



Convergence of Iterative Algorithms for Continuous Pseudocontractive Mappings

Jong Soo Jung^a

^aDepartment of Mathematics, Dong-A University, Busan 604-714, Korea

Abstract. In this paper, we prove strong convergence of a path for a convex combination of a pseudocontractive type of operators in a real reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ . Using path convergency, we establish strong convergence of an implicit iterative algorithm for a pseudocontractive mapping combined with a strongly pseudocontractive mapping in the same Banach space.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$, and let E^* be the dual space of E . The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. Let C be a nonempty closed convex subset of E .

Recall that a mapping T with domain $D(T)$ and range $R(T)$ in E is called *pseudocontractive* (Kato [9]) if for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2,$$

where $J : E \rightarrow E^*$ is the normalized duality mapping. The mapping T is said to be *strongly pseudocontractive* if for each $x, y \in D(T)$, there exists a constant $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2.$$

The class of pseudocontractive mappings is one of the most important classes of mappings in nonlinear analysis and it has been attracting mathematician's interest. In addition to generalizing the nonexpansive mappings (the mappings $T : D \rightarrow E$ for which $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in D$), the pseudocontractive ones are characterized by the fact that T is pseudocontractive if and only if $I - T$ is accretive, where a mapping A with domain $D(A)$ and range $R(A)$ in E is called accretive if the inequality

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\|,$$

2010 *Mathematics Subject Classification.* Primary: 47H10, 47H09; Secondary: 47J05, 47J20

Keywords. Pseudocontractive mapping; Path; Strongly pseudocontractive mapping; Fixed points; Weakly continuous duality mapping; Reflexive Banach space

Received: 18 September 2014; Accepted: 12 December 2014

Communicated by Hari M. Srivastava

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2013021600).

Email address: jungjs@dau.ac.kr; jungjs@mail.donga.ac.kr (Jong Soo Jung)

holds for every $x, y \in D(A)$ and for all $s > 0$.

Within the past 40 years or so, many authors have been devoting their study to the existence of zeros of accretive mappings or fixed points of pseudocontractive mappings and iterative construction of zeros of accretive mappings and of fixed points of pseudocontractive mappings (see [5, 11–13, 16]). Also several iterative schemes for approximating fixed points of single-valued or multi-valued nonexpansive and pseudocontractive mappings in Hilbert spaces and Banach spaces have been introduced and studied by many authors. We can refer to [2, 7, 10, 15, 17, 21] and references therein.

In 2007, considering the result of Rafiq [15] for hemicontractive mapping in a real Hilbert space H , Yao et al. [21] introduced an iterative algorithm (1.1) below for approximating fixed points of a continuous pseudocontractive mapping T without compactness assumption on its domain in a real uniformly smooth Banach space E : for arbitrary initial value $x_0 \in C$ and a fixed anchor $u \in C$,

$$x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n, \quad \forall n \geq 1, \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0, 1)$ satisfying some appropriate conditions. By using the Reich inequality ([16]) in uniformly smooth Banach spaces:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x) \rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|), \quad \forall x, y \in E,$$

where $b : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing continuous function, they proved that the sequence $\{x_n\}$ generated by the proposed iterative algorithm (1.1) converges strongly to a fixed point of T .

In particular, in 2007, by using the viscosity iterative method studied by [14, 19], Song and Chen [17] introduced a modified implicit iterative algorithm (1.2) below for a continuous pseudocontractive mapping T without compactness assumption on its domain in a real reflexive and strictly convex Banach space E having a uniformly Gâteaux differentiable norm: for arbitrary initial value $x_0 \in C$,

$$\begin{cases} x_n = \alpha_n y_n + (1 - \alpha_n) T x_n, \\ y_n = \beta_n f(x_{n-1}) + (1 - \beta_n) x_{n-1}, \end{cases} \quad \forall n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ satisfying some appropriate conditions and $f : C \rightarrow C$ is a contractive mapping (i.e., there exists $k \in (0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $\forall x, y \in C$), and proved that the sequence $\{x_n\}$ generated by the proposed iterative algorithm (1.2) converges strongly to a fixed point of T , which is the unique solution of a certain variational inequality related to f .

In 2007, Morales [12] introduced the following path for a continuous pseudocontractive mapping T in a real reflexive Banach space E having a uniformly Gâteaux differentiable norm such that every bounded closed convex subset of the space has the fixed point property for nonexpansive self-mappings:

$$x_t = (1 - t)Ax_t + tTx_t, \quad t \in (0, 1), \tag{1.3}$$

where A is a continuous bounded strongly pseudocontractive mapping with a pseudocontractive constant $k \in (0, 1)$. He proved strong convergence of the path $\{x_t : t \in (0, 1)\}$ described by (1.3) to a fixed point of T as $t \rightarrow 1^-$, which solves a certain variational inequality related to A .

