Global Well-Posedness for a Viscosity Problem of the Compressible Heisenberg Chain Equations

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Abstract. In this paper, we are concerned with the existence and uniqueness of global smooth solutions to a viscosity problem for the compressible Heisenberg chain equations in one dimension. Furthermore, we prove the global existence of weak solutions when the parameter $A$ tends to zero by compactness method.

1. Introduction

In this paper, we are concerned with the existence and uniqueness of global smooth solutions to the following periodic boundary value problem:

$$\begin{align*}
\dot{Z}_t &= -\varepsilon \dot{Z} \times (G(\dot{Z}_x) \dot{Z}_x) + \ddot{Z} \times G(\ddot{Z}_x) \dot{Z}_x, \quad x \in \mathbb{R}, \ t \in \mathbb{R}_+, \\
\dot{Z}(x, 0) &= \dot{Z}_0(x), \ \ddot{Z}(x + D, t) = \ddot{Z}(x - D, t), \ |\dot{Z}_0(x)| \equiv 1, \ x \in \mathbb{R},
\end{align*}$$

where $G(\xi) = A + B|\xi|^2$ and $\varepsilon, A, B, D > 0$ are generic constants. If $\varepsilon = 0$, the above system is reduced to the compressible Heisenberg chain equations (see [5]). Ding, Guo and Su [2] proved the existence of measure-valued solutions to the compressible Heisenberg chain equations. If $B = 0$, the system is reduced to the Landau-Lifshitz equations, one can refer to [3, 4, 6, 9] and their references for related topics.

There is a fact we will use in this paper: (1) is equivalent to the following form in the classical sense:

$$\begin{align*}
\dot{Z}_t &= \varepsilon G(\dot{Z}_x) Z_x^2 \dot{Z} + \varepsilon G(\ddot{Z}_x) \dot{Z}_x + \ddot{Z} \times G(\ddot{Z}_x) \dot{Z}_x.
\end{align*}$$

Note that if $A = 0$, (3) is in fact the one-dimensional heat flow of $p$-harmonic map with values into sphere and $p = 4$ by neglecting the last term $\ddot{Z} \times G(\ddot{Z}_x) \dot{Z}_x$ on the right hand side of (3), one can refer to [1] for the global existence of weak solutions to $p$-harmonic maps in multi-dimensions. The readers can also refer to [7] for related topics.

We first establish the existence of local smooth solution to problem (1)-(2) by difference-differential method, and then give a priori estimates for such solutions to obtain the global existence of regular solutions.
Lemma 2.2. \([8]\) Let \(u\), where \(u\) is a function in \(L^p\), for fixed \(\varepsilon\), independent of \(u\), we only consider the case \(\varepsilon \rightarrow 0\). The aim is to construct the local solution (in time interval) of (1)-(2), we apply the differential system:

\[
\frac{dZ}{dt} = \sum_{j=1}^{n} \left( \Delta_j Z_j \times \left( G(\frac{\Delta_j Z_j}{h}) \right) \right)
\]

where \(Z_j\) is the solution of the differential equation.

**Theorem 1.1.** Let \(Z_0(x) \in H^k(\Omega), k \geq 2\). Then for any given \(T > 0\), problem (1)-(2) admits a unique global regular solution \(Z(x, t)\):

\[
Z(x, t) \in \mathcal{F}(T) = \left( \bigcap_{s=0}^{\frac{T}{h}} W^s, T; H^{k-2s}(\Omega) \right) \bigcap \left( \bigcap_{s=0}^{\frac{T}{h}} H^s(0, T; H^{k+1-2s}(\Omega)) \right).
\]

**Theorem 1.2.** Let \(Z_0(x) \in W^{1,4}(\Omega)\) and \(A = 0\). Then for any given \(T > 0\), problem (1)-(2) admits a global weak solution \(Z(x, t)\) such that \(Z(x, t) \in L^\infty(0, T; W^{1,4}(\Omega)), \ Z_t(x, t) \in L^2(0, T; L^2(\Omega)), \) and (1) holds in the sense of distribution.

2. Local Smooth Solution

We need the following well known lemmas.

**Lemma 2.1.** \([8]\) Let \(p \) be a real number and \(j, m\) be integers such that \(2 \leq p \leq \infty, 0 \leq j < m\). Then

\[
\|\delta^j u_h\|_p \leq C \|u_h\|_2^{-\alpha} \left( \|\delta^m u_h\|_2 + \|u_h\|_2 \right)^{\alpha},
\]

where \(u_h = \{ u_j = u(x) | j = 0, \pm 1, \pm 2, \ldots, \pm J \}, x_j = jh, h = 2D/J, \alpha = \frac{1}{m}(j + \frac{1}{2} - \frac{1}{p})\), \(C\) is a constant which is independent of \(u_h\) and \(h\), and

\[
\|\delta^j u_h\|_p = \left( \sum_{i=0}^{J} \left| u_i \right|^p \right)^{\frac{1}{p}}, \quad \|\delta^j u_h\|_\infty = \max_{0 \leq j \leq k} \left| \Delta_j u_j \right| / h^k.
\]

**Lemma 2.2.** \([8]\) Let \(u_h = \{ u_j = u(x) | j = 0, \pm 1, \pm 2, \ldots, \pm J \}, v_h = \{ v_j = v(x) | j = 0, \pm 1, \pm 2, \ldots, \pm J \}, \) and \(u_{i+j} = u_j, v_{i+j} = v_j, \) we have

(i). \( \sum_{j=1}^{J} u_j \Delta_+ v_j = - \sum_{j=0}^{J} v_j \Delta_- u_i, \)

(ii). \( \sum_{j=1}^{J} u_j \Delta_+ v_j = - \sum_{j=0}^{J} \Delta_+ u_i \Delta_+ v_j, \)

(iii). \( \Delta_+ (u_j v_j) = u_{j+1} \Delta_+ v_j + v_{j+1} \Delta_+ u_j, \Delta_- (u_j v_j) = u_{j-1} \Delta_- v_j + v_{j-1} \Delta_- u_j, \)

where \(\Delta_+, \Delta_-\) denote the forward and backward difference respectively.

