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# Restricted Uniform Density and Corresponding Convergence Methods

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**Abstract.** Recently, the idea of asymptotic density of order $\alpha$ has been introduced by Bhunia et. al. in [7]. In the present paper we introduce the notion of uniform density of order $\alpha$ and define and study related convergence methods so-called $I_u$-convergence of order $\alpha$ and uniform strong $p$-Cesàro convergence of order $\alpha$. Furthermore, some examples are displayed here to show that these concepts are not same with the concepts defined in [7].

## 1. Introduction and Preliminaries

The notion of uniform density was introduced and studied in [18], [9] and [10]. It has a relation with natural density and is used in various parts of mathematics, in particular in number theory and ergodic theory, see for example [19, 23, 35]. In this paper we first make a new approach to the notion of uniform density by defining it in restricted form. Then we study on related convergence methods.

Now we remind some basic notations and definitions used in this paper.

Let $A \subset N$. If $m, n \in N$, by $A(m,n)$ we denote the cardinality of the set $A \cap [m,n]$. The upper and lower asymptotic (natural) density of the subset $A$ is defined by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{A(1,n)}{n} \text{ and } \underline{d}(A) = \liminf_{n\to\infty} \frac{A(1,n)}{n}.$$

If $\overline{d}(A) = \underline{d}(A)$ then we say that the natural density of $A$ exists and it is denoted by $d(A)$. Any number sequence $x = (x_k)$ is statistically convergent to the number $L$ provided that for each $\varepsilon > 0$, $d(A_\varepsilon) = 0$, where $A_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. In this case we write $st - \lim x = L$ (see [17]).

The uniform density of $A \subset N$ was introduced in [34] (also in [9] and [18]) as the following: Put

$$a_n = \min_{m\geq 0} A(m+1, m+n) \text{ and } a^n = \max_{m\geq 0} A(m+1, m+n).$$

Then existence of the limits

$$\underline{u}(A) = \lim_{n\to\infty} \frac{a_n}{n}, \; \overline{u}(A) = \lim_{n\to\infty} \frac{a^n}{n}$$

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is obvious and they are called the lower and upper uniform density of the set $A$, respectively. If $\underline{u}(A) = \overline{u}(A) = u(A)$, then $u(A)$ is called the uniform density of $A$.

Let $I$ be an ideal of subsets of $N$ (i.e. $I$ is an additive and hereditary class of sets). A sequence of real numbers $x = (x_k)$ is said to be $I$-convergent to $L$ provided that for each $\varepsilon > 0$ the set $A_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ belongs to $I$. In this case, we write $I-\lim x = L$ (see [21]). Put $I = I_d = \{A \subset \mathbb{N} : d(A) = 0\}$, then $I_d$-convergence coincides with statistical convergence. We refer readers to see [1, 6, 8, 20, 25–32, 37] for recent studies on $I$-convergence and statistical convergence.

In the case $I = I_u = \{A \subset \mathbb{N} : u(A) = 0\}$, we obtain $I_u$-convergence which was studied by Baláž and Šalát in [3] or by Pehlivan in [33] with the name of uniform statistical convergence (see also [14] and [38] for double sequences). If $x = (x_k)$ is $I_u$-convergent to $L$, we write $I_u - \lim x_k = L$.

In [7], Bhunia et al. have generalized the concept of statistical convergence by replacing $n$ with $n^\alpha$ $(0 < \alpha \leq 1)$ in the definition of natural density and they called it as statistical convergence of order $\alpha$ (see also, [11]). Let $0 < \alpha \leq 1$ be a real number. Then

$$\overline{d^\alpha}(A) = \limsup_{n\to\infty} \frac{A(1,n)}{n^\alpha} \text{ and } \underline{d^\alpha}(A) = \liminf_{n\to\infty} \frac{A(1,n)}{n^\alpha}$$

are called the upper and lower asymptotic density of order $\alpha$ of the set $A$, respectively. If the limit $\lim_n \frac{A(1,n)}{n^\alpha}$ exists then $d^\alpha(A) = \overline{d^\alpha}(A) = \underline{d^\alpha}(A)$ is said to be the asymptotic density of order $\alpha$ of the set $A$. The real number sequence $x = (x_k)$ is said to be statistically convergent of order $\alpha$ to $L$ provided that for all $\varepsilon > 0$, $d^\alpha(A_\varepsilon) = 0$, i.e.