In 2013, Jung [8] considered the following iterative algorithm for a continuous pseudocontractive mapping T in a real reflexive Banach space E having a uniformly Gâteaux differentiable norm such that every bounded closed convex subset of the space has the fixed point property for nonexpansive self-mappings or a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm: for arbitrary initial value $x_0 \in C$

$$x_n = \alpha_n Ax_n + \beta_n x_{n-1} + (1 - \alpha_n - \beta_n)Tx_n, \quad \forall n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ and A is a continuous bounded strongly pseudocontractive mapping with a pseudocontractive constant $k \in (0, 1)$. He also proved strong convergence of the sequences generated by proposed iterative algorithm (1.4) to a fixed point of T , which solves a certain variational inequality related to A .

The purpose of this paper is to continue the discussion concerning strong convergence of the path $\{x_t : t \in (0, 1)\}$ described by (1.3) and the iterative algorithm (1.4). In a real reflexive Banach space E having a weakly continuous duality mapping J_φ with gauge function φ , we establish strong convergence of the path $\{x_t : t \in (0, 1)\}$ described by (1.3) and the sequence $\{x_n\}$ by proposed iterative algorithm (1.4) to a fixed point of the mapping T , which solves a certain variational inequality related to A . The main results develop and complement the corresponding results obtained by Jung [8], Morales [12], Yao et al. [21] and Song and Chen [17] as well as Chen and Zhu [3], Moudafi [14] and Xu [20].

2. Preliminaries and Lemmas

Let C be a nonempty subset of a real Banach space E . For the mapping $T : C \rightarrow C$, we denote the fixed point set of T by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$.

By a *gauge function* we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r \rightarrow \infty} \varphi(r) = \infty$. The mapping $J_\varphi : E \rightarrow 2^{E^*}$ defined by

$$J_\varphi(x) = \{f \in E^* : \langle x, f \rangle = \|x\| \|f\|, \|f\| = \varphi(\|x\|)\}, \quad \forall x \in E$$

is called the *duality mapping* with gauge function φ . In particular, the duality mapping with gauge function $\varphi(t) = t$, denoted by J , is referred to as the *normalized duality mapping*. It is known that a Banach space E is smooth if and only if the normalized duality mapping J is single-valued. The following property of the duality mapping is also well-known:

$$J_\varphi(\lambda x) = \text{sign } \lambda \left(\frac{\varphi(|\lambda| \cdot \|x\|)}{\|x\|} \right) J(x) \quad \text{for all } x \in E \setminus \{0\}, \quad \lambda \in \mathbb{R}, \tag{2.1}$$

where \mathbb{R} is the set of all real numbers; in particular, $J(-x) = -J(x)$ for all $x \in E$ ([1, 4]).

We say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge function φ such that the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J_\varphi(x_n) \xrightarrow{*} J_\varphi(x)$. For example, every l^p space ($1 < p < \infty$) has a weakly continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$ ([1, 4, 6]). Set

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \quad \forall t \in \mathbb{R}^+.$$

Then it is known that $J_\varphi(x)$ is the subdifferential of the convex functional $\Phi(\|\cdot\|)$ at x .

We prepare the following Lemma.

Lemma 2.1. *Let E be a real reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ . Let $\{x_n\}$ be a bounded sequence of E , and let $A : E \rightarrow E$ be a continuous mapping. Let $\phi : E \rightarrow \mathbb{R}$ be defined by*

$$\phi(z) = \limsup_{n \rightarrow \infty} \langle z - Az, J_\varphi(z - x_n) \rangle$$

for $z \in E$. Then ϕ is a real valued continuous function on E .

Proof. Let $\{z_j\}$ be a sequence in E such that $z_j \rightarrow z \in E$. Then $\{z_j\}$ is bounded, $z_j \rightarrow z$, and $\varphi(\|z_j - x_n\|) < \infty$. So, by the continuity of A and the weak continuity of J_φ , we have

$$\begin{aligned} |\phi(z_j) - \phi(z)| &= \left| \limsup_{n \rightarrow \infty} \langle z_j - Az_j, J_\varphi(z_j - x_n) \rangle - \limsup_{n \rightarrow \infty} \langle z - Az, J_\varphi(z - x_n) \rangle \right| \\ &\leq \left| \limsup_{n \rightarrow \infty} \langle z_j - Az_j - (z - Az), J_\varphi(z_j - x_n) \rangle \right| + \left| \limsup_{n \rightarrow \infty} \langle z - Az, J_\varphi(z_j - x_n) - J_\varphi(z - x_n) \rangle \right| \\ &\leq \limsup_{n \rightarrow \infty} \| (z_j - Az_j) - (z - Az) \| \varphi(\|z_j - x_n\|) + \limsup_{n \rightarrow \infty} | \langle z - Az, J_\varphi(z_j - x_n) - J_\varphi(z - x_n) \rangle | \\ &\rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Therefore, ϕ is continuous on E . \square

We need the following well-known lemmas for the proof of our main result.

