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Some new fixed point theorems for contractive and nonexpansive mappings

Rajendra Pant

Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur 440010, India

Abstract. In the present paper, we obtain some new fixed point theorems for set-valued contractive and nonexpansive mappings in the setting of *ultrametric spaces*. Our theorems complement, generalize and extend some well known results of Petalas and Vidalis [A fixed point theorem in non-Archimedean vector spaces, Proc. Amer. Math. Soc **118**(1993), 819–821.], Suzuki [A new type of fixed point theorem in metric spaces, Nonlinear Anal. **71**(2009), 5313-5317.] and others.

1. Introduction and Preliminaries

Let (X, d) be a metric space and $T : X \to X$ a self-mapping. The mapping T is said to be contractive if $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$. It is well known that the contractive and nonexpansive mappings are necessarily continuous on the whole domain. Further, it is also interesting to note that these mappings (contractive and nonexpansive) need not have a fixed point in a complete metric space. Followings are the well known examples.

Example 1.1. *[6]. Let $X = (-\infty, -\infty)$ endowed with the usual metric and $T : X \to X$ defined by*

$$Tx = x + \frac{1}{1 + e^x}$$

for all $x \in X$. Notice that X is complete and T is contractive mapping but T does not have a fixed point.

Example 1.2. *Let $X = [0, \infty)$ endowed with the usual metric and $T : X \to X$ defined by*

$$Tx = 1 + x$$

for all $x \in X$. Notice that X is complete and T is a nonexpansive mapping but T does not have a fixed point.

A metric space (X, d) is said to be ultrametric space if the triangle inequality is replaced by the strong triangle inequality, i.e.,

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}$$

for all $x, y, z \in X$.

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Email address: pant.rajendra@gmail.com (Rajendra Pant)

Example 1.3. [2]. *Every discrete metric space is an ultra-metric space.*

An ultrametric space (X, d) is said to be spherically complete if every descending collections of closed balls in X has nonempty intersection.

Petalas and Vidalis in [6], studied the sufficient condition for existence of fixed points of contractive and nonexpansive mappings on ultrametric spaces and obtained the following theorems:

Theorem 1.4. *Let (X, d) be a spherically complete ultrametric space and $T : X \to X$ a contractive mapping. Then T has a unique fixed point.*

Theorem 1.5. *Let (X, d) be a spherically complete ultrametric space and $T : X \to X$ a nonexpansive mapping. Then either T has at least one fixed point or there exists a ball B of radius $r > 0$ such that $T : B \to B$ and for which $d(b, Tb) = r$ for each $b \in B$.*

A number of extensions and generalizations of Theorem 1.4 and 1.5 have appeared in [2–5] and others.

In this paper, we obtain some fixed point theorems for certain classes of contractive and nonexpansive mappings, which are not necessarily continuous. Our results extend and generalize a number of fixed point theorems including the above theorems of Petalas and Vidalis [6].

2. Contractive Mappings

In [1], Edelstein proved that every contractive self-mapping of a compact space has a fixed point. Suzuki [7], obtained the following generalization of Edelstein's theorem.

Theorem 2.1. [7]. *Let (X, d) be a compact metric space and $T : X \to X$ mapping. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y),$$

for $x, y \in X$. Then T has a unique fixed point.

A mapping satisfying the conditions of the Theorem 2.1 need not be continuous [7, example 1].

In this section we shall obtain a set-valued extension of the above theorem in a spherically complete ultrametric space.

Let (X, d) be an ultrametric space and $C(X)$ the collection of all compact subsets of X. The Hausdorff metric induced by d is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\}$$

for all $A, B \subseteq C(X)$, where $d(x, B) = \inf_{y\in B} d(x, y)$.

Let $T : X \to C(X)$ be a set-valued mapping. A point $z \in X$ is said to be a fixed point of T if $z \in Tz$.

The following theorem is our main result of this section.

Theorem 2.2. *Let (X, d) be a spherically complete ultrametric space and $T : X \to C(X)$ a set-valued mapping. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } H(Tx, Ty) < d(x, y) \tag{2.1}$$

for $x, y \in X$. Then T has a fixed point.

[0] The proof techniques have been used from [5, 6].

Proof. We define $\mathscr{B}_a := \mathcal{B}(a, r)$, the closed balls centered at a with radii $r = d(a, Ta)$. Let $\mathscr{A}$ be the collection of these balls for all $a \in X$. The relation

$$\mathscr{B}_a \leq \mathscr{B}_b \text{ iff } \mathscr{B}_b \subseteq \mathscr{B}_a$$

is a partial order. Let $\mathscr{A}_1$ be a totally ordered subfamily of $\mathscr{A}$. From the spherical completeness of X, we have

$$\bigcap_{\mathscr{B}_a \in \mathscr{A}_1} \mathscr{B}_a := B \neq \emptyset.$$

Let $b \in B$ and $\mathscr{B}_a \in \mathscr{A}_1$. Then if $x \in \mathscr{B}_b$,

$$d(x, b) \leq d(b, Tb) \leq \max\{d(b, a), d(a, Ta), H(Ta, Tb)\}.$$

Since $\frac{1}{2}d(a, b) < d(a, b)$ for all $a, b \in X$, by (2.1), we get

$$d(x, b) \leq \max\{d(b, a), d(a, Ta), d(a, b)\} = \max\{d(a, b), d(a, Ta)\}. \tag{2.2}$$

