Filomat 29:10 (2015), 2185–2197
DOI 10.2298/FIL1510185S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# One Invariant of Intrinsic Shape

**Nikita Shekutkovski[a], Aneta Velkoska[b]**

[a]*Institute of Mathematics, Faculty of Natural Sciences and Mathematics, University of St. Cyril and Methodius, Skopje, Republic of Macedonia*
[b]*University of Information Science and Technology St.Paul the Apostle, Faculty of Communication Networks and Security, Ohrid, Republic of Macedonia*

**Abstract.** Based on the intrinsic definition of shape by functions continuous over a covering and corresponding homotopy we will define proximate fundamental group. We prove that proximate fundamental group is an invariant of pointed intrinsic shape of a space.

## 1. Introduction

The notion of shape was introduced by K. Borsuk in 1968 as a more appropriate tool than homotopy, for study of spaces with a complicated local structure. In the past fifty years thousands of papers are published concerning shape theory. One of the most important invariants of (pointed) shape are shape groups. Main references about shape are the books of Borsuk [1] and of Mardešić and Segal [5]. The approaches in both books are using external elements for describing shape of a space: neighborhoods in some external space where the original space is embedded, or an inverse sequence (system) of ANRs or polyhedra.

From the early beginning of shape theory a question was raised regarding the intrinsic description of shape of a space,i.e., construction without using external spaces.

In Felt [3] is described intrinsically a shape morphism between two compact metric spaces. In the same paper is proved indirectly that this notion is the same with the original definition of [1]. The description of [1] uses external spaces, namely embedding of compact metric space in Hilbert cube and considering a sequence of continuous maps – fundamental sequence, between neighborhoods of the embedded metric compacta.

In order to achieve an intrinsic definition, in [3] are considered nets of functions $(f_{\mathcal{V}})$ indexed by coverings, each function $f_{\mathcal{V}}$ being continuous over a covering $\mathcal{V}$. However, the composition is not defined and thus it is not formed category.

Using a slightly different approach, with $\varepsilon$ - continuous functions, in Sanjurjo [6] is formed the category by intrinsic approach.

In Shekutkovski et al. [7], using the fact that in compact metric space there exists a cofinal sequence of finite coverings $\mathcal{V}_1 > \mathcal{V}_2 > \ldots$ i.e., for every covering $\mathcal{V}$ there exists $\mathcal{V}_n$ such that $\mathcal{V}_n < \mathcal{V}$, the intrinsic shape is described by sequence of $\mathcal{V}_n$ - continuous functions $(f_n)$. This approach enables easy definition

2010 *Mathematics Subject Classification*. Primary 55P55 (mandatory); Secondary 54C56 (optionally)
*Keywords*. (pointed homotopy over a covering, pointed proximate nets, intrinsic shape, path over a covering, proximate path, proximate loop, proximate fundamental group.)
Received: 31 August 2014; Accepted: 13 July 2015
Communicated by Miroslav Ćirić
*Email addresses:* nikita@pmf.ukim.mk (Nikita Shekutkovski), aneta.velkoska@uist.edu.mk (Aneta Velkoska)

of composition of shape morphisms and shape category, and for the first time intrinsic definition of strong shape.

In the same paper is proved that definition of shape morphism coincides with definition of [3]. In Shekutkovski et al. [12] and [13] is proved that categories of Sanjurjo and Shekutkovski coincide, and that are the same with original Borsuk category for compact metric spaces.

For noncompact spaces, we cannot work with sequences. Instead, nets of functions $(f_{\mathcal{V}})$ are used which are indexed by coverings from the set of coverings $CovX$.

A generalization for noncompact spaces is given in Kieboom [4] with actually the same approach as presented in this article, and it is shown that for paracompact spaces the obtained intrinsic shape coincides with the notion of [5]. There, shape of a space is obtained by external approach with an inverse system approximating original space. It is known that this approach and original Borsuk approach give the same result for metric compacta.

In this paper we form the pointed intrinsic shape category of paracompact topological spaces based on nets of functions indexed by all coverings. This category is playing the role of pointed homotopy category, and we construct the first invariant of this category called proximate fundamental group.

## 2. Pointed homotopy over a covering

First we present some notions about collections of subsets from a fixed set. Let $\mathcal{U}$ and $\mathcal{V}$ are some collections of subsets of the topological space $X$, $\mathcal{U} \prec \mathcal{V}$ means that $\mathcal{U}$ refines $\mathcal{V}$, i.e., for any set $U \in \mathcal{U}$ there exists a set $V \in \mathcal{V}$ such that $U \subset V$.

If $U \in \mathcal{U}$, then the star of $U$ is the set $st(U, \mathcal{U}) = \cup\{x \in W \mid \forall W \in \mathcal{U}, W \cap U \neq \emptyset\}$.

By $st(\mathcal{U})$ is denoted the collection of all $st(U, \mathcal{U})$, $U \in \mathcal{U}$, i.e., $st(\mathcal{U}) = \{st(U, \mathcal{U}) \mid U \in \mathcal{U}\}$.

By a covering we understand an open covering, and the set of all coverings we denote by $CovX$.

Let consider two paracompact topological spaces $X$ and $Y$. First we recall the definition of $\mathcal{V}$ - continuous function in [7] and [9].

**Definition 2.1.** *Let $\mathcal{V}$ is a covering of $Y$. A function $f : X \to Y$ is $\mathcal{V}$ - continuous at the point $x \in X$ if there exists a neighborhood $U_x$ of $x$ and $V \in \mathcal{V}$ such that $f(U_x) \subseteq V$.*

*A function $f : X \to Y$ is $\mathcal{V}$ - continuous on $X$ if it is $\mathcal{V}$ - continuous at every point $x \in X$. In this case, the family of all neighborhoods $U_x$ form a covering $\mathcal{U}$ of $X$. By this, the function $f : X \to Y$ is $\mathcal{V}$ - continuous on $X$ if there exists a covering $\mathcal{U}$ of $X$, such that for any $x \in X$ there exists a neighborhood $U \in \mathcal{U}$ of $x$, and $V \in \mathcal{V}$ such that $f(U) \subseteq V$. We denote: there exists a covering $\mathcal{U}$ such that $f(\mathcal{U}) \prec \mathcal{V}$.*

**Remark 2.1.** *When $X$ and $Y$ are paracompact, it is enough to take $\mathcal{U}$ and $\mathcal{V}$ to be locally finite coverings, since locally finite coverings are cofinal in the set of all coverings.*

Now, we define the pointed $\mathcal{V}$ - homotopy.

**Definition 2.2.** *Let $f, g : (X, x_0) \to (Y, y_0)$ are $\mathcal{V}$ - continuous functions and $f(x_0) = g(x_0) = y_0$. We say that $f$ and $g$ are* ***pointed $\mathcal{V}$ - homotopic functions*** *if there exists a function $F : (X \times I, x_0 \times I) \to (Y, y_0)$ such that:*

(1) *$F$ is $st(\mathcal{V})$ - continuous, which is $\mathcal{V}$ - continuous on $X \times \partial I$, $\partial I = \{0, 1\}$;*
(2) *$F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all points $x \in X$;*
(3) *$F(x_0, s) = f(x_0) = g(x_0) = y_0$ for all points $s \in I$.*

*When two $\mathcal{V}$ - continuous functions $f$ and $g$ are pointed $\mathcal{V}$ - homotopic we denote as $f \underset{\mathcal{V}}{\sim} g\,(rel\,\{x_0\})$.*

**Proposition 2.1.** *The relation of pointed $\mathcal{V}$ - homotopy $f \underset{\mathcal{V}}{\sim} g\,(rel\,\{x_0\})$ of $\mathcal{V}$ - continuous functions is an equivalence relation.*

*Proof.* The proof is the same as the proof of the Proposition 2.4 in [7] about unpointed homotopy. □

**Remark 2.2.** *The definition of $\mathcal{V}$ - homotopy between two functions $f, g : X \to Y$ in [4] (Definition 1.4, p. 703) requires to exist only $\mathcal{V}$ - continuous function $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all points $x \in X$.*

*However, this is not an equivalence relation, since the usual concatenation of homotopies given by the formula in the proof of Proposition 2.4, of [7] is not always a $\mathcal{V}$ - continuous function!*

**Proposition 2.2.** *Let $X, Y, Z$ are topological spaces, $x_0 \in X$, $y_0 \in Y$, $z_0 \in Z$, $g : (Y, y_0) \to (Z, z_0)$ is $\mathcal{W}$ - continuous function and $\mathcal{V}$ is a covering of $Y$, such that $g(\mathcal{V}) \prec \mathcal{W}$. If two $\mathcal{V}$ - continuous functions $f_1, f_2 : (X, x_0) \to (Y, y_0)$ are pointed $\mathcal{V}$ - homotopic functions, i.e. $f_1 \underset{\mathcal{V}}{\sim} f_2\,(rel\,\{x_0\})$, then $g \circ f_1 \underset{\mathcal{W}}{\sim} g \circ f_2\,(rel\,\{x_0\})$.*

*Proof.* By the conditions of the proposition, it follows that the compositions $g \circ f_1$, $g \circ f_2$ are also $\mathcal{W}$ - continuous function.

