New type of Lacunary Orlicz Difference Sequence Spaces Generated By Infinite Matrices

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Abstract. The main purpose of this paper is to introduce the spaces \( A[M,\Delta,p] \), \( p_0[A,M,\Delta,p] \) and \( w_0[A,M,\Delta,p] \) generated by infinite matrices defined by Orlicz functions. Some properties of these spaces are discussed. Also we introduce the concept of \( S[A,\Delta] \)–statistical convergence and derive some results between the spaces \( S[A,\Delta] \) and \( A[M,\Delta,p] \). Further, we study some geometrical properties such as order continuous, the Fatou property and the Banach-Saks property of the new space \( A[M,\Delta,p] \). Finally, we introduce the notion of \( S[A,\Delta] \)–statistical convergence of order \( \alpha \) of real number sequences and obtain some inclusion relations between the set of \( S[A,\Delta] \)–statistical convergence of order \( \alpha \).

1. Introduction

Let \( p = (p_k) \) be a bounded sequence of positive real numbers. If \( H = \sup_k p_k < \infty \), then for any complex numbers \( a_k \) and \( b_k \)

\[
|a_k + b_k|^p \leq C (|a_k|^p + |b_k|^p)
\]

where \( C = \max(1,2^{H-1}) \). Also, for any complex number \( \alpha \), (see [18])

\[
|\alpha|^p \leq \max(1,|\alpha|^H).
\]

We denote \( w, \ell_\infty, c \) and \( c_0 \), for the spaces of all, bounded, convergent, null sequences, respectively. Also, by \( \ell_1 \) and \( \ell_p \), we denote the spaces of all absolutely summable and \( p \)-absolutely summable series, respectively. Recall that a sequence \( \{x(i)\}_{i=1}^\infty \) in a Banach space \( X \) is called Schauder (or basis) of \( X \) if for each \( x \in X \) there exists a unique sequence \( \{a(i)\}_{i=1}^\infty \) of scalars such that \( x = \sum_{i=1}^\infty a(i)x(i) \), i.e. \( \lim_{n \to \infty} \sum_{i=1}^n a(i)x(i) = x \). A sequence space \( X \) with a linear topology is called a \( K \)-space if each of the projection maps \( P_i : X \to C \) defined by \( P_i(x) = x(i) \) for \( x = (x(i))_{i=1}^\infty \in X \) is continuous for each natural \( i \). A Fréchet space is a complete metric linear space and the metric is generated by a \( F \)-norm and a Fréchet space which is a \( K \)-space is called an \( FK \)-space i.e. a \( K \)-space \( X \) is called an \( FK \)-space if \( X \) is a complete linear metric space. In other words, \( X \) is an \( FK \)-space if \( X \) is a Fréchet space with continuous coordinatewise projections. All the sequence spaces mentioned above

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are FK-space except the space $c_{00}$. An FK-space $X$ which contains the space $c_{00}$ is said to have the property
AK if for every sequence $(x(i))_{i=1}^{\infty} \in X, x = \sum_{i=1}^{\infty} x(i)e(i)$ where $e(i) = (0,0,...1^{|i|_{place}},0,0,...)$.

A Banach space $X$ is said to be a Köthe sequence space if $X$ is a subspace of $w$ such that

(a) if $x \in w, y \in X$ and $|x(i)| \leq |y(i)|$ for all $i \in \mathbb{N}$, then $x \in X$ and $||x|| \leq ||y||$
(b) there exists an element $x \in X$ such that $x(i) > 0$ for all $i \in \mathbb{N}$.

We say that $x \in X$ is order continuous if for any sequence $(x_n) \in X$ such that $x_n(i) \leq |x(i)|$ for all $i \in \mathbb{N}$ and $x_n(i) \to 0$ as $n \to \infty$ we have $||x_n|| \to 0$ as $n \to \infty$ holds.

A Köthe sequence space $X$ is said to be order continuous if all sequences in $X$ are order continuous. It is easy to see that $x \in X$ order continuous if and only if $||0,0,...,0,x(n+1),x(n+2),...|| \to 0$ as $n \to \infty$.

A Köthe sequence space $X$ is said to have the Fatou property if for any real sequence $x$ and $(x_n)$ in $X$ such that $x_n \uparrow x$ coordinatewisely and $\sup_n ||x_n|| < \infty$, we have that $x \in X$ and $||x_n|| \to ||x||$ as $n \to \infty$.

A Banach space $X$ is said to have the Banach-Saks property if every bounded sequence $(x_n)$ in $X$ admits a subsequence $(z_n)$ such that the sequence $(t_k(z))$ is convergent in $X$ with respect to the norm, where

$$t_k(z) = \frac{z_1 + z_2 + ... + z_k}{k}$$

for all $k \in \mathbb{N}$.

Some of works on geometric properties of sequence space can be found in [1, 2, 16, 19].

An Orlicz function $M$ is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, convex, nondecreasing function such that $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the modulus function and characterized by Nakano [20], followed by Ruckle [24]. An Orlicz function $M$ is said to satisfy $\Delta_2$-condition for all values $u$, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

**Lemma 1.1.** An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) < \infty, \text{ for some } r > 0 \right\},$$

which is a Banach space normed by

$$||x_k|| = \inf \left\{ r > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) \leq 1 \right\}.$$  

The space $l_M$ is closely related to the space $l_p$, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

In the later stage, different Orlicz sequence spaces were introduced and studied by Esi [3, 4, 6], Esi and Et [5], Güngör and Et [15], Parashar and Choudhary [22], Tripathy and Mahanta [26], Tripathy and Hazarika [27], and many others.
2. Classes of Lacunary Orlicz Difference Sequences

The strongly almost summable sequence spaces were introduced and studied by Maddox [18], Nanda [21], Gungor et al., [12], Esi [7], Gungor and Et [15] and many authors. For matrix maps on sequence spaces we refer to [23] and for difference sequence spaces we refer to [28–31] and references therein.