Lemma 2.2. [1, 4] Let E be a real Banach space, and let φ be a continuous strictly increasing function on \mathbb{R}^+ such that $\varphi(0) = 0$ and $\lim_{r \rightarrow \infty} \varphi(r) = \infty$. Define

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau \text{ for all } t \in \mathbb{R}^+.$$

Then (i) The following inequalities hold:

$$\Phi(kt) \leq k\Phi(t), \quad 0 < k < 1,$$

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x + y) \rangle \text{ for all } x, y \in E,$$

where $j_\varphi(x + y) \in J_\varphi(x + y)$.

(ii) Assume that a sequence $\{x_n\}$ in E is weakly convergent to a point x . Then there holds the identity

$$\limsup_{n \rightarrow \infty} \Phi(\|x_n - y\|) = \limsup_{n \rightarrow \infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad x, y \in E.$$

Lemma 2.3. [18] Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\delta_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

- (i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=0}^\infty \lambda_n = \infty$ or, equivalently, $\prod_{n=0}^\infty (1 - \lambda_n) = 0$,
- (ii) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=0}^\infty \lambda_n|\delta_n| < \infty$,

Then $\lim_{n \rightarrow \infty} s_n = 0$.

Let C be a nonempty closed convex subset of a real Banach space E . Recall that $S : C \rightarrow C$ is called *accretive* if $I - S$ is pseudocontractive. If $T : C \rightarrow C$ is a pseudocontractive mapping, then $I - T$ is accretive. We denote $G = J_1 = (2I - T)^{-1}$. Then $F(G) = F(T)$ and the operator $G : R(2I - T) \rightarrow C$ is nonexpansive and single-valued, where I denotes the identity mapping.

We also need the following result which can be found in [17].

Lemma 2.4. [17] Let C be a nonempty closed convex subset of a real Banach space E , and let $T : C \rightarrow C$ be a continuous pseudocontractive mapping. We denote $G = (2I - T)^{-1}$. Then

- (i) The mapping G is nonexpansive self-mapping on C , i.e., for all $x, y \in C$, there holds

$$\|Gx - Gy\| \leq \|x - y\|, \text{ and } Ax \in C.$$

- (ii) If $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, then $\lim_{n \rightarrow \infty} \|x_n - Gx_n\| = 0$.

Let C be a nonempty subset of a Banach space E , For $x \in C$ the *inward set* of x , $I_C(x)$, is defined by

$$I_C(x) := \{x + \lambda(u - x) : u \in C, \lambda \geq 1\}.$$

A mapping $T : C \rightarrow E$ is said to satisfy the *weakly inward condition* if

$$Tx \in \overline{I_C(x)}, \quad \forall x \in C,$$

where $\overline{I_C(x)}$ denotes the closure of the inward set $I_C(x)$. On the other hand, if $Tx \in I_C(x)$ for each $x \in K$, T is said to, simply satisfy the *inward condition*. Every self-mapping satisfies trivially the weakly inward condition.

Finally, we recall that a Banach space E is said to be *strictly convex* ([1]) if the following implication holds for $x, y \in E$

$$\|x\| \leq 1, \|y\| \leq 1, \|x - y\| > 0 \Rightarrow \left\| \frac{x + y}{2} \right\| < 1.$$

The following Lemma, which is well-known, can be found in many books in the geometry of Banach spaces (see [1, 6]).

Lemma 2.5. *If E is a real Banach space such that E^* is strictly convex, then E is smooth and any duality mapping is norm-to-weak*-continuous.*

3. Iterative Algorithms

First, we need the following result, which is given in [12].

Proposition 3.1. *Let C be a nonempty closed convex subset of a real Banach space E . Let $T : C \rightarrow E$ be continuous pseudocontractive mapping, and let $A : C \rightarrow E$ be a continuous strongly pseudocontractive mapping with a constant $k \in (0, 1)$. Suppose, in addition, that T and A satisfy the weakly inward condition. Then there exists a unique path $t \mapsto x_t \in C, t \in (0, 1)$, satisfying*

$$x_t = (1 - t)Ax_t + tTx_t.$$

Our first result shows that Morales’s result [12] holds in a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ .

Theorem 3.2. *Let E be a real reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ . Let C be a nonempty closed convex subset of E , let $T : C \rightarrow E$ be a continuous pseudocontractive mapping with $F(T) \neq \emptyset$, and let $A : C \rightarrow E$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive constant $k \in (0, 1)$. Suppose that T and A satisfy the weakly inward condition. For each $t \in (0, 1)$, let $x_t \in C$ be defined by*

$$x_t = (1 - t)Ax_t + tTx_t. \tag{3.1}$$

Then the path $\{x_t\}$ converges strongly to a point p in $F(T)$ as $t \rightarrow 1^-$, which is the unique solution of the variational inequality

$$\langle (I - A)p, J_\varphi(p - q) \rangle \leq 0, \quad \forall q \in F(T). \tag{3.2}$$