To get the existence of local smooth solution of (1)-(2), we apply the differential-difference method. Our aim is to construct the local solution (in time interval) of (1)-(2) as limits of sequence \([Z_k]\) when \(h\) tends to zero. We only consider the case \(\varepsilon = 1\). Firstly, we establish the following difference-differential system:

\[
\frac{dZ_i}{dt} = -Z_i \times \left( \Delta_+ G \left( \left( \frac{\Delta_+ Z_i}{h} \right) \right) \right) + \Delta_- \left( G \left( \left( \frac{\Delta_- Z_i}{h} \right) \right) \right), \quad (4)
\]
Then we get the following lemma.

\[ \vec{Z} |_{t=0} = \vec{Z}_{0j} = \vec{Z}_0(jh), \]
\[ \vec{Z}_{j+1} = \vec{Z}_j, \]

where \( j = 0, \pm 1, \pm 2, \ldots, \pm J, \cdot \cdot \cdot \), \( h = \frac{20}{J}, \vec{Z}_j = \vec{Z}(jh, t), J > 0 \).

It is clear that the initial value problem for ordinary differential equations (4)-(6) admits a local smooth solution. For such solution, we shall give some estimates uniformly in \( h \) and then get a local smooth solution to problem (1)-(2). In this section we always denote a smooth solution of (4)-(6) by \( \vec{Z}_j, j = 0, \pm 1, \pm 2, \cdot \cdot \cdot \).

**Lemma 2.3.** If \( \vec{Z}_0(\vec{x}) \in W^{1,4}(\Omega) \), then \( \vec{Z}_j(t) \in S^2 \) for all \( t \) and there are uniform constants \( T_0 > 0, C > 0 \) independent of \( h \) such that

\[ \sup_{0 \leq t \leq T_0} (\|\delta \vec{Z}_0(\vec{x})\|_2 + \|\delta \vec{Z}_0(\vec{x})\|_4) \leq C. \]

**Proof.** Firstly, multiplying (4) by \( \vec{Z}_j \), we obtain \( \vec{Z}_j \cdot \vec{Z}_j = 0, j \in \mathbb{Z} \). Then we have

\[ |\vec{Z}_j(t)| = |Z_{0j}| \equiv 1, \quad j \in \mathbb{Z}. \]

Secondly, it follows from (4) that

\[ \frac{d}{dt} \Delta_+ \vec{Z}_j = -\Delta_+ \left( \vec{Z}_j \times \frac{\Delta}{h} \left( G \left( \frac{\Delta \vec{Z}_j}{h} \right) \right) \right) + \Delta_+ \left( \vec{Z}_j \times \frac{\Delta}{h} \left( G \left( \frac{\Delta \vec{Z}_j}{h} \right) \right) \right). \]

It yields from multiplying (9) by \( G \left( \frac{\Delta \vec{Z}_j}{h} \right) \Delta \vec{Z}_j \) and then summing over \( j \) from 0 to \( J-1 \) that

\[ \sum_{j=0}^{J-1} \frac{\Delta}{h} \vec{Z}_j \cdot \vec{Z}_j = -\sum_{j=1}^{J} \left| \vec{Z}_j \times \frac{\Delta}{h} \left( G \left( \frac{\Delta \vec{Z}_j}{h} \right) \right) \right|^2. \]

Therefore, we get

\[ A \frac{d}{dt} \|\delta Z_0\|^2 + B \frac{d}{dt} \|\delta Z_0\|^4 + \sum_{j=1}^{J} \left| \vec{Z}_j \times \frac{\Delta}{h} \left( G \left( \frac{\Delta \vec{Z}_j}{h} \right) \right) \right|^2 = 0. \]

Then we have

\[ \|\delta \vec{Z}_0(t)\|_2 + \|\delta \vec{Z}_0(t)\|_4 + \int_0^t \sum_{j=1}^{J} \left| \vec{Z}_j \times \frac{\Delta}{h} \left( G \left( \frac{\Delta \vec{Z}_j}{h} \right) \right) \right|^2 \, dt \leq C(\|\vec{Z}_0(\vec{x})\|_{W^{1,4}}). \]

Now we turn to get higher order estimates. Firstly, by noticing \( \vec{Z}_j(t) \in S^2 \), we have from (4) that

\[ \frac{d}{dt} \vec{Z}_j = \frac{\Delta}{h} \left( G \left( \frac{\Delta \vec{Z}_j}{h} \right) \right) \vec{Z}_j + \vec{Z}_j \times \frac{\Delta}{h} \left( G \left( \frac{\Delta \vec{Z}_j}{h} \right) \right). \]

Then we get the following lemma.
Lemma 2.4. If $\bar{Z}_0(x) \in H^2(\Omega)$, then there are uniform constants $T_0 > 0, C > 0$ independent of $h$ such that

$$
\sup_{0 \leq t \leq T_0} \left( \|\bar{Z}_h(t)\|_2 + \|\bar{Z}_h(t)\|_2 + \|\delta \bar{Z}_h(t)\|_{\infty} \right) + \int_0^{T_0} \left( \|\delta^3 \bar{Z}_h(t)\|_2^2 + \|\delta \bar{Z}_h(t)\|_2^2 \right) \, dt \leq C.
$$

(14)