By lacunary sequence we mean a sequence \( k_0 \) of positive integers satisfying; \( 0 = k_0 \) and \( h_r = k_r - k_{r-1} \rightarrow \infty \) as \( r \rightarrow \infty \). We denote the intervals, which \( \theta \) determines, by \( I_r = (k_{r-1}, k_r] \). Let \( A = (a_{ij}) \) be an infinite matrix of non-negative real numbers with all rows are linearly independent for all \( i, j = 1, 2, 3, ... \) and \( B_{kn}(x) = \sum_{i=1}^{\infty} a_{kn} x_{n+i} \) if the series converges for each \( k \) and \( n \). Now we define the following sequence spaces. Let \( M \) be an Orlicz function, \( p = (p_k) \) be a sequence of positive real numbers and \( \theta = (k_r) \) be a lacunary sequence, and for \( \rho > 0 \) then

\[
\tilde{w}_\theta^\rho [A, M, \Delta, p] = \left\{ x \in \omega : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M\left(\frac{\Delta B_{kn}(x)}{\rho}\right)^{p_k} = 0, \text{ uniformly on } n \right\},
\]

\[
\tilde{w}_\theta [A, M, \Delta, p] = \left\{ x \in \omega : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M\left(\frac{\Delta B_{kn}(x) - L}{\rho}\right)^{p_k} = 0, \text{ for some } L, \text{ uniformly on } n \right\}
\]

and

\[
\tilde{w}_\theta^\infty [A, M, \Delta, p] = \left\{ x \in \omega : \sup_{r} \frac{1}{h_r} \sum_{k \in I_r} M\left(\frac{\Delta B_{kn}(x)}{\rho}\right)^{p_k} < \infty, \text{ uniformly on } n \right\},
\]

where \( \Delta B_{kn}(x) = \sum_{i=1}^{\infty} (a_{ki} - a_{k+1,i}) x_{n+i} \).

**Theorem 2.1.** For any Orlicz function \( M \) and a bounded sequence \( p = (p_k) \) of positive real numbers, \( \tilde{w}_\theta^\rho [A, M, \Delta, p] \), \( \tilde{w}_\theta [A, M, \Delta, p] \) and \( \tilde{w}_\theta^\infty [A, M, \Delta, p] \) are linear spaces over the set of complex field.

**Proof.** We give the proof only for the space \( \tilde{w}_\theta^\rho [A, M, \Delta, p] \) and for other spaces follow by applying similar method. Let \( x = (x_k), y = (y_k) \in \tilde{w}_\theta^\rho [A, M, \Delta, p] \) and \( \alpha, \beta \in \mathbb{C} \). Then there exist \( \rho_1 > 0 \) and \( \rho_2 > 0 \) such that

\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M\left(\frac{\Delta B_{kn}(x)}{\rho_1}\right)^{p_k} = 0
\]

and

\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M\left(\frac{\Delta B_{kn}(y)}{\rho_2}\right)^{p_k} = 0.
\]

Define \( \rho_3 = \max \left\{ 2|\alpha| \rho_1, 2|\beta| \rho_2 \right\} \). Since the operator \( \Delta B_{kn} \) is linear and \( M \) is non-decreasing and convex, we have

\[
\frac{1}{h_r} \sum_{k \in I_r} M\left(\frac{\Delta B_{kn}(x) \alpha x + \beta y)}{\rho_3}\right)^{p_k} \leq \frac{1}{h_r} \sum_{k \in I_r} M\left(\frac{\alpha \Delta B_{kn}(x) + \beta \Delta B_{kn}(y)}{\rho_3}\right)^{p_k}
\]

and

\[
\frac{1}{h_r} \sum_{k \in I_r} M\left(\frac{\Delta B_{kn}(x)}{\rho_3}\right)^{p_k} + M\left(\frac{\beta \Delta B_{kn}(y)}{\rho_3}\right)^{p_k}
\]
\[ \frac{1}{h_{r}} \sum_{k \in I_{r}} \left[ M \left( \frac{\| \Delta B_{kn}(x) \|}{\rho_{1}} \right) + M \left( \frac{\| \Delta B_{kn}(y) \|}{\rho_{2}} \right) \right]^{p_{s}} \leq \frac{1}{h_{r}} \sum_{k \in I_{r}} \left[ M \left( \frac{\| \Delta B_{kn}(x) \|}{\rho_{1}} \right) + M \left( \frac{\| \Delta B_{kn}(y) \|}{\rho_{2}} \right) \right]^{p_{s}} \leq C \frac{1}{h_{r}} \sum_{k \in I_{r}} \left[ M \left( \frac{\| \Delta B_{kn}(x) \|}{\rho_{1}} \right) \right]^{p_{s}} + C \left( \sum_{k \in I_{r}} \left[ M \left( \frac{\| \Delta B_{kn}(y) \|}{\rho_{2}} \right) \right]^{p_{s}} \right) \rightarrow 0 \text{ as } r \rightarrow \infty, \]

where \( C = \max(1, 2^{H_{1}}) \), so \( \alpha x + \beta y \in \overline{w}_{\rho}^{p} \{A, M, \Delta, p\} \), hence it is a linear space. \( \Box \)

**Theorem 2.2.** For any Orlicz function \( M \) and a bounded sequence \( p = (p_{s}) \) of positive real numbers, \( \overline{w}_{\rho}^{p} \{A, M, \Delta, p\} \) is a topological paranormed space, paranormed by

\[ g(x) = \inf \left\{ \rho^{p_{s}} : \left[ \frac{1}{h_{r}} \sum_{k \in I_{r}} \left[ M \left( \frac{\| \Delta B_{kn}(x) \|}{\rho} \right) \right]^{p_{s}} \right]^{\frac{1}{r}} \leq 1, r = 1, 2, 3, ... \right\} \]

where \( T = \max(1, \sup_{k} p_{k} = H) \).