Now for $x \in \mathscr{B}_b$

$$d(x, a) \leq \max\{d(a, b), d(b, x)\}.$$

By the fact that $d(a, b) \leq d(a, Ta)$ and (2.2), we get

$$d(x, a) \leq \max\{d(a, b), d(b, x)\} \leq d(a, Ta).$$

Hence $x \in \mathscr{B}_a$ and $\mathscr{B}_b \subseteq \mathscr{B}_a$ for every $\mathscr{B}_a \in \mathscr{A}_1$. Thus $\mathscr{B}_b$ is an upper bound in $\mathscr{A}$ for the family $\mathscr{A}_1$. By Zorn's lemma, $\mathscr{A}$ has a maximal element, say $\mathscr{B}_z$, for some $z \in X$. We shall show that $z \in Tz$. Suppose that $z \notin Tz$. Then the compactness of Tz implies that there exists $w \in Tz$ with $w \neq z$ such that $d(w, z) = d(z, Tz)$. We show that $\mathscr{B}_w \subseteq \mathscr{B}_z$.

If $u \in \mathscr{B}_w$ then $d(w, u) \leq d(w, Tw)$. Since $w \in Tz$ and $\frac{1}{2}d(w, z) < d(w, z)$ for all $w, z \in X$, we have

$$d(w, u) \leq d(w, Tw) \leq H(w, Tw) < d(z, w) = d(z, Tz).$$

Also

$$d(u, z) \leq \max\{d(u, w), d(w, z)\} \leq d(z, Tz).$$

Therefore $u \in \mathscr{B}_z$ and $\mathscr{B}_w \subseteq \mathscr{B}_z$. But as

$$d(w, Tw) \leq H(Tw, Tz) < d(w, z),$$

$z \notin \mathscr{B}_w$, so $\mathscr{B}_w \subsetneq \mathscr{B}_z$. This contradicts the maximality of $\mathscr{B}_z$. Therefore T has a fixed point. □

When T is a single valued mapping on X, we get the following corollary, which generalizes Theorem 1.4 and extends Theorem 2.1.

Corollary 2.3. *Let (X, d) be a spherically complete ultrametric space and $T : X \to X$ a mapping. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \textit{ implies } d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

3. Nonexpansive Mappings

In [8], Suzuki introduced the following notion of *condition* (C), which is weaker than nonexpansiveness.

Definition 3.1. *Let C be a nonempty subset of a metric space (X, d) and $T : C \to C$ a mapping. The mapping T is said to satisfy the condition (C) if*

$$\frac{1}{2}d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le d(x, y) \tag{C}$$

for all $x, y \in C$.

Every nonexpansive mapping satisfies the condition (C), but the converse is not true. Further, the mapping satisfying condition (C), need not be continuous [8, Example 1].

Now we obtain a fixed point theorem for a set-valued mapping satisfying Suzuki's type condition.

Theorem 3.2. *Let (X, d) be a spherically complete ultrametric space and $T : X \to C(X)$ a set-valued mapping. Assume that*

$$\frac{1}{2}d(x, Tx) \le d(x, y) \text{ implies } H(Tx, Ty) \le d(x, y) \tag{3.1}$$

for $x, y \in X$. Then either T has at least one fixed point or there exists a ball B of radius $r > 0$ such that $T : B \to B$ and for which $d(b, Tb) = r$ for each $b \in B$.

Proof. Let $\mathscr{B}_a$ and $\mathscr{A}$ as in the proof of Theorem 2.2. We find a maximal element $\mathscr{B}_z$ of $\mathscr{A}$. For any $b \in \mathscr{B}_z$, we have

$$d(b, Tb) \le \max\{d(b, z), d(z, Tz), H(Tz, Tb)\}.$$

Since $\frac{1}{2}d(b, z) \le d(b, z)$, by (3.1), the above inequality leads to

$$d(b, Tb) \le \max\{d(b, z), d(z, Tz), d(z, b)\} = d(z, Tz).$$

Thus $\mathscr{B}_b \subseteq \mathscr{B}_z$ (since $b \in \mathscr{B}_z \cap \mathscr{B}_b$) and $Tb \in \mathscr{B}_z$. If $z \in Tz$ then z is fixed point of T.

Finally, we show that if $z \notin Tz$ then $d(b, Tb) = d(z, Tz)$. Suppose that for some $b \in \mathscr{B}_z$

$$d(b, Tb) < d(z, Tz).$$

We know that $d(b, z) \le d(z, Tz)$. Now

$$d(z, Tz) \le \max\{d(z, b), d(b, Tb), H(Tb, Tz)\}.$$

Since $\frac{1}{2}d(b, z) \le d(b, z)$, by (3.1), we get

$$d(z, Tz) \le \max\{d(z, b), d(b, Tb), d(b, z)\} = d(b, z).$$

Hence we get $d(b, Tb) < d(z, Tz) = d(b, z)$. This implies that $z \notin \mathscr{B}_z$, which is impossible from the maximality of $\mathscr{B}_z$. Thus

$$d(b, Tb) = d(z, Tz) := r \quad \forall b \in \mathscr{B}_z.$$

□

When T is a single valued mapping on X, we get the following corollary, which generalizes Theorem 1.5.

Corollary 3.3. (Compare [4, Theorem 4]). *Let (X, d) be a spherically complete ultrametric space and $T : X \to X$ a mapping. Assume that*

$$\frac{1}{2}d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le d(x, y)$$

for $x, y \in X$. Then either T has at least one fixed point or there exists a ball B of radius $r > 0$ such that $T : B \to B$ and for which $d(b, Tb) = r$ for each $b \in B$.

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