Since $f_1, f_2 : (X, x_0) \to (Y, y_0)$ are pointed $\mathcal{V}$ - homotopic, then there exists a function $F : (X \times I, x_0 \times I) \to (Y, y_0)$ such that:

(1) $F$ is $st(\mathcal{V})$ - continuous, which is $\mathcal{V}$ - continuous on $X \times \partial I$;
(2) $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$ for all points $x \in X$;
(3) $F(x_0, s) = f_1(x_0) = f_2(x_0) = y_0$ for all points $s \in I$.

Let consider a function $K : (X \times I, x_0 \times I) \to (Z, z_0)$ defined by $K(x, s) = (g \circ F)(x, s)$. Since $g(\mathcal{V}) \prec \mathcal{W}$ implies $g(st(\mathcal{V})) \prec st(\mathcal{W})$. Also, $F$ is $st(\mathcal{V})$ - continuous there exists an open covering $\mathcal{U}$, such that $F(\mathcal{U}) \prec st(\mathcal{V})$. We conclude that $(g \circ F)(\mathcal{U}) = g(F(\mathcal{U})) \prec g(st(\mathcal{V})) \prec st(\mathcal{W})$. Therefore, the function $K$ is $st(\mathcal{W})$ - continuous.

Since $F$ is $\mathcal{V}$ - continuous on $X \times \partial I$, $g(\mathcal{V}) \prec \mathcal{W}$ and $g$ is $\mathcal{W}$ - continuous function then it follows that $K = g \circ F$ is $\mathcal{W}$ - continuous on $X \times \partial I$.

If $x \in X$ is an arbitrary point, then $K(x, 0) = (g \circ F)(x, 0) = g(F(x, 0)) = g(f_1(x)) = (g \circ f_1)(x)$ and
$K(x, 1) = (g \circ F)(x, 1) = g(F(x, 1)) = g(f_2(x)) = (g \circ f_2)(x)$.

Let $s \in I$ is an arbitrary point, then
$K(x_0, s) = (g \circ F)(x_0, s) = g(F(x_0, s)) = g(f_1(x_0)) = (g \circ f_1)(x_0) = z_0 = (g \circ f_2)(x_0)$.
Therefore, we showed that the functions $g \circ f_1$, $g \circ f_2$ are pointed $\mathcal{W}$ - homotopic, i.e., $g \circ f_1 \underset{\mathcal{W}}{\sim} g \circ f_2\,(rel\,\{x_0\})$. □

**Proposition 2.3.** *Let $G : (Y \times I, y_0 \times I) \to (Z, z_0)$ be a $st(\mathcal{W})$ - continuous function and $\mathcal{W}$ - continuous on $Y \times \partial I$. Then there exists a covering $\mathcal{V}$ of $Y$, such that for each $\mathcal{V}$ - continuous function $f : (X, x_0) \to (Y, y_0)$, the function $G(f \times id) : (X \times I, x_0 \times I) \to (Z, z_0)$ is $st(\mathcal{W})$ - continuous, and $\mathcal{W}$ - continuous on $X \times \partial I$.*

*Proof.* The unpointed version of this theorem is proved for compact metric case in [7], Theorem 3.0.5 and in noncompact case the proof actually remains the same. □

## 3. Pointed proximate nets. Pointed intrinsic shape

Let consider two paracompact topological spaces $X$ and $Y$, $x_0 \in X$, $y_0 \in Y$. Now, we will define pointed proximate net from $(X, x_0)$ to $(Y, y_0)$.

**Definition 3.1.** *A pointed proximate net from $(X, x_0)$ to $(Y, y_0)$ is a family $\underline{f} = (f_{\mathcal{V}} \mid \mathcal{V} \in CovY)$ of $\mathcal{V}$ - continuous functions $f_{\mathcal{V}} : (X, x_0) \to (Y, y_0)$, such that $f_{\mathcal{W}} \underset{\mathcal{V}}{\sim} f_{\mathcal{V}}\,(rel\,\{x_0\})$ whenever $\mathcal{W} \prec \mathcal{V}$.*

**Definition 3.2.** *Two pointed proximate nets $\underline{f}$ and $\underline{g}$ from $(X, x_0)$ to $(Y, y_0)$ are pointed homotopic if $f_{\mathcal{V}} \underset{\mathcal{V}}{\sim} g_{\mathcal{V}}\,(rel\,\{x_0\})$ for all coverings $\mathcal{V} \in CovY$. We denote by $\underline{f} \sim \underline{g}\,(rel\,\{x_0\})$.*

**Proposition 3.1.** *The relation of pointed homotopy of pointed proximate nets is an equivalence relation. The pointed homotopy class of proximate net $\underline{f}$ from $(X, x_0)$ to $(Y, y_0)$ we will denote by $\left[\underline{f}\right]_{x_0}$.*

*Proof.* Let $\underline{f} = (f_{\mathcal{V}} \mid \mathcal{V} \in CovY)$ and $\underline{g} = (g_{\mathcal{V}} \mid \mathcal{V} \in CovY)$ be pointed homotopic pointed proximate nets from $(X, x_0)$ to $(Y, y_0)$. Therefore, for all coverings $\mathcal{V} \in CovY$ the $\mathcal{V}$ - continuous functions $f_{\mathcal{V}}$ and $g_{\mathcal{V}}$ are pointed $\mathcal{V}$ - homotopic. For all coverings $\mathcal{V} \in CovY$ by Proposition 2.1 the relation of pointed $\mathcal{V}$ - homotopy $f_{\mathcal{V}} \underset{\mathcal{V}}{\sim} g_{\mathcal{V}}\,(rel\,\{x_0\})$ of $\mathcal{V}$ - continuous functions is an equivalence relation. So, by the definition the relation of pointed homotopy of pointed proximate nets is an equivalence relation. □

Now let introduce a notion of composition of pointed proximate nets $\underline{f} : (X, x_0) \to (Y, y_0)$ and $\underline{g} : (Y, y_0) \to (Z, z_0)$.

Let $\underline{f} = (f_{\mathcal{V}} \mid \mathcal{V} \in CovY)$ and $\underline{g} = (g_{\mathcal{W}} \mid \mathcal{W} \in CovZ)$.

Because $g_{\mathcal{W}}$ is $\mathcal{W}$ - continuous, then by the definition there exists an open covering $\mathcal{V}$ of $Y$ such that $g_{\mathcal{W}}(\mathcal{V}) \prec \mathcal{W}$.

We define $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} : (X, x_0) \to (Z, z_0)$. This function is $\mathcal{W}$ - continuous. Although the definition depends on the choice of $\mathcal{V}$, the next Lemma shows that for two coverings $\mathcal{V}, \mathcal{V}' \in CovY$ such that $g_{\mathcal{W}}(\mathcal{V}), g_{\mathcal{W}}(\mathcal{V}') \prec \mathcal{W}$ is true that $g_{\mathcal{W}} \circ f_{\mathcal{V}} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}'}\,(rel\,\{x_0\})$.