Proof. Taking the forward difference $\Delta_v$ of (13), multiplying the resulting equations by $\frac{\Delta^2 \bar{Z}_j}{h^3}$, and then summing over $j$ from 1 to $J$, we have

$$
-\frac{1}{2} \frac{d}{dt} \sum_{j=0}^{J-1} \left| \Delta^2 \bar{Z}_j \right|^2 h = \sum_{j=1}^{J} \Delta_v \Delta_v \left( \frac{\Delta \bar{Z}_j}{h} \right) \cdot \frac{\Delta^2 \bar{Z}_j}{h^3}
$$

$$
- \sum_{j=1}^{J} \Delta_v \left( \bar{Z}_j \cdot \frac{\Delta \left( \frac{\Delta \bar{Z}_j}{h} \right)}{h} \right) \cdot \frac{\Delta^2 \bar{Z}_j}{h^3}
$$

$$
+ \sum_{j=1}^{J} \Delta_v \left( \bar{Z}_j \times \frac{\Delta^2 \left( \frac{\Delta \bar{Z}_j}{h} \right)}{h} \right) \cdot \frac{\Delta^2 \bar{Z}_j}{h^3}
$$

$$
= I + II + III.
$$

(15)

It yields from direct calculations, Lemma 2.2 and (8) that

$$
I = \sum_{j=1}^{J} \left( \frac{\Delta \bar{Z}_j}{h} \right) \left( \frac{\Delta^2 \bar{Z}_j}{h^3} \right) \cdot \frac{\Delta \bar{Z}_j}{h} + 2B \sum_{j=1}^{J} \left( \frac{\Delta^2 \bar{Z}_j}{h^3} \right) \cdot \frac{\Delta \bar{Z}_j}{h} + \sum_{j=1}^{J} \Delta_v \left( \frac{\Delta \bar{Z}_j}{h} \right) \left( \frac{\Delta \bar{Z}_j}{h} \right) \cdot \frac{\Delta^2 \bar{Z}_j}{h^3}
$$

$$
+ \sum_{j=1}^{J} \Delta_v \left( \frac{\Delta^2 \bar{Z}_j}{h^3} \right) \cdot \frac{\Delta \bar{Z}_j}{h}
$$

$$
II = - \sum_{j=1}^{J} \Delta_v \left( \bar{Z}_j \cdot \frac{\Delta \left( \frac{\Delta \bar{Z}_j}{h} \right)}{h} \right) \left( \frac{\Delta^2 \bar{Z}_j}{h^3} \right)
$$

$$
+ \sum_{j=1}^{J} \Delta_v \left( \bar{Z}_j \cdot \frac{\Delta \left( \frac{\Delta \bar{Z}_j}{h} \right)}{h} \right) \cdot \frac{\Delta \bar{Z}_j}{h}
$$

$$
= \sum_{j=1}^{J} \Delta_v \left( \bar{Z}_j \cdot \frac{\Delta \left( \frac{\Delta \bar{Z}_j}{h} \right)}{h} \right) \left( \frac{\Delta \bar{Z}_j}{h} \right) \cdot \frac{\Delta \bar{Z}_j}{h}
$$

$$
+ \sum_{j=1}^{J} \Delta_v \left( \bar{Z}_j \cdot \frac{\Delta \left( \frac{\Delta \bar{Z}_j}{h} \right)}{h} \right) \cdot \frac{\Delta \bar{Z}_j}{h}
$$

$$
= \sum_{j=1}^{J} \Delta_v \left( \bar{Z}_j \cdot \frac{\Delta \left( \frac{\Delta \bar{Z}_j}{h} \right)}{h} \right) \left( \frac{\Delta \bar{Z}_j}{h} \right) \cdot \frac{\Delta \bar{Z}_j}{h}
$$

$$
+ \sum_{j=1}^{J} \Delta_v \left( \bar{Z}_j \cdot \frac{\Delta \left( \frac{\Delta \bar{Z}_j}{h} \right)}{h} \right) \cdot \frac{\Delta \bar{Z}_j}{h}
where we have used the following interpolations

\[
- \sum_{j=1}^{l} \left[ \Delta_2 \left( \frac{\Delta Z_j}{h^2} \right) \right] \left( \Delta_2 \Delta Z_j \right) + \frac{\Delta_2^2 \Delta_1 \Delta Z_j}{h^3}
\]

\[III = \sum_{j=1}^{l} \left( \Delta_2 \left( \frac{\Delta Z_j}{h} \right) \right) \left( \Delta_2 \Delta Z_j \right) + \frac{\Delta_2^2 \Delta_1 \Delta Z_j}{h^3}
\]

\[+ \sum_{j=1}^{l} \left( \Delta_2 \left( \frac{\Delta Z_j}{h} \right) \right) \left( \Delta_2 \Delta Z_j \right) + \frac{\Delta_2^2 \Delta_1 \Delta Z_j}{h^3}
\]

\[= \sum_{j=1}^{l} \left( \Delta_2 \left( \frac{\Delta Z_j}{h} \right) \right) \left( \Delta_2 \Delta Z_j \right) + \frac{\Delta_2^2 \Delta_1 \Delta Z_j}{h^3}
\]

Note that we can estimate the second term on the R.H.S. of III by the Cauchy inequality and (8)

\[2B \sum_{j=1}^{l} \left( \Delta_2 \left( \frac{\Delta Z_j}{h} \right) \right) \left( \Delta_2 \Delta Z_j \right) + \frac{\Delta_2^2 \Delta_1 \Delta Z_j}{h^3}
\]

\[\leq \frac{2B}{3} \sum_{j=1}^{l} \left( \Delta_2 \left( \frac{\Delta Z_j}{h} \right) \right)^2 + \frac{3B}{2} \sum_{j=1}^{l} \left( \Delta_2 \Delta Z_j \right)^2 h.
\]  

(16)