Proof. By Proposition 3.1, the path $\{x_t\}$ exists. It remains to show that it converges strongly to a fixed point of T as $t \rightarrow 0$. Let $z \in F(T)$. Since T is pseudocontractive, we have

$$\begin{aligned} \langle x_t - Tx_t, J_\varphi(x_t - z) \rangle &= \langle x_t - z + Tz - Tx_t, J_\varphi(x_t - z) \rangle \\ &= \|x_t - z\|\varphi(\|x_t - z\|) - \langle Tx_t - Tz, J_\varphi(x_t - z) \rangle \\ &\geq \Phi(\|x_t - z\|) - \|x_t - z\|\varphi(\|x_t - z\|) \\ &= \Phi(\|x_t - z\|) - \Phi(\|x_t - z\|) = 0. \end{aligned}$$

On the other hand, since

$$x_t - Tx_t = \frac{1 - t}{t}(Ax_t - x_t),$$

we have

$$\langle x_t - Ax_t, J_\varphi(x_t - z) \rangle \leq 0. \tag{3.3}$$

Now we show that $\{x_t\}$ is bounded. Indeed, by strong pseudocontractivity of A , we get

$$\langle Ax_t - Az, J_\varphi(x_t - z) \rangle \leq k\|x_t - z\|\varphi(\|x_t - z\|).$$

Using (3.3), we deduce

$$\begin{aligned} \|x_t - z\|\varphi(\|x_t - z\|) &= \langle x_t - z, J_\varphi(x_t - z) \rangle \\ &= \langle x_t - Ax_t, J_\varphi(x_t - z) \rangle + \langle Ax_t - Az, J_\varphi(x_t - z) \rangle + \langle Az - z, J_\varphi(x_t - z) \rangle \\ &\leq k\|x_t - z\|\varphi(\|x_t - z\|) + \langle Az - z, J_\varphi(x_t - z) \rangle. \end{aligned}$$

Thus we obtain

$$\|x_t - z\|\varphi(\|x_t - z\|) \leq \frac{1}{1-k} \langle Az - z, J_\varphi(x_t - z) \rangle, \tag{3.4}$$

which yields

$$\|x_t - z\| \leq \frac{1}{1-k} \|Az - z\|.$$

Therefore $\{x_t : t \in (0, 1)\}$ is bounded. Since A is a bounded mapping, $\{Ax_t : t \in (0, 1)\}$ is bounded. Moreover, from (3.1), it follows that

$$\|Tx_t\| = \left\| \frac{1}{t}x_t - \frac{1-t}{t}Ax_t \right\| \leq \frac{1}{t}\|x_t\| + \frac{1-t}{t}\|Ax_t\|,$$

and so $\{Tx_t\}$ is bounded (as $t \rightarrow 1^-$).

Next assume $t_n \rightarrow 1^-$. Since E is reflexive and $\{x_{t_n}\}$ is bounded, we may assume that $x_{t_n} \rightharpoonup p$ for some $p \in C$. Since J_φ is weakly continuous, by Lemma 2.2, we have

$$\limsup_{n \rightarrow \infty} \Phi(\|x_{t_n} - x\|) = \limsup_{n \rightarrow \infty} \Phi(\|x_{t_n} - p\|) + \Phi(\|x - p\|), \quad x \in E. \tag{3.5}$$

Put

$$g(x) = \limsup_{n \rightarrow \infty} \Phi(\|x_{t_n} - x\|), \quad x \in E.$$

Then it follows from (3.5) that

$$g(x) = g(p) + \Phi(\|x - p\|), \quad x \in E. \tag{3.6}$$

From Lemma 2.4, we know that the mapping $G = (2I - T)^{-1} : C \rightarrow C$ is nonexpansive, and $F(G) = F(T)$. Since $\{Ax_t\}$ and $\{Tx_t\}$ are bounded, it follows from (3.1) that

$$\|x_n - Tx_n\| \leq (1 - t_n)\|Ax_n - Tx_n\| \rightarrow 0.$$

So, by Lemma 2.4, we have $\|x_n - Gx_n\| \rightarrow 0$. Thus

$$\begin{aligned} g(Gp) &= \limsup_{n \rightarrow \infty} \Phi(\|x_{t_n} - Gp\|) = \limsup_{n \rightarrow \infty} \Phi(\|Gx_{t_n} - Gp\|) \\ &\leq \limsup_{n \rightarrow \infty} \Phi(\|x_{t_n} - p\|) = g(p). \end{aligned} \tag{3.7}$$

By (3.6), we also get

$$g(Gp) = g(p) + \Phi(\|Gp - p\|). \tag{3.8}$$

Combining (3.7) and (3.8) yields

$$\Phi(\|Gp - p\|) \leq 0.$$

Hence $Gp = p$ and $p \in F(T) = F(G)$.