**Proof.** The subadditivity of \( g \) follows from the Theorem 2.1 by taking \( \alpha = \beta = 1 \) and it is clear that \( g(x) = g(-x) \). Since \( M(0) = 0 \), we get \( \inf \left\{ \rho^{p_{s}} \right\} = 0 \) for \( x = 0 \). Suppose that \( \chi_{k} \neq 0 \) for each \( k \in \mathbb{N} \). This implies that \( \Delta B_{kn}(x) \neq 0 \) for each \( k \) and \( n \). Let \( \varepsilon \rightarrow 0 \), then

\[ \frac{\| \Delta B_{kn}(x) \|}{\varepsilon} \rightarrow \infty. \]

It follows that

\[ \left( \frac{1}{h_{r}} \sum_{k \in I_{r}} \left[ M \left( \frac{\| \Delta B_{kn}(x) \|}{\varepsilon} \right) \right]^{p_{s}} \right)^{\frac{1}{r}} \rightarrow \infty \]

which is a contradiction. Now we prove that scalar multiplication is continuous. Let \( \lambda \) be any complex number, by definition

\[ g(\lambda x) = \inf \left\{ \rho^{p_{s}} : \left[ \frac{1}{h_{r}} \sum_{k \in I_{r}} \left[ M \left( \frac{\| \Delta B_{kn}(\lambda x) \|}{\rho} \right) \right]^{p_{s}} \right]^{\frac{1}{r}} \leq 1, r = 1, 2, 3, ... \right\} \]

\[ = \inf \left\{ \rho^{p_{s}} : \left[ \frac{1}{h_{r}} \sum_{k \in I_{r}} \left[ M \left( \frac{\| \Delta B_{kn}(x) \|}{\rho} \right) \right]^{p_{s}} \right]^{\frac{1}{r}} \leq 1, r = 1, 2, 3, ... \right\}. \]

Suppose that \( s = \frac{p_{s}}{n} \), then \( \rho = s|\lambda| \) and since \( |\lambda|^{p_{s}} \leq \max(1, |\lambda|^{H}) \) we have

\[ g(\lambda x) \leq |\lambda|^{p_{s}} \leq \max(1, |\lambda|^{H}) \inf \left\{ s^{p_{s}} : \left[ \frac{1}{h_{r}} \sum_{k \in I_{r}} \left[ M \left( \frac{\| \Delta B_{kn}(x) \|}{s} \right) \right]^{p_{s}} \right]^{\frac{1}{r}} \leq 1, r = 1, 2, 3, ... \right\} \]

which converges to zero as \( x \) converges to zero in \( \overline{w}_{\rho}^{p} \{A, M, \Delta, p\} \). Now suppose that \( \lambda_{i} \rightarrow 0 \) as \( i \rightarrow \infty \) and \( x \) is fixed in \( \overline{w}_{\rho}^{p} \{A, M, \Delta, p\} \). For arbitrary \( \varepsilon > 0 \) and let \( r_{s} \) be a positive integer such that

\[ \frac{1}{h_{r_{s}}} \sum_{k \in I_{r_{s}}} \left[ M \left( \frac{\| \Delta B_{kn}(x) \|}{\rho} \right) \right]^{p_{s}} \leq \left( \frac{\varepsilon}{2} \right)^{T} \]
for some $\rho > 0$ and $r > r_o$. This implies that

$$\left( \frac{1}{h_r} \sum_{k \in I} \left[ M \left( \frac{\left| A \varepsilon B_k \right|}{\rho} \right) \right] \right)^{\frac{1}{p}} < \frac{\varepsilon}{2}$$

for some $\rho > 0$ and $r > r_o$. Let $0 < |\lambda| < 1$. Using the convexity of Orlicz function $M$, for $r = r_o$, we get

$$\frac{1}{h_r} \sum_{k \in I} \left[ M \left( \frac{|\lambda| A \varepsilon B_k (x)}{\rho} \right) \right]^{p} \leq \frac{1}{h_r} \sum_{k \in I} \left[ M \left( \frac{|\lambda| A \varepsilon B_k (x)}{\rho} \right) \right]^{p} < \left( \frac{\varepsilon}{2} \right)^{T}.$$ 

Since $M$ is continuous everywhere in $[0, \infty)$, then we consider the function, for $r \leq r_o$

$$f(t) = \frac{1}{h_r} \sum_{k \in I} \left[ M \left( \frac{t A \varepsilon B_k (x)}{\rho} \right) \right]^{p}.$$ 

Then $f$ is continuous at zero. So there is a $\delta \in (0, 1)$ such that $|f(t)| < \left( \frac{\varepsilon}{2} \right)^{T}$ for $0 < t < \delta$. Let $A$ be such that $|\lambda| < \delta$ for $i > A$ and $r \leq r_o$

$$\left( \frac{1}{h_r} \sum_{k \in I} \left[ M \left( \frac{|\lambda| A \varepsilon B_k (x)}{\rho} \right) \right]^{p} \right)^{\frac{1}{2}} < \frac{\varepsilon}{2}$$

for $i > A$ and all $r$, so that $g(\lambda x) \to 0$ as $\lambda \to 0$. This completes the proof. 

**Theorem 2.3.** Let the sequence $p = (p_k)$ be bounded. Then $\tilde{w}^p_0 [A, M, \Delta, p] \subset \tilde{w}_0 [A, M, \Delta, p] \subset \tilde{w}^{\infty}_0 [A, M, \Delta, p]$.

**Proof.** Let $x = (x_k) \in \tilde{w}^p_0 [A, M, \Delta, p]$. Then we have

$$\frac{1}{h_r} \sum_{k \in I} \left[ M \left( \frac{A \varepsilon B_k (x)}{2^p} \right) \right]^{p} \leq C \sum_{k \in I} \frac{1}{2^p} \left[ M \left( \frac{|A \varepsilon B_k (x)|}{\rho} \right) \right]^{p} + \frac{C}{h_r} \sum_{k \in I} \frac{1}{2^p} \left[ M \left( \frac{|\lambda|}{\rho} \right) \right]^{p} \leq C \sum_{k \in I} \frac{1}{2^p} \left[ M \left( \frac{|A \varepsilon B_k (x)|}{\rho} \right) \right]^{p} + C \max \left( 1, \sup \left[ M \left( \frac{|\lambda|}{\rho} \right) \right] \right),$$

where $H = \sup p_k < \infty$ and $C = \max \left( 1, 2^{q-1} \right)$. Thus we have $x = (x_k) \in \tilde{w}_0 [A, M, \Delta, p]$. The inclusion $\tilde{w}_0 [A, M, \Delta, p] \subset \tilde{w}^{\infty}_0 [A, M, \Delta, p]$ is obvious. 

**Theorem 2.4.** If $0 < p_k < q_k$ and $\left( \frac{p_k}{q_k} \right)$ is bounded, then $\tilde{w}_0 [A, M, \Delta, p] \subset \tilde{w}_0 [A, M, \Delta, q]$.

**Proof.** If we take $M \left( \frac{|A \varepsilon B_k (x)|}{\rho} \right) = w_k$ for all $k \in \mathbb{N}$, then using the same technique employed in the proof of Theorem 2.9 of Güngör et al., [12]. 