In conclusion, we have from the above calculation, Lemma 2.1, and the Cauchy inequality that

\[
\frac{d}{dt} \| \delta^2 Z_h \|^2 + A \| \delta^3 Z_h \|^2
\]

\[\leq \frac{A}{2} \| \delta^3 Z_h \|^2 + C \| \delta^2 Z_h \|_\infty \| \delta^3 Z_h \|_\infty (\| \delta^2 Z_h \|^2 + 1) + C \left( \| \delta^2 Z_h \|_\infty + 1 \right) (\| \delta^2 Z_h \|^2 + 1)
\]

\[\leq \frac{3A}{4} \| \delta^3 Z_h \|^2 + C \| \delta^2 Z_h \|^3 + C,
\]  

(17)

where we have used the following interpolations

\[
\| \delta^2 Z_h \|_\infty \leq C \| \delta^2 Z_h \|^2 \left( \| \delta^2 Z_h \|^2 + \frac{\| \delta Z_h \|^2}{2D} \right)^{\frac{1}{2}}
\]

(18)

\[
\| \delta^3 Z_h \|_\infty \leq C \| \delta^3 Z_h \|^2 \left( \| \delta^3 Z_h \|^2 + \frac{\| \delta^2 Z_h \|^2}{2D} \right)^{\frac{1}{2}}
\]

(19)
Following by Gronwall’s inequality, we have there exist a $T_0 > 0$ independent of $h$, such that
\[
\sup_{0 \leq t \leq T_0} \| \delta^2 \tilde{Z}_h(t) \|^2_2 + \int_0^{T_0} \| \delta^2 \tilde{Z}_h(t) \|^2_2 \, dt \leq C.
\]

Finally, by noticing that
\[
\Delta \left( G \left( \frac{\Delta \tilde{Z}_j}{h} \right) \right) = G \left( \frac{\Delta, \tilde{Z}_{j-1}}{h^2} \right) + B \left( \frac{\Delta, \tilde{Z}_{j-1}}{h^2} \right) \frac{\Delta, \tilde{Z}_j}{h} + B \left( \frac{\Delta, \tilde{Z}_j}{h^2} \right) \frac{\Delta, \tilde{Z}_j}{h}.
\]

Then we get (14) by (13), (18), (19), Lemma 2.3 and the Cauchy inequality. □

By the similar method as in the proof of Lemma 2.3 and Lemma 2.4 and using the induction argument, one gets the following Lemma.

Lemma 2.5. Let $\tilde{Z}_0(x) \in H^k(\Omega)$ $(k \geq 2)$. There are constants $T_0 > 0, C > 0$ independent of $h$ such that
\[
\sup_{0 \leq t \leq T_0} \left\| \delta^{k-2} \tilde{Z}_{ht}(t) \right\|_2^2 + \int_0^{T_0} \left\| \delta^{k+2} \tilde{Z}_{ht}(t) \right\|_2 \, dt \leq C.
\]

From Lemma 2.5, a priori estimates for solutions to the differential-difference equation (4)-(6), we conclude that there exists a generic constant $T_0 > 0$ such that problem (1)-(2) admits a smooth solution in $\Omega \times [0, T_0]$ by the standard procedure. This result is stated as follows.

Theorem 2.6. Let $\tilde{Z}_0(x) \in H^k(\Omega)$ $(k \geq 2)$. Then problem (1)-(2) admits at least one local smooth solution $\tilde{Z}(x, t)$:
\[
\tilde{Z}(x, t) \in F(T_0) = \left( \bigcap_{s=0}^{[\frac{T}{h}]} \mathcal{W}^{p,\infty} \left( 0, T_0; H^{k-2s}(\Omega) \right) \right) \bigcap \left( \bigcap_{s=0}^{[\frac{T}{h}]} \mathcal{H}^p \left( 0, T_0; H^{k+1-2s}(\Omega) \right) \right).
\]

3. A > 0: Global Smooth Solution

In section 2, we have obtained a local smooth solution for (1)-(2). In this section, we intend to prove the existence of global smooth solution to problem (1)-(2) by deriving the global (in time) estimates for given $A$ and $B$. In the following, we always suppose $\tilde{Z}(x, t)$ is a global smooth solution of problem (1)-(2). For $p \geq 1$, denote by $L^p = L^p(\Omega)$ the $L^p$ space with the norm $\| \cdot \|_{L^p}$. For $k \geq 1$ and $p \geq 1$, denote by $W^{k,p} = W^{k,p}(\Omega)$ the Sobolev space whose norm is $\| \cdot \|_{W^{k,p}}$, $H^k = W^{k,2}(\Omega)$.

Lemma 3.1. Let $\tilde{Z}_0(x) \in W^{1,4}(\Omega)$, and suppose that $\tilde{Z}(x, t)$ is a global smooth solution of problem (1)-(2). Then for any given $T > 0$, we have
\[
|\tilde{Z}(x, t)| = 1, \quad \forall (x, t) \in \Omega \times [0, T].
\]

(22)

\[
\sup_{0 \leq t \leq T} \left( \frac{A}{2} \| \tilde{Z}_x(t) \|_{L^2}^4 + \frac{B}{4} \| \tilde{Z}_x(t) \|_{L^4}^4 \right) + \int_0^T \| \tilde{Z} \times (G(\tilde{Z}_x)\tilde{Z}_x)(\cdot, t) \|_{L^2}^2 \, dt = \frac{A}{2} \| \tilde{Z}_x(t) \|_{L^2}^4 + \frac{B}{4} \| \tilde{Z}_x(t) \|_{L^4}^4.
\]

(23)

\[
\int_0^T \| \tilde{Z}_x(t) \|_{L^2}^2 \, dt \leq A \| \tilde{Z}_x(t) \|_{L^2}^2 + B \| \tilde{Z}_x(t) \|_{L^4}^4.
\]