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{n^\alpha} = 0.$$

In this case we write $S^\alpha - \lim x = L$ and the set of all statistically convergent sequences of order $\alpha$ is denoted by $m_0^\alpha$.

## 2. Main Results

In this section we first define the notion of uniform density of order $\alpha$ of a subset of positive integers.

**Definition 2.1.** *Let $A \subset \mathbb{N}$ and $0 < \alpha \leq 1$ be a real number. Then the limits*

$$\underline{u}^\alpha(A) = \lim_{n\to\infty} \frac{\min\limits_{m\geq 0} A(m+1, m+n)}{n^\alpha} \text{ and } \overline{u}^\alpha(A) = \lim_{n\to\infty} \frac{\max\limits_{m\geq 0} A(m+1, m+n)}{n^\alpha}$$

*exist and they are called the lower and upper uniform density of order $\alpha$ of the set $A$, respectively. If $\underline{u}^\alpha(A) = \overline{u}^\alpha(A)$ then $u^\alpha(A) = \underline{u}^\alpha(A)$ is called the uniform density of order $\alpha$ of $A$.*

We can easily see that $I = I_{u^\alpha} = \{A \subset \mathbb{N} : u^\alpha(A) = 0\}$ is an admissible ideal in $\mathbb{N}$. Also, we can easily see that for each $A \subset \mathbb{N}$ the inequalities

$$\underline{u}^\alpha(A) \leq \underline{d^\alpha}(A) \leq \overline{d^\alpha}(A) \leq \overline{u}^\alpha(A)$$

hold. Hence if $u^\alpha(A)$ exists then also $d^\alpha(A)$ and $u^\alpha(A) = d^\alpha(A)$ exists. The converse is not true. For example for the set

$$A = \bigcup_{k=1}^{\infty} \left\{10^k + 1, 10^k + 2, ..., 10^k + k\right\} \tag{1}$$

it is easy to see that $d^\alpha(A) = 0$, $\underline{u}^\alpha(A) = 0$, $\overline{u}^\alpha(A) = \infty$ for $\alpha = 1/2$.

**Definition 2.2.** *The real number sequence $x = (x_k)$ is said to be $I_u$-convergent to $L$ of order $\alpha$ if for each $\varepsilon > 0$ the set $A_\varepsilon$ belongs to $I_{u^\alpha}$, i.e. $u^\alpha(A_\varepsilon) = 0$, where $A_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. In this case we write $I_{u^\alpha} - \lim x_k = L$.*

In Theorem 3 of [3], the authors claim that any sequence $x = (x_k)$ is $I_u$-convergent to $L$ if and only if there exists a set $K \subset \mathbb{N}$ with $u(K) = 1$ and the subsequence $(x_n)_{n\in K}$ converges to $L$. But it is pointed out in [5] that Theorem 3 of [3] is not true. However we can prove the following for $0 < \alpha < 1$.

Note that from now on, we apply our new concepts to obtain similar results proved in [7].

**Theorem 2.3.** *If the sequence $x = (x_k)$ is $I_u$-convergent to $L$ of order $\alpha$ $(0 < \alpha < 1)$ then there is a set $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subset \mathbb{N}$ such that $\overline{u}^\alpha(K) = \infty$ and $\lim_{n\to\infty} x_{k_n} = L$.*

**Proof.** Suppose that $I_{u^\alpha} - \lim x_k = L$. Put $K_j = \{k \in \mathbb{N} : |x_k - L| < 1/j\}$ $(j = 1, 2, ...)$. Then by definition of $I_u$-convergence of order $\alpha$, we have $\overline{u}^\alpha\left(\mathbb{N}\backslash K_j\right) = 0$ and it is easy to check that $\overline{u}^\alpha\left(K_j\right) = \infty$ $(j = 1, 2, ...)$. Also, by the definition of $K_j$ we have

$$K_1 \supset K_2 \supset \cdots \supset K_j \supset K_{j+1} \supset \cdots \tag{2}$$