**Lemma 3.1.** *If $\underline{f}$ is pointed proximate net and $\mathcal{V}, \mathcal{V}' \in CovY$ such that $g_{\mathcal{W}}(\mathcal{V}), g_{\mathcal{W}}(\mathcal{V}') \prec \mathcal{W}$, $\mathcal{W} \in CovZ$. Then $g_{\mathcal{W}} \circ f_{\mathcal{V}} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}'}\,(rel\,\{x_0\})$.*

*Proof.* Let $\mathcal{V}'' \in CovY$ be a common refinement of $\mathcal{V}$ and $\mathcal{V}'$, i.e., $\mathcal{V}'' \prec \mathcal{V}, \mathcal{V}'$. Since $\underline{f}$ is pointed proximate net by the definition follows that $f_{\mathcal{V}''} \underset{\mathcal{V}}{\sim} f_{\mathcal{V}}\,(rel\,\{x_0\})$ and $f_{\mathcal{V}''} \underset{\mathcal{V}'}{\sim} f_{\mathcal{V}'}\,(rel\,\{x_0\})$. By Proposition 2.2 it follows that $g_{\mathcal{W}} \circ f_{\mathcal{V}''} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}}\,(rel\,\{x_0\})$ and $g_{\mathcal{W}} \circ f_{\mathcal{V}''} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}'}\,(rel\,\{x_0\})$. From the transitivity of the pointed homotopy we conclude that $g_{\mathcal{W}} \circ f_{\mathcal{V}} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}'}\,(rel\,\{x_0\})$. □

Now, we will show that the function $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} : (X, x_0) \to (Z, z_0)$ from the discussion above generates a pointed proximate net from $(X, x_0)$ to $(Z, z_0)$ $\underline{h} = (h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} \mid \mathcal{W} \in CovZ)$, i.e., we will show that for all $\mathcal{W}' \prec \mathcal{W}$ is true that $h_{\mathcal{W}'} \underset{\mathcal{W}}{\sim} h_{\mathcal{W}}\,(rel\,\{x_0\})$.

Let $\mathcal{W}' \prec \mathcal{W}$ and since $\underline{g}$ is a pointed proximate net then $g_{\mathcal{W}'} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}}\,(rel\,\{y_0\})$ by a pointed homotopy $G$, which is $st(\mathcal{W})$ - continuous function and $\mathcal{W}$ - continuous on $Y \times \partial I$.

By Proposition 2.3 there exists a $\mathcal{V}''$ of $Y$, such that for each $\mathcal{V}''$ - continuous function $f_{\mathcal{V}''} : (X, x_0) \to (Y, y_0)$, the function $G\,(f_{\mathcal{V}''} \times id) : (X \times I, x_0 \times I) \to (Z, z_0)$ is $st(\mathcal{W})$ - continuous on $(X \times I, x_0 \times I)$, and $\mathcal{W}$ - continuous on $X \times \partial I$ .

It follows $g_{\mathcal{W}'} \circ f_{\mathcal{V}''} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}''}\,(rel\,\{x_0\})$.

Now, consider $h_{\mathcal{W}'} = g_{\mathcal{W}'} \circ f_{\mathcal{V}'}$ and $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}}$ for some $\mathcal{V}' \in CovY$, $g_{\mathcal{W}'}(\mathcal{V}') \prec \mathcal{W}'$ and a covering $\mathcal{V} \in CovY$, $g_{\mathcal{W}}(\mathcal{V}) \prec \mathcal{W}$.

By Lemma 3.1, since $g_{\mathcal{W}}(\mathcal{V}), g_{\mathcal{W}}(\mathcal{V}'') \prec \mathcal{W}$ it follows that $g_{\mathcal{W}} \circ f_{\mathcal{V}} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}''}\,(rel\,\{x_0\})$.

Now, consider a covering $\mathcal{V}_1$ of $Y$, such that $\mathcal{V}_1 \prec \mathcal{V}', \mathcal{V}''$. Since $g_{\mathcal{W}'}(\mathcal{V}_1), g_{\mathcal{W}'}(\mathcal{V}') \prec \mathcal{W}'$, by Lemma 3.1, it follows that $g_{\mathcal{W}'} \circ f_{\mathcal{V}_1} \underset{\mathcal{W}'}{\sim} g_{\mathcal{W}'} \circ f_{\mathcal{V}'}\,(rel\,\{x_0\})$.

Because $\mathcal{W}' \prec \mathcal{W}$ then $g_{\mathcal{W}'} \circ f_{\mathcal{V}'} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}'} \circ f_{\mathcal{V}_1}\,(rel\,\{x_0\})$.

By Proposition 2.2 since $\underline{f}$ is a pointed proximate net i.el., $f_{\mathcal{V}_1} \underset{\mathcal{V}''}{\sim} f_{\mathcal{V}''}\,(rel\,\{x_0\})$ and $g_{\mathcal{W}'}(\mathcal{V}'') \prec \mathcal{W}$, then is true that $g_{\mathcal{W}'} \circ f_{\mathcal{V}_1} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}'} \circ f_{\mathcal{V}''}\,(rel\,\{x_0\})$.

Therefore $g_{\mathcal{W}'} \circ f_{\mathcal{V}'} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}'} \circ f_{\mathcal{V}_1}\,(rel\,\{x_0\}) \underset{\mathcal{W}}{\sim} g_{\mathcal{W}'} \circ f_{\mathcal{V}''}\,(rel\,\{x_0\}) \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{V}''}\,(rel\,\{x_0\})$, i.e., we showed that $h_{\mathcal{W}'} \underset{\mathcal{W}}{\sim} h_{\mathcal{W}}\,(rel\,\{x_0\})$.

Now we will give the following definition:

**Definition 3.3.** *Let $[\underline{f}]_{x_0}$ and $[\underline{g}]_{y_0}$ are two pointed homotopy classes of pointed proximate nets. We define a composition of pointed homotopy classes $[\underline{f}]_{x_0}$ and $[\underline{g}]_{y_0}$ by $[\underline{g}]_{y_0} \circ [\underline{f}]_{x_0} = [\underline{g} \circ \underline{f}]_{x_0}$.*

From the discussion above in order to show that this composition is well defined we have only to show that if $\underline{f} \sim \underline{f}^{'}$ $(rel\{x_0\})$ and $\underline{g} \sim \underline{g}^{'}$ $(rel\{x_0\})$ then $\underline{h} \sim \underline{h}^{'}$ $(rel\{x_0\})$, where $\underline{h}$ and $\underline{h}^{'}$ are the compositions of pointed proximate nets $\underline{f}$ and $\underline{g}$, $\underline{f}^{'}$ and $\underline{g}^{'}$, respectively.

Since $\underline{g} \sim \underline{g}^{'}$ $(rel\{y_0\})$ by a homotopy then for every $\mathcal{W} \in CovZ$ is true that $g_{\mathcal{W}} \underset{\mathcal{W}}{\sim} g^{'}_{\mathcal{W}}\,(rel\{y_0\})$ and by Proposition 2.3 there exists a covering $\mathcal{U} \in CovY$, $g_{\mathcal{W}}(\mathcal{U}) < \mathcal{W}$, $g^{'}_{\mathcal{W}}(\mathcal{U}) < \mathcal{W}$ such that for $\mathcal{U}$ - continuous function $f_{\mathcal{U}} : (X, x_0) \to (Y, y_0)$ it is true that $g_{\mathcal{W}} \circ f_{\mathcal{U}} \underset{\mathcal{W}}{\sim} g^{'}_{\mathcal{W}} \circ f_{\mathcal{U}}\,(rel\{x_0\})$.

From the definition of the composition of two pointed proximate nets there exist coverings $\mathcal{V}$ and $\mathcal{V}'$ of $Y$ such $g_{\mathcal{W}}(\mathcal{V}) < \mathcal{W}$ and $g^{'}_{\mathcal{W}}(\mathcal{V}') < \mathcal{W}$ such $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}}$ and $h^{'}_{\mathcal{W}} = g^{'}_{\mathcal{W}} \circ f^{'}_{\mathcal{V}'}$.

Since $\underline{f} \sim \underline{f}^{'}$ $(rel\{x_0\})$ then $f_{\mathcal{U}} \underset{\mathcal{U}}{\sim} f^{'}_{\mathcal{U}}\,(rel\{x_0\})$, so by this fact, Lemma 3.1 and Proposition 2.2 we can conclude that $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} \underset{\mathcal{W}}{\sim} g_{\mathcal{W}} \circ f_{\mathcal{U}}\,(rel\{x_0\}) \underset{\mathcal{W}}{\sim} g^{'}_{\mathcal{W}} \circ f_{\mathcal{U}}\,(rel\{x_0\}) \underset{\mathcal{W}}{\sim} g^{'}_{\mathcal{W}} \circ f^{'}_{\mathcal{V}'}\,(rel\{x_0\}) = h^{'}_{\mathcal{W}}$, i.e., $h_{\mathcal{W}} \underset{\mathcal{W}}{\sim} h^{'}_{\mathcal{W}}\,(rel\{x_0\})$ for all $\mathcal{W} \in CovZ$.

Therefore, $\underline{h} \sim \underline{h}^{'}$ $(rel\{x_0\})$.

By the definition of the composition of pointed proximate nets and $\mathcal{U}$ - continuous function the following Theorem is valid.