(24)

\[
8B^2 \int_0^T \| \tilde{Z}_x \| (\tilde{Z}_x \cdot \tilde{Z}_x)(\cdot, t) \|_{L^2}^2 \, dt \leq \frac{A}{2} \| \tilde{Z}_x(t) \|_{L^2}^4 + \frac{B}{4} \| \tilde{Z}_x(t) \|_{L^4}^4.
\]

(25)
Proof. Multiplying (1) by \( \tilde{Z}(x, t) \), we have \( \tilde{Z}(x, t) \cdot \tilde{Z}_t(x, t) = 0 \). This implies (22). Then differentiating (1) with respect to \( x \), multiplying the resulting equation by \( G(\tilde{Z}_x)\tilde{Z}_x \), and then integrating it over \( \Omega \), we have

\[
\frac{A}{2} \frac{d}{dt} \|\tilde{Z}_x\|_{l_2}^2 + \frac{B}{4} \frac{d}{dt} \|\tilde{Z}_x\|_{l_4}^4 + \|\tilde{Z} \times (G(\tilde{Z}_x)\tilde{Z}_x)\|_{l_2}^2 = 0. \tag{26}
\]

This implies the basic energy equality

\[
\frac{A}{2} \|\tilde{Z}_x(\cdot, t)\|_{l_2}^2 + \frac{B}{4} \|\tilde{Z}_x(\cdot, t)\|_{l_4}^4 + \int_0^t \|\tilde{Z} \times (G(\tilde{Z}_x)\tilde{Z}_x)\|_{l_2}^2 \, dt = \frac{A}{2} \|\tilde{Z}_{0x}\|_{l_2}^2 + \frac{B}{4} \|\tilde{Z}_{0x}\|_{l_4}^4. \tag{27}
\]

Note that

\[
\|\tilde{Z} \times (G(\tilde{Z}_x)\tilde{Z}_x)\|_{l_2}^2 = \|\tilde{Z} \times (\tilde{Z} \times (G(\tilde{Z}_x)\tilde{Z}_x))\|_{l_2}^2. \tag{28}
\]

This, together with (1), yields that

\[
\int_0^t \|\tilde{Z}_x\|_{l_2}^2 \, dt \leq A\|\tilde{Z}_{0x}\|_{l_2}^2 + B\|\tilde{Z}_{0x}\|_{l_4}^4. \tag{29}
\]

Observing that

\[
\|\tilde{Z} \times (G(\tilde{Z}_x)\tilde{Z}_x)\|_{l_2}^2 = \|\tilde{Z} \times (\tilde{Z} \times (G(\tilde{Z}_x)\tilde{Z}_x))\|_{l_2}^2
\]
\[
= \|\tilde{Z} \cdot (G(\tilde{Z}_x)\tilde{Z}_x)\tilde{Z} - (G(\tilde{Z}_x)\tilde{Z}_x)\tilde{Z}\|_{l_2}^2
\]
\[
= \|\tilde{G}(\tilde{Z}_x)\tilde{Z}_x\|_{l_2}^2 - \|\tilde{Z} \cdot (G(\tilde{Z}_x)\tilde{Z}_x)\tilde{Z}\|_{l_2}^2
\]
\[
= \int_\Omega \tilde{G}^2(\tilde{Z}_x)|\tilde{Z}_x|^2 + 4B\tilde{G}^2(\tilde{Z}_x)|\tilde{Z}_x|^2 + 4BG(\tilde{Z}_x)|\tilde{Z}_x| \cdot |\tilde{Z}_x|^2 \, dx - \int_\Omega \tilde{G}^2(\tilde{Z}_x)(\tilde{Z} \cdot \tilde{Z}_x)^2 \, dx
\]
\[
\geq \int_\Omega (4B\tilde{G}^2(\tilde{Z}_x)|\tilde{Z}_x| \cdot |\tilde{Z}_x|^2 + 4BG(\tilde{Z}_x)|\tilde{Z}_x| \cdot |\tilde{Z}_x|^2) \, dx
\]
\[
\geq \int_\Omega 8B|\tilde{Z}_x|^2 |\tilde{Z}_x| |\tilde{Z}_x|^4 \, dx. \tag{30}
\]

Therefore, (25) follows.

**Remark 3.2.** From the proof of (30), we have

\[
\|\tilde{Z} \times (G(\tilde{Z}_x)\tilde{Z}_x)\|_{l_2}^2 \geq \int_\Omega \tilde{G}^2(\tilde{Z}_x)|\tilde{Z}_x|^2 \, dx - \int_\Omega \tilde{G}^2(\tilde{Z}_x)(\tilde{Z} \cdot \tilde{Z}_x)^2 \, dx
\]
\[
\geq \int_\Omega \tilde{G}^2(\tilde{Z}_x)|\tilde{Z}_x|^2 \, dx - \int_\Omega \tilde{G}^2(\tilde{Z}_x)(\tilde{Z} \cdot \tilde{Z}_x)^2 \, dx
\]
\[
\geq \int_\Omega G^2(\tilde{Z}_x)|\tilde{Z}_x|^2 \, dx - 2A^2 \int_\Omega |\tilde{Z}_x|^4 \, dx - 2B^2 \int_\Omega |\tilde{Z}_x|^8 \, dx
\]
\[
\geq \int_\Omega (A^2 + B^2)|\tilde{Z}_x|^2 |\tilde{Z}_x|^4 \, dx - 2A^2 \int_\Omega |\tilde{Z}_x|^4 \, dx - 2B^2 \int_\Omega |\tilde{Z}_x|^8 \, dx - \int_\Omega |\tilde{Z}_x|^4 \, dx. \tag{31}
\]