**Corollary 2.5.** The following statements are valid.

(i) If $0 < \inf \lambda_k \leq 1$ for all $k \in \mathbb{N}$, then $\tilde{w}_0 [A, M, \Delta, p] \subset \tilde{w}_0 [A, M, \Delta]$.

(ii) If $1 \leq p_k \leq \sup \lambda_k = H < \infty$ for all $k \in \mathbb{N}$, then $\tilde{w}_0 [A, M, \Delta] \subset \tilde{w}_0 [A, M, \Delta, p]$.

The proof of the following result is a routine work, so we omitted.

**Proposition 2.6.** Let $M$ be an Orlicz function satisfies $\Delta_2$-condition. Then $\tilde{w}^p_0 [A, \Delta, p] \subset \tilde{w}^p_0 [A, M, \Delta, p]$, $\tilde{w}_0 [A, \Delta, p] \subset \tilde{w}_0 [A, M, \Delta, p]$ and $\tilde{w}^{\infty}_0 [A, \Delta, p] \subset \tilde{w}^{\infty}_0 [A, M, \Delta, p]$.
3. New Sequence Space of Order $\alpha$

In this section let $\alpha \in (0, 1]$ be any real number, $\theta = (k_r)$ be a lacunary sequence, and $p$ be a positive real number such that $1 < p < \infty$. Now we define the following sequence space.

$$\widetilde{w}_{\theta}^{\alpha} [A, \Delta] (p) = \left\{ x \in w : \sup_r \frac{1}{h_r^p} \sum_{k \in I_r} |\Delta B_{kn} (x)|^p < \infty, \text{ uniformly on } n. \right\}$$

Special cases:

(a) For $p = 1$ we have $\widetilde{w}_{\theta}^{\alpha} [A, \Delta] (p) = \widetilde{z}_{\theta}^{\alpha} [A, \Delta]$.

(b) For $\alpha = 1$ and $p = 1$ we have $\widetilde{w}_{\theta}^{\alpha} [A, \Delta] (p) = \widetilde{z}_{\theta}^{\alpha} [A, \Delta]$.

**Theorem 3.1.** Let $\alpha \in (0, 1]$ and $p$ be a positive real number such that $1 \leq p < \infty$. Then the sequence space $\widetilde{w}_{\theta}^{\alpha} [A, \Delta] (p)$ is a BK-space normed by

$$||x||_a = \sup_r \frac{1}{h_r^p} \left( \sum_{k \in I_r} |\Delta B_{kn} (x)|^p \right)^{\frac{1}{p}}.$$

**Proof.** The proof of the result is straightforward, so omitted. \( \square \)

**Theorem 3.2.** Let $\alpha \in (0, 1]$ and $p$ be a positive real number such that $1 \leq p < \infty$. Then $\widetilde{w}_{\theta}^{\alpha} [A, \Delta] \subseteq \widetilde{w}_{\theta} [A, \Delta] (p)$.

**Proof.** The proof of the result is straightforward, so omitted. \( \square \)

**Theorem 3.3.** Let $\alpha$ and $\beta$ be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $p$ be a positive real number such that $1 \leq p < \infty$. Then $\widetilde{w}_{\theta}^{\alpha} [A, \Delta] (p) \subseteq \widetilde{w}_{\theta}^{\beta} [A, \Delta] (p)$.

**Proof.** The proof of the result is straightforward, so omitted. \( \square \)

**Theorem 3.4.** Let $\alpha$ and $\beta$ be fixed real numbers with $0 < \alpha \leq \beta \leq 1$ and $p$ be a positive real number such that $1 \leq p < \infty$. For any two lacunary sequences $\theta = (h_r)$ and $\phi = (l_r)$ for all $r$, then $\widetilde{w}_{\theta}^{\alpha} [A, \Delta] (p) \subset \widetilde{w}_{\phi}^{\beta} [A, \Delta] (p)$ if and only if $\sup \left( \frac{h_r}{l_r^p} \right) < \infty$.

**Proof.** Let $x = (x_r) \in \widetilde{w}_{\theta}^{\alpha} [A, \Delta] (p)$ and $\sup_{r \geq 1} \left( \frac{h_r}{l_r^p} \right) < \infty$. Then

$$\sup_r \frac{1}{h_r^p} \sum_{k \in I_r} |\Delta B_{kn} (x)|^p < \infty$$

and there exists a positive number $K$ such that $h_r^p \leq K l_r^p$ and so that $\frac{1}{l_r} \leq \frac{K}{h_r^p}$ for all $r$. Therefore, we have

$$\frac{1}{l_r^p} \sum_{k \in I_r} |\Delta B_{kn} (x)|^p \leq \frac{K}{h_r^p} \sum_{k \in I_r} |\Delta B_{kn} (x)|^p.$$ 

Now taking supremum over $r$, we get

$$\sup \frac{1}{l_r^p} \sum_{k \in I_r} |\Delta B_{kn} (x)|^p \leq \sup \frac{K}{h_r^p} \sum_{k \in I_r} |\Delta B_{kn} (x)|^p$$

and hence $x \in \widetilde{w}_{\phi}^{\beta} [A, \Delta] (p)$. 

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Next suppose that $\overline{w}^{\infty}_{0a} [A, \Delta] (p) \subset \overline{w}^{\infty}_{\psi} [A, \Delta] (p)$ and $\sup_r \left( \frac{b_r^p}{r} \right) = \infty$. Then there exists an increasing sequence $(r_i)$ of natural numbers such that $\lim_r \left( \frac{b_r^p}{r} \right) = \infty$. Let $L$ be a positive real number, then there exists $i_0 \in \mathbb{N}$ such that $b_{r_i}^p > L$ for all $r_i \geq i_0$. Then $h_{r_i}^p > L_{r_i}^p$ and so $\frac{1}{r_i} > \frac{b_{r_i}^p}{r_i}$. Therefore we can write

$$\frac{1}{r_i} \sum_{k \in I_r} |AB_k (x)|^p > \frac{L}{h_{r_i}^p} \sum_{k \in I_r} |AB_k (x)|^p$$

for all $r_i \geq i_0$.