**Theorem 3.1.** *Let* $\left[\underline{f}\right]_{x_0} : (X, x_0) \to (Y, y_0)$, $\left[\underline{g}\right]_{y_0} : (Y, y_0) \to (Z, z_0)$ *and* $\left[\underline{h}\right]_{z_0} : (Z, z_0) \to (W, w_0)$ *are three pointed homotopy classes of pointed proximate nets. Then* $\left[\underline{h}\right]_{z_0} \circ \left(\left[\underline{g}\right]_{y_0} \circ \left[\underline{f}\right]_{x_0}\right) = \left(\left[\underline{h}\right]_{z_0} \circ \left[\underline{g}\right]_{y_0}\right) \circ \left[\underline{f}\right]_{x_0}$.

In this way we proved that the topological pointed spaces and pointed homotopy classes of pointed proximate nets form category of pointed intrinsic shape. We say that pointed topological spaces $(X, x_0)$ and $(Y, y_0)$ has same pointed intrinsic shape if they are isomorphic in this category.

## 4. Homotopy of $\mathcal{U}$ - paths

Let $X$ be a topological space and $I = [0, 1]$. Now, we recall some definitions introduced in Shekutkovski et al. [11].

**Definition 4.1.** *Let $\mathcal{U}$ be an open covering of the space $X$ and $x_0, x_1 \in X$ are fixed points. The $st(\mathcal{U})$ - continuous function $k_{\mathcal{U}} : I \to X$ which is $\mathcal{U}$ - continuous on $\partial I = \{0, 1\}$ and $k_{\mathcal{U}}(0) = x_0$,$k_{\mathcal{U}}(1) = x_1$ is called $\mathcal{U}$ - path with endpoints $x_0$ and $x_1$.*

**Definition 4.2.** *Let $\mathcal{U}$ be an open covering of the space $X$ and $k_{\mathcal{U}}, l_{\mathcal{U}} : I \to X$ are $\mathcal{U}$ - paths with endpoints $x_0$ and $x_1$. We say that the $\mathcal{U}$ - paths $k_{\mathcal{U}}$ and $l_{\mathcal{U}}$ are $\mathcal{U}$ - homotopic paths if there exists a function $F : I \times I \to X$ such that:*

(I) *$F$ is $st^2(\mathcal{U})$ - continuous;*
(II) *$F$ is $st(\mathcal{U})$ - continuous on $\partial I^2 = \partial(I \times I)$;*
(III) *$F$ is $\mathcal{U}$ - continuous on $\partial^2 I^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$;*

*and satisfies the usual conditions for homotopy of paths relative endpoints*

(IV) *$F(t, 0) = k_{\mathcal{U}}(t)$ and $F(t, 1) = l_{\mathcal{U}}(t)$ for all points $t \in I$;*
(V) *$F(0, s) = k_{\mathcal{U}}(0) = l_{\mathcal{U}}(0) = x_0$ and $F(1, s) = k_{\mathcal{U}}(1) = l_{\mathcal{U}}(1) = x_1$ for all elements $s \in I$.*

*When two $\mathcal{U}$ - paths $k_{\mathcal{U}}$ and $l_{\mathcal{U}}$ with same endpoints are $\mathcal{U}$ - homotopic we denote as $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}\,(rel\{0, 1\})$, i.e., $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}$.*

**Proposition 4.1.** *The relation of $\mathcal{U}$ - homotopy $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}\,(rel\{0, 1\})$ of $\mathcal{U}$ - paths is an equivalence relation.*

*Proof.* It is enough to prove transitivity of the relation. Let $k_{\mathcal{U}}, l_{\mathcal{U}}, p_{\mathcal{U}} : I \to X$ are $\mathcal{U}$ - paths in $X$ such that $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}\,(rel\,\{0,1\})$ and $l_{\mathcal{U}} \underset{\mathcal{U}}{\sim} p_{\mathcal{U}}\,(rel\,\{0,1\})$. Then there exist $\mathcal{U}$ - homotopies relative endpoints $K : I \times I \to X$ and $L : I \times I \to X$ connecting the $\mathcal{U}$ - paths $k_{\mathcal{U}}$ and $l_{\mathcal{U}}$, $l_{\mathcal{U}}$ and $p_{\mathcal{U}}$, respectively.

We define a function $H : I \times I \to X$ by:

$$H(t,s) = \begin{cases} K(t,2s) = K \circ f(t,s), & 0 \le s \le \frac{1}{2} \\ L(t,2s-1) = L \circ g(t,s), & \frac{1}{2} \le s \le 1 \end{cases},$$

where the continuous functions $f$ and $g$ are defined by:

$f : I \times \left[0, \frac{1}{2}\right] \to I \times I$, $f(t,s) = (t,2s)$ and $g : I \times \left[\frac{1}{2}, 1\right] \to I \times I$, $g(t,s) = (t, 2s-1)$.

By Theorem 2.2 [7], since the compositions $K \circ f$ and $L \circ g$ are $st^2(\mathcal{U})$ - continuous on $I \times \left[0, \frac{1}{2}\right]$ and $I \times \left[\frac{1}{2}, 1\right]$, respectively and $st(\mathcal{U})$ - continuous on $I \times \left\{\frac{1}{2}\right\}$ the function $H$ is $st^2(\mathcal{U})$ - continuous on $I \times I$.

By the definition of the function $H$ and the facts that $K$ and $L$ are $st(\mathcal{U})$ - continuous on $\partial I^2$ it follows that the function $H$ is $st(\mathcal{U})$ - continuous on $\partial I^2$. Also, considering the definition of the function $H$ since $K$ and $L$ are $\mathcal{U}$ - continuous at the points $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ then the function $H$ is also $\mathcal{U}$ - continuous at these points.

Furthermore, $H(t,0) = K(t,0) = k_{\mathcal{U}}(t)$ and $H(t,1) = L(t,1) = p_{\mathcal{U}}(t)$ for all $t \in I$ and

$$H(0,s) = \begin{cases} K(0,2s), & 0 \le s \le \frac{1}{2} \\ L(0,2s-1), & \frac{1}{2} \le s \le 1 \end{cases} = \begin{cases} k_{\mathcal{U}}(0), & 0 \le s \le \frac{1}{2} \\ l_{\mathcal{U}}(0), & \frac{1}{2} \le s \le 1 \end{cases} = x_0,$$

So, $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} p_{\mathcal{U}}\,(rel\,\{0,1\})$, i.e., the relation of $\mathcal{U}$ - homotopy relative endpoints is transitive. □

Let consider an open covering $\mathcal{U}$ of the space $X$, and two $\mathcal{U}$ - paths $k_{\mathcal{U}}, l_{\mathcal{U}} : I \to X$ such that $k_{\mathcal{U}}(1) = l_{\mathcal{U}}(0)$. We define a concatenation by:

$$(k_{\mathcal{U}} * l_{\mathcal{U}})(t) = \begin{cases} k_{\mathcal{U}}(2t), & 0 \le t \le \frac{1}{2} \\ l_{\mathcal{U}}(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

By Theorem 2.2 in [7] the concatenation is well defined and $st(\mathcal{U})$ - continuous function. Also by the definition of $\mathcal{U}$ - paths $k_{\mathcal{U}}, l_{\mathcal{U}} : I \to X$ the concatenation $k_{\mathcal{U}} * l_{\mathcal{U}}$ is $\mathcal{U}$ - continuous on $\partial I = \{0,1\}$. Therefore, $k_{\mathcal{U}} * l_{\mathcal{U}}$ is $\mathcal{U}$ - path.

The proofs of the following two theorems are presented in [11].

**Theorem 4.1.** *Let $k^0_{\mathcal{U}}, k^1_{\mathcal{U}} : I \to X$, $l^0_{\mathcal{U}}, l^1_{\mathcal{U}} : I \to X$ are $\mathcal{U}$ - paths such that $k^0_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k^1_{\mathcal{U}}\,(rel\,\{0,1\})$, $l^0_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l^1_{\mathcal{U}}\,(rel\,\{0,1\})$ and the concatenations $k^0_{\mathcal{U}} * l^0_{\mathcal{U}}$ and $k^1_{\mathcal{U}} * l^1_{\mathcal{U}}$ are defined. Then $k^0_{\mathcal{U}} * l^0_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k^1_{\mathcal{U}} * l^1_{\mathcal{U}}\,(rel\,\{0,1\})$.*

**Theorem 4.2.** *Let $k_{\mathcal{U}}, l_{\mathcal{U}}, p_{\mathcal{U}} : I \to X$ are $\mathcal{U}$ - paths in $X$ and the concatenations $k_{\mathcal{U}} * l_{\mathcal{U}}$ and $l_{\mathcal{U}} * p_{\mathcal{U}}$ are defined, $k_{\mathcal{U}}(1) = l_{\mathcal{U}}(0)$ and $l_{\mathcal{U}}(1) = p_{\mathcal{U}}(0)$. Then $(k_{\mathcal{U}} * l_{\mathcal{U}}) * p_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} * (l_{\mathcal{U}} * p_{\mathcal{U}})\,(rel\,\{0,1\})$.*

Let $X$ be a topologic space and $x_0 \in X$. The constant $\mathcal{U}$ - path $c_{x_0} : I \to X$ is defined by $c_{x_0}(t) = x_0$, for all $t \in I$.