Let $\{t_n\}$ be a sequence in $(0, 1)$ such that $t_n \rightarrow 1^-$ and $x_{t_n} \rightarrow p$ as $n \rightarrow \infty$. Then the argument above shows that $p \in F(T)$. We next show that $x_{t_n} \rightarrow p$. Replacing z in (3.4) with p , we have

$$\Phi(\|x_{t_n} - p\|) = \|x_{t_n} - p\|\varphi(\|x_{t_n} - p\|) \leq \frac{1}{1-k} \langle Ap - p, J_\varphi(x_{t_n} - p) \rangle.$$

Noting that $x_{t_n} \rightarrow p$ implies $J_\varphi(x_{t_n} - p) \rightarrow 0$, we get

$$\Phi(\|x_{t_n} - p\|) \rightarrow 0.$$

Hence $x_{t_n} \rightarrow p$.

Finally, we prove that the entire net $\{x_t\}$ converges strongly to p . To this end, we assume that two sequences $\{t_n\}$ and $\{s_n\}$ are such that

$$x_{t_n} \rightarrow p \quad \text{and} \quad x_{s_n} \rightarrow q.$$

By (3.3),

$$\langle x_{t_n} - Ax_{t_n}, J_\varphi(x_{t_n} - q) \rangle \leq 0 \quad \text{and} \quad \langle x_{s_n} - Ax_{s_n}, J_\varphi(x_{s_n} - p) \rangle \leq 0.$$

Since a Banach space E has a weakly continuous duality mapping J_φ , J_φ is single valued and weak-to-weak* continuous. So, we deduce for any $z \in F(T)$,

$$\|x_{s_n} - Ax_{s_n} - (q - Aq)\| \rightarrow 0 \quad (s_n \rightarrow 1^-)$$

and

$$\begin{aligned} & |\langle x_{s_n} - Ax_{s_n}, J_\varphi(x_{s_n} - z) \rangle - \langle q - Aq, J_\varphi(q - z) \rangle| \\ &= |\langle x_{s_n} - Ax_{s_n} - (q - Aq), J_\varphi(x_{s_n} - z) \rangle + \langle q - Aq, J_\varphi(x_{s_n} - z) - J_\varphi(q - z) \rangle| \\ &\leq \|x_{s_n} - Ax_{s_n} - (q - Aq)\| \|J_\varphi(x_{s_n} - z)\| + |\langle q - Aq, J_\varphi(x_{s_n} - z) - J_\varphi(q - z) \rangle| \rightarrow 0 \quad \text{as } s_n \rightarrow 1^-. \end{aligned}$$

Therefore, we get

$$\langle q - Aq, J_\varphi(q - z) \rangle = \lim_{s_n \rightarrow 1^-} \langle x_{s_n} - Ax_{s_n}, J_\varphi(x_{s_n} - z) \rangle \leq 0. \tag{3.9}$$

Interchange p and z in (3.9) to obtain

$$\langle q - Aq, J_\varphi(q - p) \rangle \leq 0. \tag{3.10}$$

Interchange q and p in (3.10) to obtain

$$\langle p - Ap, J_\varphi(p - q) \rangle \leq 0. \tag{3.11}$$

Adding (3.10) and (3.11) yields

$$\langle (p - q) - (Ap - Aq), J_\varphi(p - q) \rangle \leq 0.$$

That is,

$$\|p - q\|\varphi(\|p - q\|) \leq k\|p - q\|\varphi(\|p - q\|).$$

This implies

$$(1 - k)\Phi(\|p - q\|) \leq 0.$$

Hence $p = q$.

Again, it follow from (3.3) that

$$\langle (I - A)x_t, J_\varphi(x_t - q) \rangle \leq 0, \quad \forall q \in F(T). \tag{3.12}$$

We notice that the definition of the weak continuity of the duality mapping J_φ implies that E is smooth. Thus E^* is strictly convex for E is reflexive. Noting that J_φ is norm-to-weak*-continuous by Lemma 2.5 and taking the limit as $t \rightarrow 1^-$ in (3.12), we obtain

$$\langle (I - A)p, J_\varphi(p - q) \rangle \leq 0, \quad \forall q \in F(T).$$

This means that p is the unique solution of the variational inequality (3.2). This completes the proof. \square

Using Theorem 3.2, we establish the following main result.

Theorem 3.3. *Let E be a real reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ . Let C be a nonempty closed convex subset of E , let $T : C \rightarrow C$ be a continuous pseudocontractive mapping with $F(T) \neq \emptyset$, and let $A : C \rightarrow C$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive constant $k \in (0, 1)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ satisfying the following conditions:*

(C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \beta_n = 0$;

(C2) $\sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha_n + \beta_n} = \infty$.

For arbitrary initial value $x_0 \in C$, let the sequence $\{x_n\}$ be defined by

$$x_n = \alpha_n Ax_n + \beta_n x_{n-1} + (1 - \alpha_n - \beta_n)Tx_n, \quad \forall n \geq 1. \tag{3.13}$$

Then $\{x_n\}$ converges strongly to a fixed point p of T , which is the unique solution of the variational inequality (3.2).