Now taking supremum over $r_i \geq i_0$ then we get

$$\sup_{r_i \geq i_0} \frac{1}{r_i} \sum_{k \in I_r} |AB_k (x)|^p > \sup_{r_i \geq i_0} \frac{L}{h_{r_i}^p} \sum_{k \in I_r} |AB_k (x)|^p. \tag{3}$$

Since the relation (3) holds for all $L \in \mathbb{R}^+$ (we may take the number $L$ sufficiently large), we have

$$\sup_{r_i \geq i_0} \frac{1}{r_i} \sum_{k \in I_r} |AB_k (x)|^p = \infty$$

but $x = (x_l) \in \overline{w}^{\infty}_{0a} [A, \Delta, p]$ with

$$\sup_r \left( \frac{b_r^p}{r} \right) < \infty.$$

Therefore $x \notin \overline{w}^{\infty}_{0a} [A, \Delta] (p)$ which contradicts that $\overline{w}^{\infty}_{0a} [A, \Delta] (p) \subset \overline{w}^{\infty}_{\psi} [A, \Delta] (p)$. Hence $\sup_{r \geq 1} \left( \frac{b_r^p}{r} \right) < \infty$. $\square$

**Corollary 3.5.** Let $\alpha$ and $\beta$ be fixed real numbers with $0 < \alpha \leq \beta \leq 1$ and $p$ be a positive real number such that $1 \leq p < \infty$. For any two lacunary sequences $\theta = (h_r)$ and $\phi = (l_r)$ for all $r \geq 1$, then

(a) $\overline{w}^{\infty}_{0a} [A, \Delta] (p) = \overline{w}^{\infty}_{\phi} [A, \Delta] (p)$ if and only if $0 < \inf_r \left( \frac{b_r^p}{r} \right) < \sup_r \left( \frac{b_r^p}{r} \right) < \infty$.

(b) $\overline{w}^{\infty}_{0a} [A, \Delta] (p) = \overline{w}^{\infty}_{\psi} [A, \Delta] (p)$ if and only if $0 < \inf_r \left( \frac{b_r^p}{r} \right) < \sup_r \left( \frac{b_r^p}{r} \right) < \infty$.

(c) $\overline{w}^{\infty}_{0a} [A, \Delta] (p) = \overline{w}^{\infty}_{\psi} [A, \Delta] (p)$ if and only if $0 < \inf_r \left( \frac{b_r^p}{r} \right) < \sup_r \left( \frac{b_r^p}{r} \right) < \infty$.

We state the following results without proof.

**Theorem 3.6.** $\ell_p [A, \Delta] \subset \overline{w}^{\infty}_{0a} [A, \Delta] (p) \subset \ell_\infty [A, \Delta]$.

Proof. The proof of the result is straightforward, so omitted. $\square$

**Theorem 3.7.** If $0 < p < q$, then $\overline{w}^{\infty}_{0a} [A, \Delta] (p) \subset \overline{w}^{\infty}_{0a} [A, \Delta] (q)$.

Proof. The proof of the result is straightforward, so omitted. $\square$

### 4. Some Geometric Properties of the New Space

In this section we study some of the geometric properties like order continuous, the Fatou property and the Banach-Saks property of type $p$ in this new sequence space.

**Theorem 4.1.** The space $\overline{w}^{\infty}_{0a} [A, \Delta] (p)$ is order continuous.
Theorem 4.2. The space \( \tilde{w}_0^{\alpha} [A, \Delta] (p) \) has the Fatou property.

Proof. Let \( x \) be a real sequence and \( (x_i) \) be any nondecreasing sequence of non-negative elements form \( \tilde{w}_0^{\alpha} [A, \Delta] (p) \) such that \( x_j(i) \to x(i) \) as \( j \to \infty \) coordinatewisely and \( \sup_j \|x_j\|_\alpha < \infty \).

Let us denote \( T = \sup_j \|x_j\|_\alpha \). Since the supremum is homogeneous, then we have

\[
\frac{1}{T} \sup_r \left( \frac{1}{r^\alpha} \left( \sum_{k \in \mathbb{N}} |\Delta B_{k\alpha} (x_j(i))|^p \right)^{\frac{1}{p}} \right) \leq \frac{1}{r^\alpha} \left( \sum_{k \in \mathbb{N}} \frac{|\Delta B_{k\alpha} (x_j(i))|^p}{\|x_j\|_\alpha} \right)^{\frac{1}{p}} = \frac{1}{\|x_j\|_\alpha} = 1.
\]

Also by the assumptions that \( (x_j) \) is non-decreasing and convergent to \( x \) coordinatewisely and by the Beppo-Levi theorem, we have

\[
\frac{1}{T} \lim_{j \to \infty} \sup_r \left( \frac{1}{r^\alpha} \left( \sum_{k \in \mathbb{N}} |\Delta B_{k\alpha} (x_j(i))|^p \right)^{\frac{1}{p}} \right) \leq \frac{1}{r^\alpha} \left( \sum_{k \in \mathbb{N}} \frac{|\Delta B_{k\alpha} (x(i))|^p}{T} \right)^{\frac{1}{p}} \leq 1,
\]

whence

\[
\|x\|_\alpha \leq T = \sup_j \|x_j\|_\alpha = \lim_{j \to \infty} \|x_j\|_\alpha < \infty.
\]

Therefore \( x \in \tilde{w}_0^{\alpha} [A, \Delta] (p) \). On the other hand, since \( 0 \leq x_j \) for any natural number \( j \) and the sequence \( (x_j) \) is non-decreasing, we obtain that the sequence \( \|x_j\|_\alpha \) is bounded form above by \( \|x\|_\alpha \). Therefore \( \lim_{j \to \infty} \|x_j\|_\alpha \leq \|x\|_\alpha \) which contradicts the above inequality proved already, yields that \( \|x\|_\alpha = \lim_{j \to \infty} \|x_j\|_\alpha \). □

Theorem 4.3. The space \( \tilde{w}_0^{\alpha} [A, \Delta] (p) \) has the Banach-Saks property.

Proof. The proof of the result follows from the standard technique. □

5. Lacunary Statistical Convergence

The notion of statistical convergence was introduced by Fast [8] and studied various authors (see [7, 9, 25]). The notion of lacunary statistical convergence was introduced by Fridy and Orhan [10] and has been investigated for the real case in [11]. For more details on lacunary statistical convergence we refer to [13, 14] and many others. In this section, we define the concept of \( \bar{S}_0 [A, \Delta] \)-statistical convergence and establish the relationship of \( \bar{S}_0 [A, \Delta] \) with \( \tilde{w}_0 [A, \Delta] \). Also we introduce the notion of \( \bar{S}_0 [A, \Delta] \)-statistical convergence of order \( \alpha \) of real number sequences and obtain some inclusion relations between the set \( S [A, \Delta] \)-statistical convergence of order \( \alpha \).

