ part of the $(\mathcal{E}, \mathcal{M})$-factorization of g_s for each $s \in \widetilde{\mathcal{F}S}$. But the $(\mathcal{E}, \mathcal{M})$-diagonalization property implies that $m_s \leq m$ for each $\widetilde{\mathcal{F}S}$, hence $n \leq m$. Thus, h factors through m. Consequently, $c_{\mathbf{K}}c_{\mathbf{K}}(m) \leq \bigvee\{x \in \tilde{K};$ there is an $\mathcal{F}$-net $\langle U, h\rangle$ in K such that h factors through m and $\langle U, h\rangle \xrightarrow{\pi} x\} \cong c_{\mathbf{K}}(m)$. $\square$

Definition 3.16. Let $\mathcal{S}$ be a construct with concrete products, let $\mathcal{F} : \mathcal{S} \to Set$ be the forgetful functor, and let $\mathcal{F}S \neq \emptyset$ for each $S \in Obj\mathcal{S}$. An object $(K, \pi) \in Lim_{\mathcal{F}}$ is said to fulfil the *condition of iterated limits* if the following is valid: Let $S \in Obj\mathcal{S}$ and let $\langle T_s, g_s\rangle$ be an $\mathcal{F}$-net in K for each $s \in \mathcal{F}S$. Put $U = S \times \prod_{s \in S} T_s$ and let $h : \mathcal{F}U \to K$ be the map given by $h(s, t) = g_s(t(s))$. If $\langle T_s, g_s\rangle \xrightarrow{\pi} x_s$ for each $s \in \mathcal{F}S$ and $\langle S, f\rangle \xrightarrow{\pi} x$, where $f : \mathcal{F}S \to K$ is the map given by $f(s) = x_s$, then $\langle U, h\rangle \xrightarrow{\pi} x$.

Example 3.17. If $\mathcal{S} = Dir$ (see Example 2.2(2)) and $\mathcal{F} : \mathcal{S} \to Set$ is the forgetful functor, then the condition of iterated limits from Definition 3.16 coincides with the condition of iterated limits introduced by J.L. Kelley in [9].

As the condition of iterated limits (for constructs) clearly implies the weak condition of iterated limits, we have

Corollary 3.18. *Let $\mathcal{S}$ be a construct with concrete products, let $\mathcal{F} : \mathcal{S} \to Set$ be the forgetful functor, and let $\mathcal{F}S \neq \emptyset$ for each $S \in Obj\mathcal{S}$. Let $\mathcal{M}$ be the class of all injective maps in Set. If each object of $Lim_{\mathcal{F}}$ fulfils the condition of iterated limits, then the $\mathcal{M}$-natural closure operator on $Lim_{\mathcal{F}}$ is idempotent.*

Definition 3.19. Let $\mathcal{S}$ be a construct, let $\mathcal{F} : \mathcal{S} \to Set$ be the forgetful functor, and let $\mathcal{F}S \neq \emptyset$ for each $S \in Obj\mathcal{S}$. Then $\mathcal{S}$ is called *fine* if for each $\mathcal{S}$-object S there is a map $i \mapsto S^{(i)}$ of $\mathcal{F}S$ into the class of all subobjects of S such that

(1) $j \in \mathcal{F}S^{(i)} \Rightarrow \mathcal{F}S^{(j)} \subseteq \mathcal{F}S^{(i)}$,
(2) $\mathcal{F}(\prod\limits_{k \in K} S_k)^{(v)} \subseteq \mathcal{F}\prod\limits_{k \in K} S_k^{(v(k))}$ whenever $S_k \in Obj\mathcal{S}$ for each $k \in K$,
(3) given $S, T \in Obj\mathcal{S}$, a map $h : S \to T$ is an $\mathcal{S}$-morphism if for any $j \in \mathcal{F}T$ there exists $i \in \mathcal{F}S$ such that $h(\mathcal{F}S^{(i)}) \subseteq \mathcal{F}T^{(j)}$.

Example 3.20. (1) The category Dir of directed sets and cofinal maps is fine: For any directed set $S = (\mathcal{F}S, \leq)$ and any $i \in \mathcal{F}S$, the directed subset $S^{(i)}$ of S is given by $S^{(i)} = ([i), \leq)$ (where $[i) = \{s \in S;\ s \geq i\}$).

(2) The category Set^+ of non-empty sets is fine: For any non-empty set S and any $i \in S$ the subset $S^{(i)}$ is given by $S^{(i)} = S$.

Theorem 3.21. *Let $\mathcal{S}$ be a construct with concrete products, let $\mathcal{F} : \mathcal{S} \to Set$ be the forgetful functor, and let $\mathcal{F}S \neq \emptyset$ for each $S \in Obj\mathcal{S}$ and let $\mathcal{S}$ be fine. If every object of $Lim_{\mathcal{F}}$ fulfils the condition of iterated limits, then for every pair $\mathbf{K}, \mathbf{L} \in ObjLim_{\mathcal{F}}$ we have:*

(1) if $\mathbf{K} \neq \mathbf{L}$, then $c_{\mathbf{K}} \neq c_{\mathbf{L}}$,
(2) if $\mathbf{K} = (K, \pi)$, $\mathbf{L} = (L, \varrho)$ and $\varphi : K \to L$ is a map with $\varphi(c_{\mathbf{K}}(m)) \subseteq c_{\mathbf{L}}(\varphi(m))$ for each $m \subseteq K$, then φ is continuous w.r.t. π and ϱ.

Proof. The proof is analogous to that of Theorem 5.5 in [11]. $\square$

Remark 3.22. Let $\mathcal{S}$ be a construct with concrete products, let $\mathcal{F} : \mathcal{S} \to Set$ be the forgetful functor and let $\mathcal{F}S \neq \emptyset$ for each $S \in Obj\mathcal{S}$. Let $\widehat{Lim_{\mathcal{F}}}$ denote the full subcategory of $Lim_{\mathcal{F}}$ whose objects are precisely the $\mathcal{F}$-limit spaces fulfilling the condition of iterated limits. Theorem 3.21 states that the "large" category of all categories $\widehat{Lim_{\mathcal{F}}}$ can be fully concretely embedded into the "large" category of all categories with an idempotent closure operator. Note that, in difference to Theorem 3.6, the assumptions of Theorem 3.21 are satisfied in the case when $\mathcal{S}$ is the construct Dir of directed sets and cofinal maps.

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