Let $\left(\Delta_j\right)$ be a strictly increasing sequence of positive real numbers. Let us choose an arbitrary number $s_1 \in K_1$. By (2) there exists a number $s_2 > s_1$, $s_2 \in K_2$ such that for each $n \geq s_2$ we have $n^{-\alpha}\max_{m\geq 0} K_2(m+1, m+n) > \Delta_2$. Further choose an $s_3 > s_2$, $s_3 \in K_3$ such that for each $n \geq s_3$ we have $n^{-\alpha}\max_{m\geq 0} K_3(m+1, m+n) > \Delta_3$. In this way we can find an increasing sequence of positive integers $s_1 < s_2 < \cdots < s_j < \cdots$ such that

$$s_j \in K_j \ \ (j = 1, 2, ...) \text{ and } \frac{\max\limits_{m\geq 0} K_j(m+1, m+n)}{n^\alpha} > \Delta_j \tag{3}$$

for each $n \geq s_j$. Now define the set $K$ as follows:

Each positive integer of $[1, s_1]$ belongs to $K$ and suppose that any positive integer of $\left[s_j, s_{j+1}\right]$ belongs to $K$ if and only if it belongs to $K_j$ $(j = 1, 2, ...)$. According to (2) and (3) it follows that for each $n$, $s_j \leq n < s_{j+1}$ we have

$$\frac{\max\limits_{m\geq 0} K(m+1, m+n)}{n^\alpha} \geq \frac{\max\limits_{m\geq 0} K_j(m+1, m+n)}{n^\alpha} > \Delta_j.$$

From this it is obvious that $\overline{u}^\alpha(K) = \infty$.

Let $\varepsilon > 0$ be given and choose $j$ such that $1/j < \varepsilon$. Let $n \geq s_j$, $j \in K$. Then, there exists a number $r \geq j$ such that $s_r \leq n < s_{r+1}$. From the definition of $K$, $n \in K_r$ and hence

$$|x_n - L| < \frac{1}{r} \leq \frac{1}{j} < \varepsilon\,.$$

Thus, $\lim_{n\to\infty} x_{k_n} = L$ and this completes the proof. □

**Remark 2.4.** *The converse of Theorem 2.3 is not necessarily true. For instance, let $A \subset \mathbb{N}$ be the set given by (1) and consider the sequence $x = (x_k)$ defined by*

$$x_k = \begin{cases} 1, & k \in A \\ 0, & k \notin A. \end{cases}$$

*Then $\lim_{n\to\infty, k_n\in A} x_{k_n} = 1$ and $A$ has the property $\overline{u}^\alpha(A) = \infty$ for $\alpha = \frac{1}{2}$ but $x$ is not $I_u$-convergent of order $\alpha$. Since $S^\alpha - \lim x = 0$ for all $\alpha$, $0 < \alpha \leq 1$, this example also shows that $m_1^\alpha \neq m_0^\alpha$. That is, the inclusion $m_1^\alpha \subset m_0^\alpha$ is proper.*

**Theorem 2.5.** *Let $0 < \alpha \leq \beta \leq 1$. Then $m_1^\alpha \subset m_1^\beta$ and the inclusion is strict for any $\alpha < \beta$.*

**Proof.** Let $0 < \alpha \leq \beta \leq 1$. Then

$$\frac{\max_{m\geq 0} A_\varepsilon(m+1, m+n)}{n^\beta} \leq \frac{\max_{m\geq 0} A_\varepsilon(m+1, m+n)}{n^\alpha}$$

for each $\varepsilon > 0$ and this implies that $m_1^\alpha \subset m_1^\beta$. To verify the strictness of inclusion consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & k \in P \\ 0, & k \notin P. \end{cases} \tag{4}$$

where $P$ is the set of all prime numbers. Let $\alpha = 1/2$ and $\beta = 1$. For the reason that $u^1(P) = u(P) = 0$ (see [10]), $x$ is $I_u$-convergent to zero. The well-known prime number theorem tells us that the number of primes less than $n$ is approximately $n/\log n$. Therefore

$$\frac{\min_{m\geq 0} A_\varepsilon(m+1, m+n)}{n^{1/2}} \approx \frac{\sqrt{n}}{\log n} \to \infty, \text{ as } n \to \infty.$$