**Definition 4.3.** *Let $X$ be a topologic space and $k_{\mathcal{U}} : I \to X$ is $\mathcal{U}$ - path in $X$. The $\mathcal{U}$ - path in $X$, $k^{-1}_{\mathcal{U}} : I \to X$, defined by $k^{-1}_{\mathcal{U}}(t) = k_{\mathcal{U}}(1-t)$ is called inverse $\mathcal{U}$ - path of the $\mathcal{U}$ - path $k_{\mathcal{U}}$. Notice that $(k^{-1}_{\mathcal{U}})^{-1}_{\mathcal{U}} = k_{\mathcal{U}}$.*

The proofs of the following three theorems follow the line of construction of the standard fundamental group (for example Shekutkovski [10]).

**Theorem 4.3.** *Let $k_{\mathcal{U}} : I \to X$ is $\mathcal{U}$ - path with endpoints $x_0$ and $x_1$. Then*

a) $k_{\mathcal{U}} * c_{x_1} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}\,(rel\,\{1,0\})$

b) $c_{x_0} * k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}\,(rel\,\{1,0\})$.

*Proof.* a) First let represent the square $I \times I$ as union of two closed sets $A_1$ and $A_2$, i.e $I \times I = A_1 \cup A_2$, where $A_1 = \left\{(t,s) \mid s \in I, 0 \le t \le \frac{s+1}{2}\right\}, A_2 = \left\{(t,s) \mid s \in I, \frac{s+1}{2} \le t \le 1\right\}$.

Let consider the following function defined by $a(t,s) = k_{\mathcal{U}} \circ f(t,s)$, where $f(t,s) = \frac{2t}{s+1}$.

Now, we define a function $H : I \times I \to X$ by:

$$H(t,s) = \begin{cases} a(t,s), & (t,s) \in A_1 \\ x_1, & (t,s) \in A_2. \end{cases}$$

The function $f$ defined on $A_1$ is continuous. The $\mathcal{U}$ - path $k_{\mathcal{U}}$ is $st(\mathcal{U})$ - continuous. So the function $a = k_{\mathcal{U}} \circ f$ is $st(\mathcal{U})$ - continuous on $A_1$.

If $(t,s) \in A_1 \cap A_2 = \left\{\left(\frac{s+1}{2}, s\right) \mid s \in I\right\}$, then $a(t,s) = k_{\mathcal{U}}(1) = x_1$.

By Theorem 2.2 [7] since $a$ and constant $\mathcal{U}$ - path $c_{x_1}$ are $st(\mathcal{U})$ - continuous and equal on $A_1 \cap A_2$. The function $H$ is $st^2(\mathcal{U})$ - continuous on $I \times I$.

The $\mathcal{U}$ - path $k_{\mathcal{U}}$ and constant $\mathcal{U}$ - path are $\mathcal{U}$ - continuous on $\partial I = \{0,1\}$. By the definition of the function $a$ and constant $\mathcal{U}$ - path $c_{x_1}$ are $\mathcal{V}$ - continuous functions at the vertices of the sets $A_1$ and $A_2$, respectively.

By the definition of the function $H$ and the fact that $a$ and constant $\mathcal{U}$ - path $c_{x_1}$ are $st(\mathcal{U})$ continuous on $\partial A_1$ and $\partial A_2$, and $\mathcal{U}$ - continuous at the vertices of the sets $A_1$ and $A_2$, it follows that the function $H$ is $st(\mathcal{U})$ - continuous on $\partial I^2$.

Considering the definition of the function $H$ since $a$ and constant $\mathcal{U}$ - path $c_{x_1}$ are $\mathcal{U}$ - continuous at the points $(0,0)$, $(0,1)$ and $(1,0)$, $(1,1)$, respectively, the function $H$ is $\mathcal{U}$ - continuous on

$$\partial^2 I^2 = \{(0,0), (0,1), (1,0), (1,1)\}.$$

If $s = 0$

$$H(t,0) = \begin{cases} k_{\mathcal{U}}(2t), & 0 \le t \le \frac{1}{2} \\ x_1, & \frac{1}{2} \le t \le 1 \end{cases} = (k_{\mathcal{U}} * c_{x_1})(t) \text{ for all } t \in I.$$

If $s = 1$

$$H(t,1) = \begin{cases} k_{\mathcal{U}}(t), & 0 \le t \le 1 \\ x_1, & 1 \le t \le 1 \end{cases} = k_{\mathcal{U}}(t) \text{ for all } t \in I.$$

Let $s \in I$ is an arbitrary point. If $t = 0$ then $H(0,s) = k_{\mathcal{U}}(0) = x_0$. If $t = 1$ then $H(1,s) = x_1$. Therefore, we showed that $k_{\mathcal{U}} * c_{x_1} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}\,(rel\,\{0,1\})$, as required.

b) First let represent the square $I \times I$ as union of two closed sets $B_1$ and $B_2$, i.e., $I \times I = B_1 \cup B_2$, where

$$B_1 = \left\{(t,s) \mid s \in I, 0 \le t \le \frac{1-s}{2}\right\}, B_2 = \left\{(t,s) \mid s \in I, \frac{1-s}{2} \le t \le 1\right\}.$$

Let consider the following function defined by $b(t,s) = k_{\mathcal{U}} \circ g(t,s)$ where $g(t,s) = \dfrac{2t-1+s}{s+1}$.

Now, we define a function $K : I \times I \to X$ by:

$$K(t,s) = \begin{cases} x_1, & (t,s) \in B_1 \\ b(t,s), & (t,s) \in B_2. \end{cases}$$

With similar discussion as in a) can be obtained that the function $K$ is pointed $\mathcal{U}$ - homotopy relative endpoints connecting the $\mathcal{U}$ - paths $c_{x_0} * k_{\mathcal{U}}$ and $k_{\mathcal{U}}$. □

**Theorem 4.4.** *Let $k_{\mathcal{U}}, l_{\mathcal{U}} : I \to X$ are $\mathcal{U}$ - paths in $X$ such that $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}\,(rel\,\{0,1\})$. Then $k_{\mathcal{U}}^{-1} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}^{-1}\,(rel\,\{0,1\})$.*

*Proof.* Because $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}\,(rel\,\{0,1\})$ there exists a function $K : I \times I \to X$ connecting the $\mathcal{U}$ - paths $k_{\mathcal{U}}$ and $l_{\mathcal{U}}$.

Let define a function $H : I \times I \to X$ by: $H(t,s) = K(1-t,s)$.

All conditions (I) - (III) from the Definition 4.2 are valid for the function $H$ by its definition.
Now, if $s = 0$ then $H(t,0) = K(1-t,0) = k_{\mathcal{U}}(1-t) = k_{\mathcal{U}}^{-1}(t)$ for all $t \in I$; If $s = 1$ then $H(t,1) = H(t,1) = K(1-t,1) = l_{\mathcal{U}}(1-t) = l_{\mathcal{U}}^{-1}(t)$ for all $t \in I$.
Let $s \in I$ is an arbitrary point. If $t = 0$ then $H(0,s) = K(1-0,s) = K(1,s) = k_{\mathcal{U}}(1) = k_{\mathcal{U}}^{-1}(0)$. If $t = 1$ then $H(1,s) = K(1-1,s) = K(0,s) = l_{\mathcal{U}}(0) = l_{\mathcal{U}}^{-1}(1)$.