It yields from the one dimensional Sobolev embedding \( W^{1,1}(\Omega) \hookrightarrow L^{\infty}(\Omega) \) that

\[
\|\tilde{Z}_x|^4\|_{l_\infty} \leq c_1\|\tilde{Z}_x|^4\|_{l_2} + c_1\|\tilde{Z}_x|^4\|_{l_2},
\]
\[
\leq c_1\|\tilde{Z}_x|^4\|_{l_2} + 2c_1\|\tilde{Z}_x|^2 |\tilde{Z}_x| \cdot |\tilde{Z}_x|\|_{l_2},
\]
\[
\leq c_1\|\tilde{Z}_x|^4\|_{l_2} + 2c_1\|\tilde{Z}_x|^2 |\tilde{Z}_x| \cdot |\tilde{Z}_x|\|_{l_2} + c_1\|\tilde{Z}_x|^2 |\tilde{Z}_x| \cdot |\tilde{Z}_x|\|_{l_2}. \tag{32}
\]
where $c_\varepsilon$ is a Sobolev constant. If we assume that $A < 1$, then by using Lemma 3.1, (31) and (32), we can conclude that
\begin{equation}
A^2 \int_0^T \|\bar{Z}_{xx}^e(t)\|^2_{\Omega} \, dt \leq C(B, c_\varepsilon, \|\bar{Z}_{0x}\|_{W^{1,4}}, T),
\end{equation}
where the generic constant $C(B, c_\varepsilon, \|\bar{Z}_{xx}\|_{W^{1,4}})$ is independent of $A$. This remark will be useful in the next section for the sake of establishing the global existence of weak solutions when $A \to 0$.

In the following lemmas, denote by $C$ the uniform constants depending on $A$ and $B$, but independent of $Z(x, t)$.

**Lemma 3.3.** Let $Z_0(x) \in H^2(\Omega)$, and suppose that $Z(x, t)$ is a global smooth solution of problem (1)-(2). Then for any given $T > 0$, we have
\begin{equation}
\sup_{0 \leq t \leq T} (\|Z_x(t)\|_{L^2} + \|\bar{Z}_{xx}(t)\|_{L^2} + \|\bar{Z}_x(t)\|_{L^4} + \|(G(Z_x)Z_x)(t)\|_{L^2} + \int_0^T \|Z_{tt}(t)\|_{L^2}^2 + \|\bar{Z}_{xxx}(t)\|_{L^2}^2) \, dt \leq C.
\end{equation}

**Proof.** Differential (3) with respect to $t$, then multiplying the resulting equation by $\bar{Z}_t$ and integrating it over $\Omega$, one has
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_\Omega |\bar{Z}_t|^2 \, dx = \int_\Omega (G(Z_x)Z_x)_t \cdot \bar{Z}_t \, dx + \int_\Omega G(Z_x)|\bar{Z}_{tt}|^2 \, dx + \int_\Omega (\bar{Z} \times (G(Z_x)Z_x))_t \cdot \bar{Z}_t \, dx.
\end{equation}

We get from the first term on the right-hand side of (35) that
\begin{align*}
\int_\Omega (G(Z_x)Z_x)_t \cdot \bar{Z}_t \, dx &= - \int_\Omega (G(Z_x)Z_x)_t \cdot \bar{Z}_t \, dx \\
&= - \int_\Omega [2B(Z_x \cdot Z_{xx})Z_x + (A + B|Z_x|^2)Z_{xx}] \cdot \bar{Z}_t \, dx \\
&= -2B \int_\Omega |Z_x \cdot Z_{tt}|^2 \, dx - A \int_\Omega |Z_{tt}|^2 \, dx - B \int_\Omega |Z_x|^2 |Z_{tt}|^2 \, dx.
\end{align*}

For the second term on the right-hand side of (35), we have
\begin{equation}
\int_\Omega G(Z_x)|\bar{Z}_{tt}|^2 |\bar{Z}_t|^2 \, dx \leq (A\|Z_x\|_{L^4} + B\|\bar{Z}_{xx}\|_{L^2}^2)\|\bar{Z}_t\|_{L^2}^2 \leq C(1 + \|\bar{Z}_t(\bar{Z}_x \cdot \bar{Z}_{xx})\|_{L^2}^2)\|\bar{Z}_t\|_{L^2}^2,
\end{equation}

where we have used Lemma 3.1, (32) and $\|\bar{Z}_x\|_{L^4} \leq C\|\bar{Z}_x\|_{L^2}^{\frac{1}{2}}$.

We finally deal with the third term as follows,
\begin{align*}
\int_\Omega (\bar{Z} \times (G(Z_x)Z_x))_t \cdot \bar{Z}_t \, dx &= - \int_\Omega [\bar{Z} \times (G(Z_x)Z_x)]_t \cdot \bar{Z}_t \, dx - \int_\Omega [\bar{Z}_x \times (G(Z_x)Z_x)]_t \cdot \bar{Z}_t \, dx \\
&= -2B \int_\Omega (\bar{Z}_x \cdot Z_{tt})(\bar{Z} \times \bar{Z}_x) \cdot \bar{Z}_t \, dx - \int_\Omega G(Z_x)(\bar{Z}_x \times \bar{Z}_t) \cdot \bar{Z}_t \, dx \\
&\leq \frac{2B}{3} \int_\Omega |\bar{Z}_x|^2 |Z_{tt}|^2 \, dx + \frac{3B}{2} \int_\Omega |\bar{Z}_x \cdot Z_{tt}|^2 \, dx + \frac{B}{6} \int_\Omega |\bar{Z}_{tt}|^2 |Z_x|^2 \, dx + C \int_\Omega G^2(\bar{Z}_x)|\bar{Z}_t|^2 \, dx \\
&= \frac{5B}{6} \int_\Omega |\bar{Z}_x|^2 |Z_{tt}|^2 \, dx + \frac{3B}{2} \int_\Omega |\bar{Z}_x \cdot Z_{tt}|^2 \, dx + C(1 + \|\bar{Z}_t\|_{L^4}^4) \int_\Omega |\bar{Z}_t|^2 \, dx.
\end{align*}
In conclusion, we have

\[ \frac{d}{dt} \| \vec{Z} \|_{L^2}^2 + A \int_\Omega |\vec{Z}|^2 \, dx + B \int_\Omega |\vec{Z}|^2 \, dx \leq C(\|\vec{Z}\|_{L^2}^2 + 1)\|\vec{Z}\|_{L^2}^2. \]