Thus $x$ is not $I_u$-convergent of order $\alpha$ for $\alpha = 1/2$. □

**Corollary 2.6.** *If a sequence is $I_u$-convergent of order $\alpha$ to $L$ for some $0 < \alpha \leq 1$, then it is $I_u$-convergent to $L$, that is $m_1^\alpha \subset m_1$ and the inclusion is strict.*

Lorentz [22] defined that a sequence $x = (x_k)$ is almost convergent to a number $L$ if every Banach limit of $x$ is equal to $L$, which is equivalent to the condition

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=m+1}^{m+n} x_k = L, \text{ uniformly in } m.$$

On the other hand, in [3] Baláž and Šalát introduced the concept of uniformly strong $p$-Cesàro convergence which is a generalization of the notion of strong almost convergence (see [24]). The sequence $x$ is said to be uniformly strong $p$-Cesàro convergent $(0 < p < \infty)$ to $L$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=m+1}^{m+n} |x_k - L|^p = 0, \text{ uniformly in } m.$$

By $uw_p$, denote the set of all sequences which are uniformly strong $p$ -Cesàro convergent.

Next, we generalize this method by replacing $1/n$ with $1/n^\alpha, 0 < \alpha \leq 1$.

**Definition 2.7.** *Let $0 < p < \infty$ and $0 < \alpha \leq 1$. Then a sequence $x = (x_k)$ is said to be uniformly strong p-Cesàro convergent of order $\alpha$ to a number L if*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{k=m+1}^{m+n} |x_k - L|^p = 0$$

*uniformly in m. By $uw_p^\alpha$ we denote the set of all sequences which are uniformly strong p-Cesàro convergent of order $\alpha$.*

**Theorem 2.8.** *Let $0 < p < \infty$ and $0 < \alpha \leq 1$. If a sequence $x = (x_k)$ is uniformly strong p-Cesàro convergent of order $\alpha$ to L then it is $I_u$-convergent of order $\alpha$ to L.*

**Proof.** Suppose that $x$ is uniformly strong $p$-Cesàro convergent of order $\alpha$ to $L$ and let $\varepsilon > 0$. Then for each $m \geq 0$ we have

$$\begin{aligned}\frac{1}{n^\alpha}\sum_{k=m+1}^{m+n}|x_k - L|^p &\geq \frac{1}{n^\alpha}\sum_{\substack{k=m+1\\|x_k-L|\geq\varepsilon}}^{m+n}|x_k - L|^p\\ &\geq \varepsilon^p\frac{\max\limits_{m\geq 0} A_\varepsilon(m+1,m+n)}{n^\alpha}\\ &= \varepsilon^p\frac{a^n}{n^\alpha}.\end{aligned}$$

This implies that $\lim_{n\to\infty}\dfrac{a^n}{n^\alpha} = 0$ and $u^\alpha(A_\varepsilon) = 0$, i.e. $I_{u^\alpha} - \lim x_k = L$. □

**Remark 2.9.** *The converse of Theorem 2.8 holds for $\alpha = 1$ in case of bounded sequences, which is proved in [3]. However, we claim that it does not hold for $0 < \alpha < 1$ even in case of bounded sequences. To show this we have to construct a sequence such that it is bounded and $I_u$-convergent of order $\alpha$ but not uniformly strong p-Cesàro convergent of order $\alpha$. We leave it as an open problem.*

**Theorem 2.10.** *Let $0 < p < \infty$ and $0 < \alpha \leq \beta \leq 1$. Then $uw_p^\alpha \subset uw_p^\beta$ and the inclusion remains strict for any $\alpha < \beta$.*

**Proof.** Let $x = (x_k) \in uw_p^\alpha$. Then for given $\alpha$ and $\beta$ such that $0 < \alpha \leq \beta \leq 1$ and a positive number $p$, we have

$$\frac{1}{n^\beta}\sum_{k=m+1}^{m+n}|x_k - L|^p \leq \frac{1}{n^\alpha}\sum_{k=m+1}^{m+n}|x_k - L|^p$$

for each $m \geq 0$, which gives that $uw_p^\alpha \subset uw_p^\beta$.