Proof. We divide the proof into several steps as follows.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $q \in F(T)$. Then, noting that

$$\begin{aligned} x_n - q &= \alpha_n(Ax_n - q) + \beta_n(x_{n-1} - q) + (1 - \alpha_n - \beta_n)(Tx_n - q), \\ \langle Tx_n - q, J(x_n - q) \rangle &\leq \|x_n - q\|\varphi(\|x_n - q\|) \end{aligned} \tag{3.14}$$

and

$$\langle Ax_n - Aq, J(x_n - q) \rangle \leq k\|x_n - q\|\varphi(\|x_n - q\|), \tag{3.15}$$

we have

$$\begin{aligned} \|x_n - q\|\varphi(\|x_n - q\|) &= \langle \alpha_n[(Ax_n - Aq) + (Aq - q)] + \beta_n(x_{n-1} - q) + (1 - \alpha_n - \beta_n)(Tx_n - q), J_\varphi(x_n - q) \rangle \\ &\leq \alpha_n k \|x_n - q\|\varphi(\|x_n - q\|) + \alpha_n \|Aq - q\|\varphi(\|x_n - q\|) \\ &\quad + \beta_n \|x_{n-1} - q\|\varphi(\|x_n - q\|) + (1 - \alpha_n - \beta_n) \|x_n - q\|\varphi(\|x_n - q\|), \end{aligned}$$

which implies

$$\|x_n - q\| \leq (1 - \alpha_n(1 - k) - \beta_n)\|x_n - q\| + \alpha_n \|Aq - q\| + \beta_n \|x_{n-1} - q\|.$$

So, we obtain

$$\begin{aligned} \|x_n - q\| &\leq \frac{\alpha_n}{(1 - k)\alpha_n + \beta_n} \|Aq - q\| + \frac{\beta_n}{(1 - k)\alpha_n + \beta_n} (x_{n-1} - q) \\ &= \frac{(1 - k)\alpha_n}{(1 - k)\alpha_n + \beta_n} \frac{\|Aq - q\|}{1 - k} + \frac{\beta_n}{(1 - k)\alpha_n + \beta_n} \|x_{n-1} - p\| \\ &\leq \max\left\{ \|x_{n-1} - q\|, \frac{\|Aq - q\|}{1 - k} \right\}. \end{aligned}$$

By induction, we have

$$\|x_n - q\| \leq \max\left\{ \|x_0 - q\|, \frac{1}{1 - k} \|Aq - q\| \right\} \text{ for } n \geq 1.$$

Hence $\{x_n\}$ is bounded. Since A is a bounded mapping, $\{Ax_n\}$ is bounded. From (3.13) it follows that

$$\|Tx_n\| = \frac{1}{1 - \alpha_n - \beta_n} (\|x_n\| + \alpha_n \|Ax_n\| + \beta_n \|x_{n-1}\|),$$

and so $\{Tx_n\}$ is bounded ($n \rightarrow \infty$).

Step 2. We show that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. In fact, by (3.13) and the condition (C1)

$$\|x_n - Tx_n\| \leq \alpha_n \|Ax_n - Tx_n\| + \beta_n \|x_{n-1} - Tx_n\| \rightarrow 0.$$

Step 3. We show that $\limsup_{n \rightarrow \infty} \langle Ap - p, J_\varphi(x_n - p) \rangle \leq 0$, where $p = \lim_{t \rightarrow 1^-} x_t$ with $x_t \in C$ being defined by $x_t = (1 - t)Ax_t + tTx_t$ in Theorem 3.2. We know from Theorem 3.2 that p is the unique solution of the variational inequality (3.2). Using the equality

$$x_t - x_n = t(Tx_t - x_n) + (1 - t)(Ax_t - x_n)$$

and the inequality

$$\langle Tx - Ty, J_\varphi(x - y) \rangle \leq \|x - y\|\varphi(\|x - y\|), \quad \forall x, y \in C,$$

we obtain

$$\begin{aligned} \|x_t - x_n\|\varphi(\|x_t - x_n\|) &= t\langle Tx_t - x_n, J_\varphi(x_t - x_n) \rangle + (1 - t)\langle Ax_t - x_n, J_\varphi(x_t - x_n) \rangle \\ &= t[\langle Tx_t - Tx_n, J_\varphi(x_t - x_n) \rangle + \langle Tx_n - x_n, J_\varphi(x_t - x_n) \rangle] \\ &\quad + (1 - t)\langle Ax_t - x_t, J_\varphi(x_t - x_n) \rangle + (1 - t)\|x_t - x_n\|\varphi(\|x_t - x_n\|) \\ &\leq t\|x_t - x_n\|\varphi(\|x_t - x_n\|) + t\|Tx_n - x_n\|\varphi(\|x_t - x_n\|) \\ &\quad + (1 - t)\langle Ax_t - x_t, J_\varphi(x_t - x_n) \rangle + (1 - t)\|x_t - x_n\|\varphi(\|x_t - x_n\|) \end{aligned}$$

and hence

$$\langle x_t - Ax_t, J_\varphi(x_t - x_n) \rangle \leq \frac{t\|Tx_n - x_n\|}{1 - t}\varphi(\|x_t - x_n\|).$$