Therefore, we showed that $k_{\mathcal{U}}^{-1} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}}^{-1}\,(rel\,\{0,1\})$ as required. □

**Theorem 4.5.** *Let $k_{\mathcal{U}} : I \to X$ is $\mathcal{U}$ - path in $X$ such that $k_{\mathcal{U}}(0) = x_0$ and $k_{\mathcal{U}}(1) = x_1$. Then is true that $k_{\mathcal{U}} * k_{\mathcal{U}}^{-1} \underset{\mathcal{U}}{\sim} c_{x_0}\,(rel\,\{0,1\})$.*

*Proof.* By the definition of concatenation:

$$\left(k_{\mathcal{U}} * k_{\mathcal{U}}^{-1}\right)(t) = \begin{cases} k_{\mathcal{U}}(2t), & 0 \le t \le \frac{1}{2} \\ k_{\mathcal{U}}^{-1}(2t-1), & 0 \le t \le \frac{1}{2} \end{cases} = \begin{cases} k_{\mathcal{U}}(2t), & 0 \le t \le \frac{1}{2} \\ k_{\mathcal{U}}(2-2t), & 0 \le t \le \frac{1}{2}. \end{cases}$$

Let represent the square $I \times I$ as union of two closed sets $A$ and $B$, i.e $I \times I = A \cup B$, where

$$A = \left\{(t,s) \mid s \in I, 0 \le t \le \frac{1}{2}\right\}, B = \left\{(t,s) \mid s \in I, \frac{1}{2} \le t \le 1\right\}.$$

We consider the following functions defined by:

$a(t,s) = k_{\mathcal{U}} \circ f(t,s)$, where $f(t,s) = 2t(1-s)$ and $b(t,s) = k_{\mathcal{U}} \circ g(t,s)$, where $g(t,s) = (2-2t)(1-s)$.
Now define a function $H : I \times I \to X$ by:

$$H(t,s) = \begin{cases} a(t,s), & (t,s) \in A \\ b(t,s), & (t,s) \in B. \end{cases}$$

We can verify all conditions (I) - (III) from the Definition 4.2 for the function $H$ with similar discussion as the proof of the Theorem 4.3.
Now, if $s = 0$ then

$$H(t,0) = \begin{cases} k_{\mathcal{U}}(2t), & 0 \le t \le \frac{1}{2} \\ k_{\mathcal{U}}(2-2t), & \frac{1}{2} \le t \le 1 \end{cases} = \left(k_{\mathcal{U}} * k_{\mathcal{U}}^{-1}\right)(t) \text{ for all } t \in I.$$

If $s = 1$ then

$$H(t,1) = k_{\mathcal{U}}(0) = x_0 \text{ for all } t \in I.$$

Let $s \in I$ is an arbitrary point. If $t = 0$ then $H(0,s) = k_{\mathcal{U}}(0) = x_0$, and if $t = 1$ then $H(1,s) = k_{\mathcal{U}}(0) = \left(k_{\mathcal{U}} * k_{\mathcal{U}}^{-1}\right)(1)$.

Therefore, we showed that $k_{\mathcal{U}} * k_{\mathcal{U}}^{-1} \underset{\mathcal{U}}{\sim} c_{x_0}\,(rel\,\{0,1\})$, as required. □

## 5. Proximate fundamental group

Proximate fundamental group is definied in [11]. Now, we recall the definition and prove that it is invariant of pointed shape cathegory.

**Definition 5.1.** *Let $\mathcal{U}$ is an open covering of the space $X$ and $x_0 \in X$ is a fixed point. The $\mathcal{U}$ - path $k_{\mathcal{U}} : I \to X$ such that $k_{\mathcal{U}}(0) = k_{\mathcal{U}}(1) = x_0$ is called $\mathcal{U}$ - loop in $x_0$.*

The homotopy class of $\mathcal{U}$ - loops in $x_0$, $k_{\mathcal{U}} : I \to X$ we will denote by $[k_{\mathcal{U}}]_{x_0}$.

**Definition 5.2.** *A proximate loop in $x_0$ (over $CovX$) is a family $\underline{k} = (k_{\mathcal{U}} \mid \mathcal{U} \in CovX)$ such that $k_{\mathcal{V}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} \,(rel\,\{0,1\})$ for all $\mathcal{V} \prec \mathcal{U}$.*

We can denote the proximate loop also by $\underline{k} = (k_{\mathcal{U}})_{\mathcal{U} \in CovX}$.

**Definition 5.3.** *Two proximate loops $\underline{k}$ and $\underline{l}$ in $x_0$ are homotopic over all coverings if $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}} \,(rel\,\{0,1\})$ for all $\mathcal{U} \in CovX$. We denote that by $\underline{k} \sim \underline{l}\,(rel\,\{0,1\})$.*

**Proposition 5.1.** *The relation $\underline{k} \sim \underline{l}\,(rel\,\{0,1\})$ is an equivalence relation. The homotopy class of proximate loop $\underline{k}$ in $x_0$ is denoted by $\left[\underline{k}\right]_{x_0}$.*

*Proof.* Let $\underline{k} = (k_{\mathcal{U}} \mid \mathcal{U} \in CovX)$ and $\underline{l} = (l_{\mathcal{U}} \mid \mathcal{U} \in CovX)$ be two homotopic proximate loops in $x_0$. Therefore, $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}} \,(rel\,\{0,1\})$ for all coverings $\mathcal{U} \in CovX$. For all coverings $\mathcal{U} \in CovX$ by Proposition 4.1 the relation of $\mathcal{U}$ - homotopy relative endpoints $k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}} \,(rel\,\{0,1\})$ of $\mathcal{U}$ - loops is an equivalence relation. So, the relation of homotopy of proximate loops is an equivalence relation. □

We consider the following set:

$$prox\pi_1(X, x_0) = \left\{ \left[\underline{k}\right]_{x_0} \mid \underline{k} \text{ is proximate loop in } x_0 \right\}.$$

In this set we define an operation "$*$" by: $\left[\underline{k}\right]_{x_0} * \left[\underline{l}\right]_{x_0} = \left[\underline{k} * \underline{l}\right]_{x_0}$, where $\underline{k} * \underline{l}$ is defined as: $\underline{k} * \underline{l} = (k_{\mathcal{U}} * l_{\mathcal{U}} \mid \mathcal{U} \in CovX)$.

We will show that this operation is well defined.

First we will find that $\underline{k} * \underline{l}$ is proximate loop in $x_0$. By the definition of the composition of two $\mathcal{U}$ - loops for all $\mathcal{U} \in CovX$ the function $k_{\mathcal{U}} * l_{\mathcal{U}}$ is $\mathcal{U}$ - loop in $x_0$. Now, let consider any $\mathcal{V} \prec \mathcal{U}$. Since $\underline{k}$ and $\underline{l}$ are proximate loops then $k_{\mathcal{V}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} \,(rel\,\{0,1\})$ and $l_{\mathcal{V}} \underset{\mathcal{U}}{\sim} l_{\mathcal{U}} \,(rel\,\{0,1\})$, so by Proposition 1.3 (iii) [4] and Theorem 4.1 is true that $k_{\mathcal{V}} * l_{\mathcal{V}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} * l_{\mathcal{U}} \,(rel\,\{0,1\})$. Therefore, $\underline{k} * \underline{l}$ is proximate loop in $x_0$.

Now, by Theorem 4.1 if $k^0_{\mathcal{U}}, k^1_{\mathcal{U}} : I \to X$, $l^0_{\mathcal{U}}, l^1_{\mathcal{U}} : I \to X$ are $\mathcal{U}$ - loops in $x_0$ such that $k^0_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k^1_{\mathcal{U}} \,(rel\,\{0,1\})$, $l^0_{\mathcal{U}} \underset{\mathcal{U}}{\sim} l^1_{\mathcal{U}} \,(rel\,\{0,1\})$ then is true that $k^0_{\mathcal{U}} * l^0_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k^1_{\mathcal{U}} * l^1_{\mathcal{U}} \,(rel\,\{0,1\})$.

Therefore, the operation "$*$" in the set $prox\pi_1(X, x_0)$ is well defined.