To verify the strictness of inclusion we give two examples. First, consider the sequence $x = (x_k)$ defined by

$$x_{m+k} = \begin{cases} 1, & k = l^j,\\ 0, & k \neq l^j, l \in \mathbb{N}\end{cases}$$

for each $m \geq 0$, where $j$ is any positive integer such that $\alpha < \frac{1}{j} < \beta$. Then it is easy to see that

$$\frac{1}{n^\beta}\sum_{k=m+1}^{m+n}|x_k - L|^p = \frac{1}{n^\beta}\sum_{k=1}^{n}|x_{m+k} - 0|^p \leq \frac{n^{1/j}}{n^\beta} = \frac{1}{n^{\beta-1/j}}.$$

Since $\dfrac{1}{n^{\beta-1/j}} \to 0$ as $n \to \infty$, $x \in uw_p^\beta$. But $x \notin uw_p^\alpha$.

The another example can be given as in proof of Theorem 2.5. For the sequence $x$ defined by (4), $x \in uw_p^1$ but $x \notin uw_p^{1/2}$. □

Let $m$ be set of all bounded real sequences endowed with the supremum norm. Then we have the following result.

**Theorem 2.11.** *For any fixed $\alpha$, $0 < \alpha \leq 1$, the set $m_1^\alpha \cap m$ is a closed linear subspace of $m$.*

**Proof.** It is obvious that $m_1^\alpha \cap m$ is a linear subspace of $m$. To show that it is closed, let $x^{(k)} = \left(x_j^{(k)}\right)$ $(k = 1, 2, ...)$ be any sequence in $m_1^\alpha \cap m$ and $x^{(k)} \to x = \left(x_j\right)$ in $m$ . Then $\left\|x^{(k)} - x\right\|_\infty \to 0$ as $k \to \infty$. Since $m_1^\alpha \subset m_1$

(Corollary 2.6) and $m_1 \cap m$ is closed in $m$ (see [4]), we have $x \in m_1 \cap m$. Also, since $x^{(k)} \in m_1^{\alpha} \subset m_1$, $x^{(k)}$ is $I_u$-convergent to some number $L_k$ for all $k \in \mathbb{N}$. We will verify that the sequence $(L_k)$ is convergent to number $L$ and the sequence $x = \left(x_j\right)$ is $I_u$-convergent of order $\alpha$ to $L$. Now we have

$$|L_k - L_r| \leq \left|\frac{1}{n}\sum_{j=m+1}^{m+n} x_j^{(k)} - L_k\right| + \left|\frac{1}{n}\sum_{j=m+1}^{m+n} x_j^{(r)} - L_r\right| + \left|\frac{1}{n}\sum_{j=m+1}^{m+n} x_j^{(k)} - \frac{1}{n}\sum_{j=m+1}^{m+n} x_j^{(r)}\right| \tag{5}$$

for all $j \in \mathbb{N}$. Let $\varepsilon > 0$. Since $x^{(k)} \to x$ in $m$, there exists $n_0 \in \mathbb{N}$ such that for $k, r > n_0$ we have $\left\|x^{(k)} - x^{(r)}\right\|_{\infty} < \frac{\varepsilon}{3}$. Since $x^{(k)}$ and $x^{(r)}$ is $I_u$-convergent to $L_k$ and $L_r$, they are also almost convergent to these numbers (see Theorem 1 of [3]), so there exists an $p_0$ such that the first and second summand in (5) is less than $\frac{\varepsilon}{3}$ for $n > p_0$, $m = 1, 2, ...$, respectively. On the other hand, we have