Therefore, by Step 2 and $\limsup_{n \rightarrow \infty} \varphi(\|x_t - x_n\|) < \infty$, we have

$$\limsup_{n \rightarrow \infty} \langle x_t - Ax_t, J_\varphi(x_t - x_n) \rangle \leq \limsup_{n \rightarrow \infty} \frac{t\|Tx_n - x_n\|}{1 - t}\varphi(\|x_t - x_n\|) = 0.$$

Thus, by Lemma 2.1, we conclude

$$\limsup_{n \rightarrow \infty} \langle Ap - p, J_\varphi(x_n - p) \rangle = \lim_{t \rightarrow 1^-} \limsup_{n \rightarrow \infty} \langle Ax_t - x_t, J_\varphi(x_n - x_t) \rangle \leq 0.$$

Step 4. We show that $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$, where $p = \lim_{t \rightarrow 1^-} x_t$ with $x_t \in C$ being defined by $x_t = (1 - t)Ax_t + tTx_t$ and $p \in F(T)$ is the unique solution of the variational inequality (3.2) by Theorem 3.2. To this end, first we note that

$$x_n - p = \alpha_n(Ax_n - Ap) + \beta_n(x_{n-1} - p) + \alpha_n(Ap - p) + (1 - \alpha_n - \beta_n)(Tx_n - p).$$

From (3.13), (3.14), (3.15) and Lemma 2.2, we have

$$\begin{aligned} \Phi(\|x_n - p\|) &\leq \Phi(\|\beta_n\langle x_{n-1} - p \rangle + \alpha_n\langle Ax_n - Ap, J_\varphi(x_n - p) \rangle \\ &\quad + (1 - \alpha_n - \beta_n)\langle Tx_n - p, J_\varphi(x_n - p) \rangle + \alpha_n\langle Ap - p, J_\varphi(x_n - p) \rangle) \\ &\leq \beta_n\Phi(\|x_{n-1} - p\|) + k\alpha_n\|x_n - p\|\varphi(\|x_n - p\|) \\ &\quad + (1 - \alpha_n - \beta_n)\|x_n - p\|\varphi(\|x_n - p\|) + \alpha_n\langle Ap - p, J_\varphi(x_n - p) \rangle \\ &= \beta_n\Phi(\|x_{n-1} - p\|) + (1 - (1 - k)\alpha_n - \beta_n)\Phi(\|x_n - p\|) + \alpha_n\langle Ap - p, J_\varphi(x_n - p) \rangle. \end{aligned}$$

This implies that

$$\begin{aligned} \Phi(\|x_n - p\|) &\leq \frac{\beta_n}{(1 - k)\alpha_n + \beta_n}\Phi(\|x_{n-1} - p\|) + \frac{\alpha_n}{(1 - k)\alpha_n + \beta_n}\langle Ap - p, J_\varphi(x_n - p) \rangle \\ &= \left(1 - \frac{(1 - k)\alpha_n}{(1 - k)\alpha_n + \beta_n}\right)\Phi(\|x_{n-1} - p\|) + \frac{(1 - k)\alpha_n}{((1 - k)\alpha_n + \beta_n)} \cdot \frac{1}{1 - k}\langle Ap - p, J_\varphi(x_n - p) \rangle \\ &= (1 - \lambda_n)\Phi(\|x_{n-1} - p\|) + \lambda_n\delta_n, \end{aligned} \tag{3.16}$$

where $\lambda_n = \frac{(1-k)\alpha_n}{(1-k)\alpha_n + \beta_n}$ and $\delta_n = \frac{1}{1-k} \langle Ap - p, J_\varphi(x_n - p) \rangle$. We observe that $0 \leq \frac{(1-k)\alpha_n}{(1-k)\alpha_n + \beta_n} \leq 1$ and $\frac{(1-k)\alpha_n}{\alpha_n + \beta_n} < \frac{(1-k)\alpha_n}{(1-k)\alpha_n + \beta_n}$. From the condition (C2) and Step 3, it is easily seen that $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Thus, applying Lemma 2.3 to (3.16), we conclude that $\lim_{n \rightarrow \infty} \Phi(\|x_n - p\|) = 0$. Hence $\lim_{n \rightarrow \infty} x_n = p$. This completes the proof. \square

Corollary 3.4. *Let E be a real reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ . Let C be a nonempty closed convex subset of E , and let $T : C \rightarrow C$ be a continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $(0, 1)$ satisfying the conditions (C1) and (C2) in Theorem 3.2 and $\gamma_n = 1 - \alpha_n - \beta_n$ for $n \geq 1$. For arbitrary initial value $x_0 \in C$ and a fixed anchor $u \in C$, let the sequence $\{x_n\}$ be generated by*

$$x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges strongly to a fixed point p of T , which is the unique solution of the variational inequality

$$\langle p - u, J_\varphi(p - q) \rangle \leq 0, \quad \forall q \in F(T).$$

Proof. Taking $Ax = u$, $\forall x \in C$ as a constant function, the result follows from Theorem 3.3. \square