**Theorem 5.1.** *The set $prox\pi_1(X, x_0)$ with the operation "$*$" is group. This group $prox\pi_1(X, x_0)$ is called **proximate fundamental group**.*

*Proof.* **Associativity:** Let $\left[\underline{k}\right]_{x_0}$, $\left[\underline{l}\right]_{x_0}$ and $\left[\underline{p}\right]_{x_0}$ are homotopy class of proximate loops in $x_0$. We should show that:

$$\left(\left[\underline{k}\right]_{x_0} * \left[\underline{l}\right]_{x_0}\right) * \left[\underline{p}\right]_{x_0} = \left[\underline{k}\right]_{x_0} * \left(\left[\underline{l}\right]_{x_0} * \left[\underline{p}\right]_{x_0}\right) \tag{1}$$

For the left side of the equation (1) is true the following identity:

$$\left(\left[\underline{k}\right]_{x_0} * \left[\underline{l}\right]_{x_0}\right) * \left[\underline{p}\right]_{x_0} = \left[\underline{k} * \underline{l}\right]_{x_0} * \left[\underline{p}\right]_{x_0} = \left[\left(\underline{k} * \underline{l}\right) * \underline{p}\right]_{x_0}, \tag{2}$$

and for the right side of (1) is true:

$$\left[\underline{k}\right]_{x_0} * \left(\left[\underline{l}\right]_{x_0} * \left[\underline{p}\right]_{x_0}\right) = \left[\underline{k}\right]_{x_0} * \left[\underline{l} * \underline{p}\right]_{x_0} = \left[\underline{k} * \left(\underline{l} * \underline{p}\right)\right]_{x_0}. \tag{3}$$

So, to show that the equation (1) is true is enough to show that $\left[\left(\underline{k} * \underline{l}\right) * \underline{p}\right]_{x_0} = \left[\underline{k} * \left(\underline{l} * \underline{p}\right)\right]_{x_0}$, i.e., that the proximate loops $\left(\underline{k} * \underline{l}\right) * \underline{p}$ and $\underline{k} * \left(\underline{l} * \underline{p}\right)$ are homotopic over all coverings.

Let $k_{\mathcal{U}}$, $l_{\mathcal{U}}$ and $p_{\mathcal{U}}$ are $\mathcal{U}$ - loops in $x_0$ for an arbitrary covering $\mathcal{U} \in CovX$. Then by Theorem 4.2 $(k_{\mathcal{U}} * l_{\mathcal{U}}) * p_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}} * (l_{\mathcal{U}} * p_{\mathcal{U}})\,(rel\,\{0,1\})$ for any covering $\mathcal{U} \in Cov\,(X)$. Therefore, $\left(\underline{k} * \underline{l}\right) * \underline{p} \sim \underline{k} * \left(\underline{l} * \underline{p}\right)(rel\,\{0,1\})$, i.e., $\left[\left(\underline{k} * \underline{l}\right) * \underline{p}\right]_{x_0} = \left[\underline{k} * \left(\underline{l} * \underline{p}\right)\right]_{x_0}$.

So, the associative law for the operation "$*$" in the set $prox\pi_1\,(X, x_0)$ is true.

**Identity element :** It is the homotopy class $\left[\underline{c_{x_0}}\right]_{x_0}$ of the constant proximate loop in $x_0$ defined by the constant $\mathcal{U}$ - loop $c_{x_0}$ in $x_0$.

Let $k_{\mathcal{U}}$ is $\mathcal{U}$ - loop in $x_0$ for an arbitrary covering $\mathcal{U} \in CovX$. Then for an arbitrary covering $\mathcal{U} \in CovX$ by Theorem 4.3 $k_{\mathcal{U}} * c_{x_0} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}\,(rel\,\{0,1\})$ and $c_{x_0} * k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} k_{\mathcal{U}}\,(rel\,\{0,1\})$.

Therefore, $\underline{k} * \underline{c_{x_0}} \sim \underline{k}\,(rel\,\{0,1\})$ and $\underline{c_{x_0}} * \underline{k} \sim \underline{k}\,(rel\,\{0,1\})$, i.e., $\left[\underline{k} * \underline{c_{x_0}}\right]_{x_0} = \left[\underline{k}\right]_{x_0}$ and $\left[\underline{c_{x_0}} * \underline{k}\right]_{x_0} = \left[\underline{k}\right]_{x_0}$.

By the definition of the operation "$*$" in the set $prox\pi_1\,(X, x_0)$ the following identities are true:
$\left[\underline{k}\right]_{x_0} * \left[\underline{c_{x_0}}\right]_{x_0} = \left[\underline{k} * \underline{c_{x_0}}\right]_{x_0} = \left[\underline{k}\right]_{x_0}$ and $\left[\underline{c_{x_0}}\right]_{x_0} * \left[\underline{k}\right]_{x_0} = \left[\underline{c_{x_0}} * \underline{k}\right]_{x_0} = \left[\underline{k}\right]_{x_0}$.

**Inverse element:** An inverse element of a homotopy class $\left[\underline{k}\right]_{x_0}$ of a proximate loop in $x_0$ is the homotopy class $\left[\underline{k}^{-1}\right]_{x_0}$ of the proximate loop $\underline{k}^{-1} = \left(k_{\mathcal{U}}^{-1} \mid \mathcal{U} \in CovX\right)$ defined by the inverse $\mathcal{U}$ - loop of the $\mathcal{U}$ - loop $k_{\mathcal{U}}$ in $x_0$. For any covering $\mathcal{U} \in CovX$ by Theorem 4.5 $k_{\mathcal{U}} * k_{\mathcal{U}}^{-1} \underset{\mathcal{U}}{\sim} c_{x_0}\,(rel\,\{0,1\})$ and $k_{\mathcal{U}}^{-1} * k_{\mathcal{U}} \underset{\mathcal{U}}{\sim} c_{x_0}\,(rel\,\{0,1\})$.

So, $\left[\underline{k}\right]_{x_0} * \left[\underline{k}^{-1}\right]_{x_0} = \left[\underline{k} * \underline{k}^{-1}\right] = \left[\underline{c_{x_o}}\right]_{x_0}$ and $\left[\underline{k}^{-1}\right]_{x_0} * \left[\underline{k}\right]_{x_0} = \left[\underline{k}^{-1} * \underline{k}\right] = \left[\underline{c_{x_o}}\right]_{x_0}$.

Therefore, the set $prox\pi_1\,(X, x_0)$ with the operation "$*$" is a group. □

Let $X$ and $Y$ be topological spaces, and $\underline{f} = (f_{\mathcal{V}} \mid \mathcal{V} \in CovY)$ is a pointed proximate net from $(X, x_0)$ to $(Y, y_0)$.

Now, to the proximate net $\underline{f}$ we can associate an **induced function** $f_{prox} : prox\pi_1(X, x_0) \to prox\pi_1(Y, y_0)$ defined in the following way:

Let $[\underline{k}]_{x_0} \in prox\pi_1\,(X, x_0)$, where $\underline{k} = (k_{\mathcal{U}} \mid \mathcal{U} \in CovX)$ is a proximate loop in $x_0$. Since the proximate loop is a proximate net, if we define a proximate net $\underline{p} = (p_{\mathcal{V}} \mid \mathcal{V} \in CovY)$ as a composition of proximate nets $\underline{k} = (k_{\mathcal{U}} \mid \mathcal{U} \in CovX)$ and $\underline{f} = (f_{\mathcal{V}} \mid \mathcal{V} \in CovY)$, i.e., $\underline{p} = \underline{f} \circ \underline{k} = (p_{\mathcal{V}} = f_{\mathcal{V}} \circ k_{\mathcal{U}} \mid \mathcal{V} \in CovY)$, we obtain a proximate loop in $y_0$. Finally, we define:

$$f_{prox}([\underline{k}]_{x_0}) = [\underline{p}]_{y_0}.$$

Let $\underline{k^0}$ and $\underline{k^1}$, are proximate loops in $x_0$ from the same homotopy class of proximate loop $\left[\underline{k}\right]_{x_0}$. So there exists a homotopy $\underline{K}$ between the proximate loops $\underline{k^0}$ and $\underline{k^1}$. Then the proximate loops $\underline{f} \circ \underline{k^0}$ and $\underline{f} \circ \underline{k^1}$ are homotopic by a homotopy $\underline{f} \circ \underline{K}$. Therefore the induced function $f_{prox}$ is well defined.