Then following by Gronwall’s inequality and (25), we get

\[ \sup_{0 \leq t \leq T} \| \vec{Z} \|_{L^2} + A \int_0^T \int_\Omega |\vec{Z}|^2 \, dx \, dt + B \int_0^T \int_\Omega |\vec{Z}|^2 \, dx \, dt \leq C. \]  

Multiply (3) by \( \vec{Z}_{xx} \), and then integrating it over \( \Omega \), we have

\begin{align*}
2B \int_\Omega |\vec{Z}_x \cdot \vec{Z}_{xx}|^2 \, dx &+ \int_\Omega (A + B|\vec{Z}_x|^2)|\vec{Z}_{xx}|^2 \, dx \\
&= \int_\Omega \vec{Z}_x \cdot \vec{Z}_{xx} \, dx + \int_\Omega G(\vec{Z}_x)|\vec{Z}_{xx}|^4 \, dx - 2B \int_\Omega (\vec{Z}_x \cdot \vec{Z}_{xx})(\vec{Z} \times \vec{Z}_x) \cdot \vec{Z}_{xx} \, dx \\
&\leq \frac{A}{2} \int_\Omega |\vec{Z}_{xx}|^2 \, dx + C \int_\Omega |\vec{Z}|^2 \, dx + \|\vec{Z}_{xx}\|_{L^4}^4 \int_\Omega (A + B|\vec{Z}_x|^2) \, dx \\
&\quad + \frac{3B}{2} \int_\Omega |\vec{Z}_x \cdot \vec{Z}_{xx}|^2 \, dx + \frac{2B}{3} \int_\Omega |\vec{Z}_{xx}|^2 \, dx \\
&\leq \frac{A}{2} \int_\Omega |\vec{Z}_{xx}|^2 \, dx + C + \frac{5B}{3} \int_\Omega |\vec{Z}_x \cdot \vec{Z}_{xx}|^2 \, dx + \frac{2B}{3} \int_\Omega |\vec{Z}_{xx}|^2 \, dx \\
&\leq \frac{A}{2} \int_\Omega |\vec{Z}_{xx}|^2 \, dx + C + \frac{5B}{3} \int_\Omega |\vec{Z}_x \cdot \vec{Z}_{xx}|^2 \, dx + \frac{2B}{3} \int_\Omega |\vec{Z}_{xx}|^2 \, dx.
\end{align*}

Then we conclude that \( \sup_{0 \leq t \leq T} \| \vec{Z}_{xx}(\cdot, t) \|_2 \leq C \), and then the estimate about \( \| \vec{Z}_{x} \|_\infty \) follows by one dimensional Sobolev embedding.

Finally, multiplying (3) by \( (G(\vec{Z}_x)\vec{Z})_{xx} \), and then integrating over \( \Omega \), we have

\[ \int_\Omega \vec{Z}_t \cdot (G(\vec{Z}_x)\vec{Z})_{xx} \, dx = \int_\Omega \|G(\vec{Z}_x)\vec{Z}_{xx}\|^2 \, dx - \int_\Omega G(\vec{Z}_x)^2|\vec{Z}_{xx}|^4 \, dx \\
= \int_\Omega \|G(\vec{Z}_x)\vec{Z}_{xx}\|^2 \, dx - \int_\Omega A^2 \left( |\vec{Z}_{xx}|^4 + B|\vec{Z}_{x}|^8 + 2AB|\vec{Z}_x|^6 \right) \, dx. \]  

Then (34) follows from the interpolation inequalities. \( \square \)

**Lemma 3.4.** Let \( \vec{Z}_0(x) \in H^k(\Omega), k \geq 2 \) and suppose that \( \vec{Z}(x, t) \) is a global smooth solution of problem (1)-(2). Then for any given \( T > 0 \), there is \( C > 0 \) such that

\[ \sup_{0 \leq t \leq T} \| \vec{Z}_{xx}^k \cdot \vec{Z}(\cdot, t) \|_{L^2} + \int_0^T \| \vec{Z}_{xx}^{k-2} \vec{Z}(\cdot, t) \|_{L^2}^2 \, dt \leq C, \quad 0 \leq s \leq [k/2]. \]  

Combining the local existence obtained in section 2 and the global in time estimates in Lemma 3.4, we can get the existence of global smooth solution to problem (1)-(2) in the following sense.
Theorem 3.5. Let \( \tilde{Z}_0(x) \in H^k(\Omega) \), \( k \geq 2 \). Then for any given \( T > 0 \), problem (1)-(2) admits at least one global smooth solution \( \tilde{Z}(x,t) \):

\[
\tilde{Z}(x,t) \in \mathcal{F}(T) = \left( \bigcap_{n=0}^{\infty} W^{n,\infty}(0,T;H^{k-2n}(\Omega)) \right) \bigcap \left( \bigcap_{n=0}^{\infty} H^n(0,T;H^{k+1-2n}(\Omega)) \right).
\]

The uniqueness of the global smooth solution can be proved by standard discussion.