Remark 3.5. 1) *Theorem 3.2 extends and improves Theorem 3.1 of Chen and Zhu [3] (and Theorem 2.1 of Moudafi [14] and Theorem 3.1 of Xu [20]) in the following aspects:*

- (a) *The nonexpansive mapping T is replaced by a continuous pseudocontractive mapping T .*
- (b) *The contraction f (and the constant u) is replaced by a continuous bounded strongly pseudocontractive mapping A .*

2) *Theorem 3.2 also develops Theorem 2 of Morales [12] to a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ .*

3) *Theorem 3.3 says that Theorem 3.2 of Jung [8] holds in a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ .*

4) *Theorem 3.3 also extends and improves Theorem 3.1 of Yao et al. [21] in the following aspects:*

- (a) *u is replaced by a continuous bounded strongly pseudocontractive mapping A .*
- (b) *The space is replaced by a space having a weakly continuous duality mapping J_φ with gauge function φ .*
- (c) *The condition $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\beta_n} = 0$ is weakened to the condition $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \beta_n = 0$.*

5) *Theorem 3.3 develops and complements Theorem 3.1 of Song and Chen [17] by replacing the contractive mapping f in the iterative algorithm (1.2) with a continuous bounded strongly pseudocontractive mapping A in a reflexive Banach space which has a weakly continuous duality mapping J_φ with gauge function φ .*

References

- [1] R.P. Agarwal, D. O'Regan, D.R. Sahu, Fixed Point Theory for Lipschitzian-type Mappings with Applications, Springer 2009.
- [2] C.E. Chidume, Global iteration schemes for strongly pseudocontractive maps, Proc. Amer. Math. Soc. 126 (1998) 2641–2649.
- [3] R. Chen and Z. Zhu, Viscosity approximation fixed points for nonexpansive and m -accretive operators, Fixed Point Theory Appl. 2006 (2006) Article ID 81325, pages 1–10, DOI 10.1155/FPTA/2006/81325.
- [4] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer Academic Publishers, Dordrecht 1990.
- [5] K. Deimling, Zeros of accretive operators, Manuscripta Math. 13 (1974) 365–374.
- [6] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory in Cambridge Studies in Advanced Mathematics, Vol. 28, Cambridge Univ. Press, Cambridge, UK, 1990.
- [7] S. Homaeipour, A. Razani, Strong convergence theorems of three-step iterations for multi-valued mappings in Banach spaces, Filomat 25(1) (2011) 173–184.
- [8] J.S. Jung, Iterative methods for pseudocontractive mappings in Banach spaces, Abstr. Appl. Anal. 2013 (2013) Article ID 643602, 7 pages. <http://dx.doi.org/10.1155/2013/643602>.

- [9] T. Kato, Nonlinear semigroup and evolution, *J. Math. Soc. Japan* 19 (1967) 508–520.
- [10] Y. Kimura, K. Satô, Two convergence theorems to fixed point of a nonexpansive mapping on the unit sphere of a Hilbert space, *Filomat* 26(5) (2012) 949–955.
- [11] R.H. Martin, Differential equations on closed subsets of Banach spaces, *Trans. Amer. Math. Soc.* 179 (1973) 399–414.
- [12] C.H. Morales, Strong convergence of path for continuous pseudo-contractive mappings, *Proc. Amer. Math. Soc.* 135 (2007) 2831–2838.
- [13] C.H. Morales, J.S. Jung, Convergence of paths for pseudocontractive mappings in Banach spaces, *Proc. Amer. Math. Soc.* 128 (2000) 3411–3419.
- [14] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.* 241 (2000) 46–55.
- [15] A. Rafiq, On Mann iteration in Hilbert spaces, *Nonlinear Anal.* 66 (2007) 2230–2236.
- [16] S. Reich, An iterative procedure for constructing zero of accretive sets in Banach spaces, *Nonlinear Anal.* 2 (1978) 85–92.
- [17] Y. Song, R. Chen, Convergence theorems of iterative algorithms for continuous pseudocontractive mappings, *Nonlinear Anal.* 67 (2007) 486–497.
- [18] H.K. Xu, Iterative algorithms for nonlinear operators, *J. Lond. Math. Soc.* 66 (2002) 240–256.
- [19] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* 298 (2004) 279–291.
- [20] H.K. Xu, Strong convergence of an iterative method for nonexpansive and accretive operators, *J. Math. Anal. Appl.* 314 (2006) 631–643.
- [21] Y. Yao, C.Y. Liou, R. Chen, Strong convergence of an iterative algorithm for pseudocontractive mapping in Banach spaces, *Nonlinear Anal.* 67 (2007) 3311–3317.