Definition 5.1. [8] A sequence \( x = (x_k) \) is said to be statistically convergent to \( L \), if for every \( \varepsilon > 0 \)

\[
\lim_{n} \frac{1}{n} \sum_{k \leq n} |x_k - L| \leq \varepsilon
\]

In this case we write \( S \)-lim \( x = L \) or \( x_k \to L(S) \).
Definition 5.2. [10] Let \( \theta = (k_r) \) be a lacunary sequence. A sequence \( x = (x_k) \) is said to be lacunary statistically convergent or \( S_0 \)-convergent to \( L \), if for every \( \epsilon > 0 \)

\[
\lim_{r \to \infty} \frac{1}{h_r} |\{ k \in I_r : |x_k - L| \geq \epsilon \}| = 0.
\]

In this case we write \( S_0 - \lim x = L \) or \( x_k \to L(S_0) \) and \( S_0 = \{ x \in w : S_0 - \lim x = L \ \text{for some} \ L \} \).

Definition 5.3. Let \( \theta = (k_r) \) be a lacunary sequence. A sequence \( x = (x_k) \) is said to be \( S_\theta [A, \Delta] \)-convergent to \( L \), if for every \( \epsilon > 0 \)

\[
\lim_{r \to \infty} \frac{1}{h_r} |\{ k \in I_r : |\Delta B_{k_n} (x) - L| \geq \epsilon \}| = 0.
\]

In this case we write \( S_\theta [A, \Delta] - \lim x = L \) or \( x_k \to L(S_\theta [A, \Delta]) \).

Theorem 5.4. Let \( \theta = (k_r) \) be a lacunary sequence.

(a) If \( x_k \to L(\widehat{w}_0 [A, \Delta]) \) then \( x_k \to L(S_\theta [A, \Delta]) \),

(b) If \( x \in l_\infty [A, \Delta] \) and \( x_k \to L(S_\theta [A, \Delta]) \), then \( x_k \to L(\widehat{w}_0 [A, \Delta]) \),

(c) \( \widehat{w}_0 [A, \Delta] \cap l_\infty [A, \Delta] = \widehat{S}_\theta [A, \Delta] \cap l_\infty [A, \Delta] \), where

\[
l_\infty [A, \Delta] = \left\{ x \in w : \sup_{k,n} |\Delta B_{k_n} (x)| < \infty \right\}.
\]

Proof. (a) Suppose that \( \epsilon > 0 \) and \( x_k \to L(\widehat{w}_0 [A, \Delta]) \), then we have

\[
\sum_{k \in I_r} |\Delta B_{k_n} (x) - L| \geq \sum_{k \in I_r, \ |\Delta B_{k_n} (x) - L| \geq \epsilon} |\Delta B_{k_n} (x) - L| \geq e |\{ k \in I_r : |\Delta B_{k_n} (x) - L| \geq \epsilon \}|
\]

Therefore \( x_k \to L(S_\theta [A, \Delta]) \).

(b) Suppose that \( x \in l_\infty [A, \Delta] \) and \( x_k \to L(S_\theta [A, \Delta]) \), i.e., for some \( K > 0 \), \( |\Delta B_{k_n} (x) - L| \leq K \) for all \( k \) and \( n \). Given \( \epsilon > 0 \), we get

\[
\frac{1}{h_r} \sum_{k \in I_r, \ |\Delta B_{k_n} (x) - L| \geq \epsilon} |\Delta B_{k_n} (x) - L| + \frac{1}{h_r} \sum_{k \in I_r, \ |\Delta B_{k_n} (x) - L| < \epsilon} |\Delta B_{k_n} (x) - L|
\]

\[
\leq \frac{K}{h_r} |\{ k \in I_r : |\Delta B_{k_n} (x) - L| \geq \epsilon \}| + \epsilon,
\]

as \( r \to \infty \), the right side goes to zero, which implies that \( x_k \to L(\widehat{w}_0 [A, \Delta]) \).

(c) Follows from (a) and (b). \( \square \)

Definition 5.5. Let \( 0 < \alpha \leq 1 \) be given. A sequence \( x = (x_k) \) is said to be almost statistically \( [A, \Delta] \)-convergent of order \( \alpha \) if \( S_\alpha [A, \Delta] \)-convergent of order \( \alpha \) if there is a real number \( L \) such that for every \( \epsilon > 0 \)

\[
\lim_{n \to \infty} \frac{1}{n^\alpha} |\{ k \leq n : |\Delta B_{k_n} (x) - L| \geq \epsilon \}| = 0.
\]

In this case we write \( S_\alpha [A, \Delta] - \lim x = L \) or \( x_k \to L(S_\alpha [A, \Delta]) \).
Definition 5.6. Let \( \theta = (k) \) be a lacunary sequence and \( 0 < \alpha \leq 1 \) be given. A sequence \( x = (x_k) \) is said to be \( S^\alpha_0 [A, \Delta] \)–convergent of order \( \alpha \) if there is a real number \( L \) such that for every \( \varepsilon > 0 \)

\[
\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} |\Delta B_{kn}(x) - L| \geq \varepsilon = 0.
\]

(4)

In this case we write \( S^\alpha_0 [A, \Delta] \)–\( \lim x = L \) or \( x_k \to L(S^\alpha_0 [A, \Delta]) \).

Theorem 5.7. Let \( 0 < \alpha \leq 1 \) and \( x = (x_k) \) and \( (y) = (y_k) \) be sequences of real numbers.

(a) If \( S^\alpha_0 [A, \Delta] \)–\( \lim x_k = x_0 \) and \( c \in \mathbb{C} \), then \( S^\alpha [A, \Delta] \)–\( \lim c(x_k) = cx_0 \);

(b) If \( S^\alpha_0 [A, \Delta] \)–\( \lim x_k = x_0 \) and \( S^\alpha [A, \Delta] \)–\( \lim y_k = y_0 \), then \( S^\alpha [A, \Delta] \)–\( \lim (x_k + y_k) = x_0 + y_0 \).