$$\left|\frac{1}{n}\sum_{j=m+1}^{m+n} x_j^{(k)} - \frac{1}{n}\sum_{j=m+1}^{m+n} x_j^{(r)}\right| \leq \frac{1}{n}\sum_{j=m+1}^{m+n} \left|x_j^{(k)} - x_j^{(r)}\right| \leq \frac{1}{n}\sum_{j=m+1}^{m+n} \left\|x^{(k)} - x^{(r)}\right\|_{\infty} < \frac{\varepsilon}{3}.$$

Consequently from (5) it follows that $|L_k - L_r| < \varepsilon$ for all $k, r > n_0$. This means that $(L_k)$ is a Cauchy sequence of real numbers. Thus there exists $L$ such that $\lim_{k\to\infty} L_k = L$. Further let $\eta > 0$ and set $A_\eta = \left\{j \in \mathbb{N} : \left|x_j - L\right| \geq \eta\right\}$. Choose an $r$ such that $\left\|x^{(r)} - x\right\|_{\infty} < \frac{\eta}{3}$ and $\|L_r - L\|_{\infty} < \frac{\eta}{3}$ holds at the same time. Also set $B_\eta = \left\{j \in \mathbb{N} : \left|x_j^{(r)} - L_r\right| \geq \frac{\eta}{3}\right\}$. For an arbitrary $j \in \mathbb{N}$ we have

$$\left|x_j - L\right| \leq \left|x_j - x_j^{(r)}\right| + \left|x_j^{(r)} - L_r\right| + |L_r - L| < \frac{2\eta}{3} + \left|x_j^{(r)} - L_r\right| \tag{6}$$

According to (6), $j \in A_\eta$ implies $j \in B_\eta$, thus $A_\eta \subset B_\eta$. Using the fact that $u^{\alpha}\left(B_\eta\right) = 0$, we get $u^{\alpha}\left(A_\eta\right) = 0$ and therefore $I_{u^{\alpha}} - \lim x_j = L$. This shows that $x = \left(x_j\right) \in m_1^{\alpha}$ and so $m_1^{\alpha}$ is closed in $m$. □

## 3. Concluding Remarks

In this section we suggest new different ideas for further study.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to $\infty$ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \quad \lambda_1 = 1.$$

Then by using such type sequences and the idea given by Altin et al. (see [2]), we can define a generalization of uniform density as the following:

Let $A \subset \mathbb{N}$ and $0 < \alpha \leq 1$ be a real number. Then the limits

$$\underline{u}_\lambda^{\alpha}(A) = \lim_{n\to\infty} \frac{\min_{m\geq 0} A\left(m + \lambda_n - n + 1, m + n\right)}{\lambda_n^{\alpha}}$$

and

$$\overline{u}_\lambda^{\alpha}(A) = \lim_{n\to\infty} \frac{\max_{m\geq 0} A\left(m + \lambda_n - n + 1, m + n\right)}{\lambda_n^{\alpha}}$$

exist. These are called the lower and upper uniform $\lambda$-density of order $\alpha$ of the set $A$, respectively. If $\underline{u}_\lambda^{\alpha}(A) = \overline{u}_\lambda^{\alpha}(A)$ then $u_\lambda^{\alpha}(A) = \underline{u}_\lambda^{\alpha}(A)$ is called the uniform $\lambda$-density of order $\alpha$ of $A$. Then with the help of uniform $\lambda$-density of order $\alpha$, one can define the corresponding method $I_{u_\lambda}$-convergence of order $\alpha$.

On the other hand the concept of double uniform density for any subset of $\mathbb{N} \times \mathbb{N}$ has been introduced by Das and Savaş in [14] (see also [38]). Hence our results can be reformulated in the light of the concept of double uniform density of order $\alpha$.

Finally, we note that some sequence spaces can also be defined through the corresponding convergence methods of these densities of order $\alpha$ (for instance see [12, 13, 15, 16, 36]).

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