4. \( A = 0 \): Global Weak Solution

Lemma 4.1. \((11)\) Let \( p \geq 2 \). Then there holds for all \( a,b \in \mathbb{R}^k \)

\[
(\|a\|^{p-2}a - \|b\|^{p-2}b) \cdot (a-b) \geq 2^{p-2}\|a-b\|^p.
\]

Proof of Theorem 1.2 For any \( \tilde{Z}_0(x) \in W^{1,4}(\Omega) \), we can construct a approximate sequence \( \{\tilde{Z}^{(k)}_0(x)\}_{k=1}^{\infty} \) such that \( \tilde{Z}^{(k)}_0(x) \in H^k(\Omega) \) \( (k \geq 2) \) and \( \tilde{Z}_0^{(k)}(x) \to \tilde{Z}_0(x) \) in \( W^{1,4}(\Omega) \). Suppose that \( \{\tilde{Z}^{(k)}(x,t)\}_{k=1}^{\infty} \) is the sequence of regular solutions corresponding to the initial data \( \{\tilde{Z}_0^{(k)}(x)\}_{k=1}^{\infty} \). Then by Lemma 3.1 in Section three, we have from the estimates uniform in the parameter \( A \) that

\[
\{\tilde{Z}^{(k)}(x,t)\}_{k=1}^{\infty} \text{ is a bounded set in } L^\infty(0,T;W^{1,4}(\Omega)),
\]

\[
\{\partial_t\tilde{Z}^{(k)}(x,t)\}_{k=1}^{\infty} \text{ is a bounded set in } L^2(0,T;L^2(\Omega)).
\]

Then one can pass to a subsequence, without changing notation, to get that as \( k \to \infty \),

\[
\tilde{Z}^{(k)} \rightharpoonup \tilde{Z} \text{ weakly* in } L^\infty(0,T;W^{1,4}(\Omega)),
\]

\[
\tilde{Z}^{(k)}_t \rightharpoonup \tilde{Z}_t \text{ weakly in } L^2(0,T;L^2(\Omega)).
\]

By the compactness argument, one have

\[
\tilde{Z}^{(k)} \to \tilde{Z} \text{ strongly in } C(Q_T), \ Q_T = \Omega \times [0,T].
\]

Now we claim that

\[
\tilde{Z}^{(k)}_x \to \tilde{Z}_x \text{ strongly in } L^4(Q_T).
\]

In fact, by Lemma 4.1, we have

\[
\int_{Q_T} |\tilde{Z}^{(k)}_x - \tilde{Z}_x|^4 \, dx \, dt \leq \frac{1}{4} \int_{Q_T} \left( |\tilde{Z}^{(k)}_x|^2 |\tilde{Z}_x|^2 - |\tilde{Z}_x|^2 \tilde{Z}_x \cdot (\tilde{Z}^{(k)}_x - \tilde{Z}_x) \right) \, dx \, dt
\]

\[
= \frac{1}{4} \int_{Q_T} |\tilde{Z}^{(k)}_x|^2 |\tilde{Z}_x| \cdot (\tilde{Z}^{(k)}_x - \tilde{Z}_x) - \frac{1}{4} \int_{Q_T} |\tilde{Z}_x|^2 \tilde{Z}_x \cdot (\tilde{Z}^{(k)}_x - \tilde{Z}_x) \, dx \, dt
\]

\[
= I_1 + I_2.
\]

Note that \( |\tilde{Z}_x|^2 \tilde{Z}_x \in L^\infty(0,T;L^2) \) and \( L^2 = (L^4)^* \) (the dual space of \( L^4 \)) and by the weak convergence of \( \{\tilde{Z}^{(k)}(x,t)\}_{k=1}^{\infty} \), we have

\[
I_2 \to 0, \ k \to \infty.
\]

On the other hand, recalling (3), Lemma 3.1 and Remark 3.2, we get

\[
I_1 = -\frac{1}{4} \langle (\tilde{Z}^{(k)}_x^2 \tilde{Z}_x)_x, \tilde{Z}^{(k)} - \tilde{Z} \rangle
\]
\[
\begin{align*}
&= -\frac{1}{4}\left(\left(\frac{A}{B}\right)_{x}^{(k)} Z_{x}^{(k)} v_{Z}^{(k)} + \frac{A}{B} v_{Z}^{(k)} Z_{x}^{(k)} - \tilde{Z}_{y} \right) + \frac{1}{4} \left(\frac{A}{B} Z_{x}^{(k)} - Z_{y} \right) \\
&= -\frac{1}{4B} \left( Z_{x}^{(k)} - G\left(Z_{x}^{(k)} \right) Z_{x}^{(k)} Z_{x}^{(k)} - Z_{x}^{(k)} \times (G\left(Z_{x}^{(k)} \right) Z_{x}^{(k)}), x, Z_{x}^{(k)} - \tilde{Z} \right) + \frac{1}{4} \left(\frac{A}{B} Z_{x}^{(k)} - \tilde{Z}_{y} \right) \\
&\leq C_{B} \left\| \tilde{Z}_{l}^{(k)} - \tilde{Z}_{l} \right\|_{L^{2}(Q_{l})} \left( \left\| Z_{x}^{(k)} \right\|_{L^{2}(Q_{l})} + \left\| G\left(Z_{x}^{(k)} \right) Z_{x}^{(k)} \right\|_{L^{2}(Q_{l})} + \left\| Z_{x}^{(k)} \times (G\left(Z_{x}^{(k)} \right) Z_{x}^{(k)}), \right\|_{L^{2}(Q_{l})} + A^{2} \right\| \tilde{Z}_{l}^{(k)} \right\|_{L^{2}(Q_{l})} \right) \\
&\leq C_{B} \left\| \tilde{Z}_{l}^{(k)} - \tilde{Z}_{l} \right\|_{L^{2}(Q_{l})} \rightarrow 0, \quad k \rightarrow \infty,
\end{align*}
\]

where \( \langle \cdot, \cdot \rangle \) represents the inner product in \( L^{2}(Q_{l}) \) space. Thus we finish the proof of the claim.

Finally, by the convergence (47)-(50), we can easily conclude that (1) holds in the sense of distribution.

\[\square\]

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References