Proof. (a) For \( \varepsilon = 0 \), the result is trivial. Suppose that \( \varepsilon \neq 0 \), then for every \( \varepsilon > 0 \) the result follows from the following inequality

\[
\frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(cx) - cx_0| \geq \varepsilon\| = \frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(x) - x_0| \geq \varepsilon|c|\|
\]

(b) For every \( \varepsilon > 0 \). The result follows from the from the following inequality.

\[
\frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(x + y) - (x_0 + y_0)| \geq \varepsilon\| = \frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(x) - x_0| \geq \varepsilon\| + \frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(y) - y_0| \geq \varepsilon\|
\]

\( \square \)

Theorem 5.8. Let \( 0 < \alpha \leq 1 \) and \( x = (x_k) \) and \( (y) = (y_k) \) be sequences of real numbers.

(a) If \( S^\alpha_0 [A, \Delta] \)–\( \lim x_k = x_0 \) and \( c \in \mathbb{C} \), then \( S^\alpha [A, \Delta] \)–\( \lim c(x_k) = cx_0 \);

(b) If \( S^\alpha_0 [A, \Delta] \)–\( \lim x_k = x_0 \) and \( S^\alpha [A, \Delta] \)–\( \lim y_k = y_0 \), then \( S^\alpha [A, \Delta] \)–\( \lim (x_k + y_k) = x_0 + y_0 \).

Proof. (a) For \( \varepsilon = 0 \), the result is trivial. Suppose that \( \varepsilon \neq 0 \), then for every \( \varepsilon > 0 \) the result follows from the following inequality

\[
\frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(cx) - cx_0| \geq \varepsilon\| = \frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(x) - x_0| \geq \varepsilon|c|\|
\]

(b) For every \( \varepsilon > 0 \). The result follows from the from the following inequality.

\[
\frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(x + y) - (x_0 + y_0)| \geq \varepsilon\| = \frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(x) - x_0| \geq \varepsilon\| + \frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(y) - y_0| \geq \varepsilon\|
\]

\( \square \)

Theorem 5.9. If \( 0 < \alpha < \beta \leq 1 \), then \( S^\alpha_0 [A, \Delta] \subset S^\beta_0 [A, \Delta] \) and the inclusion is strict.

Proof. The proof of the result follows from the following inequality.

\[
\frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(x) - L| \geq \varepsilon\| = \frac{1}{h_r^\alpha} \|k \leq n : |\Delta B_{kn}(x) - L| \geq \varepsilon|c|\|
\]

To prove the inclusion is strict, let \( \theta \) be given and we consider a sequence \( x = (x_k) \) be defined by

\[
\Delta B_{kn}(x_k) = \begin{cases} \lfloor \sqrt[n]{k} \rfloor, & \text{if } k = 1, 2, 3, ... \lfloor \sqrt[n]{r} \rfloor; \\
0, & \text{otherwise.}
\end{cases}
\]

Then we have \( x \in S^\alpha_0 [A, \Delta] \) for \( \frac{1}{2} < \beta \leq 1 \) but \( x \not\in S^\alpha_0 [A, \Delta] \) for \( 0 < \alpha \leq \frac{1}{2} \). \( \square \)
Corollary 5.10. If a sequence is $\tilde{S}_0^\alpha [A, \Delta]$-convergent to L then it is $\tilde{S}_0 [A, \Delta]$-convergent to L.

Theorem 5.11. Let $0 < \alpha \leq 1$ and $\theta = (k_r)$ be a lacunary sequence. If $\lim \inf, q_r > 1$, then $\tilde{S}^\alpha [A, \Delta] \subset \tilde{S}_0^\alpha [A, \Delta]$.

Proof. Suppose that $\lim \inf, q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large $r$ which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1 + \delta} \Rightarrow \left(\frac{h_r}{k_r}\right)^\alpha \geq \left(\frac{\delta}{1 + \delta}\right)^\alpha \Rightarrow \frac{1}{k_r^\alpha} \geq \frac{\delta^\alpha}{(1 + \delta)^\alpha} \frac{1}{h_r^\alpha}.$$ 

If $x_k \to L(\tilde{S}^\alpha [A, \Delta])$ then for every $\varepsilon > 0$ and for sufficiently large $r$ we have

$$\frac{1}{k_r^\alpha} ||k \leq k_r : |\Delta B_{kn} (x) - L| \geq \varepsilon|| \geq \frac{1}{k_r^\alpha} ||k \in I_r : |\Delta B_{kn} (x) - L| \geq \varepsilon|| \geq \delta^\alpha \frac{1}{(1 + \delta)^\alpha} \frac{1}{h_r^\alpha} ||k \in I_r : |\Delta B_{kn} (x) - L| \geq \varepsilon||.$$

This complete the proof of the theorem. \qed

Theorem 5.12. Let $0 < \alpha \leq 1$ and $\theta = (k_r)$ be a lacunary sequence. If $\lim \sup, q_r < \infty$, then $\tilde{S}^\alpha [A, \Delta] \subset \tilde{S} [A, \Delta]$.

Proof. If $\lim \sup, q_r < \infty$, then there exists an $K > 0$ such that $q_r < K$ for all $r$. Suppose that $x_k \to L(\tilde{S}^\alpha [A, \Delta])$ and let $M_r = ||k \in I_r : |\Delta B_{kn} (x) - L| \geq \varepsilon||$. Then form relation (4) for given $\varepsilon > 0$ there is an $r_0 \in \mathbb{N}$ such that for $0 < \alpha \leq 1$

$$\frac{M_r}{h_r^\alpha} < \varepsilon \Rightarrow \frac{M_r}{h_r^\alpha} < \varepsilon \text{ for all } r > r_0.$$ 

The rest of the proof of the theorem follows by using the similar technique of Lemma 3 [10]. \qed

Theorem 5.13. If

$$\lim \inf_{r \to \infty} \frac{h_r^\alpha}{k_r} = 0,$$

then $\tilde{S} [A, \Delta] \subset \tilde{S}^\alpha [A, \Delta]$.

Proof. For a given $\varepsilon > 0$ we have

$$\{k \leq k_r : |\Delta B_{kn} (x) - L| \geq \varepsilon\} \supset \{k \leq k_r : |\Delta B_{kn} (x) - L| \geq \varepsilon\}.$$

Then we have

$$\frac{1}{k_r^\alpha} ||k \leq k_r : |\Delta B_{kn} (x) - L| \geq \varepsilon|| \geq \frac{1}{k_r^\alpha} ||k \in I_r : |\Delta B_{kn} (x) - L| \geq \varepsilon|| = \frac{h_r^\alpha}{k_r^\alpha} \frac{1}{h_r^\alpha} ||k \in I_r : |\Delta B_{kn} (x) - L| \geq \varepsilon||.$$

By taking limit as $r \to \infty$ and from relation (5) we have

$$x_k \to L(\tilde{S} [A, \Delta]) \Rightarrow x_k \to L(\tilde{S}^\alpha [A, \Delta]).$$

\qed
Definition 5.14. Let $M$ be an Orlicz function, $p = (p_k)$ be a sequence of strictly positive real numbers, $\alpha \in (0, 1]$, $\theta = (\kappa)$ be a lacunary sequence, and for $p > 0$, now we define

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p = 0, \text{ for some } L, \text{ uniformly on } n.$$  

If $M(x) = x$ and $p_k = p$ for all $k \in \mathbb{N}$ then we shall write $\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p = 0, \text{ for some } L, \text{ uniformly on } n.$

Theorem 5.15. Let $(p_k)$ be a bounded and $0 < \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Let $\alpha, \beta \in (0, 1]$ be real numbers such that $\alpha \leq \beta$, $M$ be an Orlicz function and $\theta = (\kappa)$ be a lacunary sequence, then $\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p = 0, \text{ for some } L, \text{ uniformly on } n.$

Proof. Let $x = (x_k) \in \lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p = 0, \text{ for some } L, \text{ uniformly on } n.$

\[
\frac{1}{h_r^\beta} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p \leq \frac{1}{h_r^\beta} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p \leq \frac{1}{h_r^\beta} \sum_{k \in L_r} \min \left( \left( M(\epsilon_1) \right)^p, \left( M(\epsilon_1) \right)^p \right), \quad \epsilon_1 = \frac{\epsilon}{\rho}.
\]

From the above inequality we have $(x_k) \in \lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p = 0, \text{ for some } L, \text{ uniformly on } n.$

Corollary 5.16. Let $0 < \alpha \leq 1, M$ be an Orlicz function and $\theta = (\kappa)$ be a lacunary sequence, then $\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p = 0, \text{ for some } L, \text{ uniformly on } n.$

Theorem 5.17. Let $M$ be an Orlicz function, $x = (x_k)$ be a sequence in $l_\infty [A, \Delta]$, and $\theta = (\kappa)$ be a lacunary sequence. If $\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p = 0, \text{ for some } L, \text{ uniformly on } n.$

Proof. Suppose that $x = (x_k)$ is a in $l_\infty [A, \Delta]$ and $\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p = 0, \text{ for some } L, \text{ uniformly on } n.$ There exists $K > 0$ such that $\|A_{B_k} \| \leq K$ for all $k \in \mathbb{N}$ and $n$. For given $\epsilon > 0$ we have

\[
\frac{1}{h_r^\beta} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p \leq \frac{1}{h_r^\beta} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p \leq \max \left\{ \left( M \left( \frac{\epsilon_1}{\rho} \right) \right)^p, \left( M \left( \frac{\epsilon_1}{\rho} \right) \right)^p \right\}.
\]

Therefore we have $(x_k) \in \lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in L_r} \left( M \left( \frac{\|A_{B_k} \| - L}{\rho} \right) \right)^p = 0, \text{ for some } L, \text{ uniformly on } n.$
Theorem 5.18. Let $M$ be an Orlicz function and if $\inf_k p_k > 0$, then limit of any sequence $x = (x_k)$ in $\mathcal{w}_M [A, M, \Delta, p]$ is unique.

Proof. Let $\lim_k p_k = s > 0$. Suppose that $(x_k) \to I_1 \left(\mathcal{w}_M [A, M, \Delta, p]\right)$ and $(x_k) \to I_2 \left(\mathcal{w}_M [A, M, \Delta, p]\right)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\lim_{r \to \infty} \frac{1}{H_r^p} \sum_{k \in I_r} \left[ M \left( \frac{|\Delta B_{\rho_k}(x) - l_1|}{\rho} \right) \right]^{p_k} = 0, \text{ uniformly on } n$$

and

$$\lim_{r \to \infty} \frac{1}{H_r^p} \sum_{k \in I_r} \left[ M \left( \frac{|\Delta B_{\rho_k}(x) - l_2|}{\rho} \right) \right]^{p_k} = 0, \text{ uniformly on } n.$$

Let $\rho = \max\{2\rho_1, 2\rho_2\}$. As $M$ is nondecreasing and convex, we have

$$\frac{1}{H_r^p} \sum_{k \in I_r} \left[ M \left( \frac{|l_1 - l_2|}{\rho} \right) \right]^{p_k} \leq \frac{D}{H_r^p} \sum_{k \in I_r} \frac{1}{2^{p_k}} \left( \left[ M \left( \frac{|\Delta B_{\rho_k}(x) - l_1|}{\rho} \right) \right]^{p_k} + \left[ M \left( \frac{|\Delta B_{\rho_k}(x) - l_2|}{\rho} \right) \right]^{p_k} \right)$$

$$\frac{D}{H_r^p} \sum_{k \in I_r} \left[ M \left( \frac{|\Delta B_{\rho_k}(x) - l_1|}{\rho} \right) \right]^{p_k} + \frac{D}{H_r^p} \sum_{k \in I_r} \left[ M \left( \frac{|\Delta B_{\rho_k}(x) - l_2|}{\rho} \right) \right]^{p_k} \to 0 \text{ as } r \to \infty,$$

where $\sup_k p_k = H$ and $D = \max(1, 2^{l-1})$. Therefore we get

$$\lim_{r \to \infty} \frac{1}{H_r^p} \sum_{k \in I_r} \left[ M \left( \frac{|l_1 - l_2|}{\rho} \right) \right]^{p_k} = 0.$$

As $\lim_k p_k = s$, we have

$$\lim_{k \to \infty} \left[ M \left( \frac{|l_1 - l_2|}{\rho} \right) \right]^{p_k} = \left[ M \left( \frac{|l_1 - l_2|}{\rho} \right) \right]^{s}$$

and so $l_1 = l_2$. Hence the limit is unique. □

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