**Theorem 5.2.** *Let $X$ and $Y$ are topological spaces, $\underline{f} = (f_{\mathcal{V}} \mid \mathcal{V} \in CovY)$ is a pointed proximate net from $(X, x_0)$ to $(Y, y_0)$. Then the induced function $f_{prox} : prox\pi_1(X, x_0) \to prox\pi_1(Y, f\,(x_0))$ is homomorphism.*

*Proof.* Let $[\underline{k}]_{x_0}$, $[\underline{l}]_{x_0} \in prox\pi_1(X, x_0)$. We should show that:

$$f_{prox}\left(\left[\underline{k}\right]_{x_0} * \left[\underline{l}\right]_{x_0}\right) = f_{prox}\left(\left[\underline{k}\right]_{x_0}\right) * f_{prox}\left(\left[\underline{l}\right]_{x_0}\right)$$

Because

$$f_{prox}\left(\left[\underline{k}\right]_{x_0} * \left[\underline{l}\right]_{x_0}\right) = f_{prox}\left(\left[\underline{k} * \underline{l}\right]_{x_0}\right) = f_{prox}\left[(k_{\mathcal{U}} * l_{\mathcal{U}})_{\mathcal{U}\in CovX}\right]_{x_0} = \left[(f_{\mathcal{V}}(k_{\mathcal{U}} * l_{\mathcal{U}}))_{\mathcal{V}\in CovY}\right]_{y_0}$$

and

$$f_{prox}\left(\left[\underline{k}\right]_{x_0}\right) * f_{prox}\left(\left[\underline{l}\right]_{x_0}\right) = \left[(f_{\mathcal{V}} \circ k_{\mathcal{U}})_{\mathcal{V}\in CovY}\right]_{y_0} * \left[(f_{\mathcal{V}} \circ l_{\mathcal{U}})_{\mathcal{V}\in CovY}\right]_{y_0} = \left[((f_{\mathcal{V}} \circ k_{\mathcal{U}}) * (f_{\mathcal{V}} \circ l_{\mathcal{U}}))_{\mathcal{V}\in CovY}\right]_{y_0},$$

we should show that $\left[(f_{\mathcal{V}}(k_{\mathcal{U}} * l_{\mathcal{U}}))_{\mathcal{V}\in CovY}\right]_{y_0} = \left[((f_{\mathcal{V}} \circ k_{\mathcal{U}}) * (f_{\mathcal{V}} \circ l_{\mathcal{U}}))_{\mathcal{V}\in CovY}\right]_{y_0}$.
The equality follows since $((f_{\mathcal{V}} \circ k_{\mathcal{U}}) * (f_{\mathcal{V}} \circ l_{\mathcal{U}}))(t) =$

$$\begin{cases} (f_{\mathcal{V}} \circ k_{\mathcal{U}})(2t), & 0 \le t \le \frac{1}{2} \\ (f_{\mathcal{V}} \circ l_{\mathcal{U}})(2t-1), & \frac{1}{2} \le t \le 1 \end{cases} = \begin{cases} f_{\mathcal{V}}(k_{\mathcal{U}}(2t)), & 0 \le t \le \frac{1}{2} \\ f_{\mathcal{V}}(l_{\mathcal{U}}(2t-1)), & \frac{1}{2} \le t \le 1 \end{cases} = f_{\mathcal{V}}((k_{\mathcal{U}} * l_{\mathcal{U}})(t)). \quad \square$$

Since the proximate loop is a proximate net by Theorem 3.1 the following Theorem is valid:

**Theorem 5.3.** *Let $\underline{f} = (f_{\mathcal{V}} \,|\mathcal{V} \in CovY)$, $f_{\mathcal{V}} : (X, x_0) \to (Y, y_0)$ is $\mathcal{V}$ - continuous, and $\underline{g} = (g_{\mathcal{W}} \,|\mathcal{W} \in CovZ)$, $g_{\mathcal{W}} : (Y, y_0) \to (Z, z_0)$ is $\mathcal{W}$ - continuous, are two proximate nets. For any $[\underline{k}]_{x_0} \in prox\pi_1(X, x_0)$ is true that:*

$$\left(\underline{g} \circ \underline{f}\right)_{prox}\left([\underline{k}]_{x_0}\right) = g_{prox}\left(f_{prox}\left([\underline{k}]_{x_0}\right)\right)$$

**Theorem 5.4.** *Let $\underline{f} = (f_{\mathcal{V}} \,|\mathcal{V} \in CovY)$, $f_{\mathcal{V}} : (X, x_0) \to (Y, y_0)$ is $\mathcal{V}$ - continuous, and $\underline{f'} = \left(f'_{\mathcal{V}} \,|\mathcal{V} \in CovY\right)$, $f'_{\mathcal{V}} : (X, x_0) \to (Y, y_0)$ is $\mathcal{V}$ - continuous, are two proximate nets. For any proximate loop in $x_0$ if $\underline{f}$ and $\underline{f'}$ are homotopic then proximate loops in $y_0$, $\underline{f} \circ \underline{k}$ and $\underline{f'} \circ \underline{k}$ are homotopic.*

*Proof.* If $\underline{f}$ and $\underline{f'}$ are homotopic there exists a homotopy $\underline{F}$ connecting $\underline{f}$ and $\underline{f'}$. For a covering $\mathcal{V}$ of $Y$ we choose a covering $\mathcal{U}$ of $X$ as in Proposition 2.3. Then $\underline{L} = (L_{\mathcal{V}})$, where $L_{\mathcal{V}} = F_{\mathcal{V}}(k_{\mathcal{U}} \times id) : I \times I \to Y$ is a proximate net. Since
$L_{\mathcal{V}}(t, 0) = F_{\mathcal{V}}k_{\mathcal{U}}(t)$ and $L_{\mathcal{V}}(t, 1) = F_{\mathcal{V}}k'_{\mathcal{U}}(t)$ for all $t \in I$, and $L_{\mathcal{V}}(0, s) = F_{\mathcal{V}}(x_0, s) = y_0$ and $L_{\mathcal{V}}(1, s) = F_{\mathcal{V}}(x_0, s) = y_0$ for all $s \in I$, we have only to check the conditions (I), (II), (III) of Definition 4.2.

(I) By Proposition 2.3 the function $k_{\mathcal{U}} \times id : I \times I \to X \times I$ is $st(\mathcal{U})$ - continuous. And $F_{\mathcal{V}} : X \times I \to Y$ is $st(\mathcal{V})$ - continuous. It follows $L_{\mathcal{V}}$ is $st^2(\mathcal{V})$ - continuous.

(II) For $(0, s)$ from $\partial I^2 = \partial(I \times I)$, since $k_{\mathcal{U}} \times id$ is $\mathcal{U}$ - continuous at point $(0, s)$ and $F_{\mathcal{V}}$ is $st(\mathcal{V})$ - continuous at $(x_0, s) = (k_{\mathcal{U}} \times id)(0, s)$, it follows $L_{\mathcal{V}}$is $st(\mathcal{V})$ - continuous at point $(0, s)$. Similar for $(1, s)$.
For $(t, 0)$ from $\partial I^2 = \partial(I \times I)$, since $k_{\mathcal{U}} \times id$ is $st(\mathcal{U})$ - continuous at point $(t, 0)$ and $F_{\mathcal{V}}$ is $\mathcal{V}$ - continuous at $(k_{\mathcal{U}}(t), 0) = (k_{\mathcal{U}} \times id)(t, 0)$ it follows $L_{\mathcal{V}}$is $st(\mathcal{V})$ - continuous at point $(t, 0)$. Similar for $(t, 1)$.

(III) For $(0, 0)$ from $\partial^2 I^2$, since $k_{\mathcal{U}} \times id$ is $\mathcal{U}$ - continuous at point $(0, 0)$ and $F_{\mathcal{V}}$ is $\mathcal{V}$ - continuous at $(x_0, 0) = (k_{\mathcal{U}} \times id)(0, 0)$, it follows $L_{\mathcal{V}}$is $\mathcal{V}$ - continuous at $(0, 0)$. Similar for all other points $(1, 0)$, $(0, 1)$ and $(1, 1)$ from $\partial\partial^2 I^2$.

We proved that $\underline{L} = (L_{\mathcal{V}})$ is homotopy connecting $\underline{f} \circ \underline{k}$ and $\underline{f'} \circ \underline{k}$ as required. $\square$

By Theorems 5.2, 5.3 and 5.4 we obtain the following result

**Theorem 5.5.** *Associating $prox\pi_1(X, x_0)$ to a pointed topological space $(X, x_0)$ and associating to a proximate net $[\underline{f}]_{x_0}$ the homomorphism $f_{prox} : prox\pi_1(X, x_0) \to prox\pi_1(Y, f(x_0))$ we obtain a functor from category of pointed intrinsic shape to category of groups.*

*Proof.* Let consider the functor defined above from the category of pointed intrinsic shape to the category of groups.

By Theorems 5.2 and 5.4 this functor is well defined. By Theorem 5.3 it preserves composition of morphisms.

At last, we have to show that it preserves the identity morphisms.

Let $\left[\underline{f}\right]_{x_0}$ be an arbitrary morphism in the category of pointed intrinsic shape from $(X, x_0)$ to $(Y, y_0)$. We consider the pointed homotopy class $\left[\underline{1_X}\right]_{x_0}$ of pointed proximate net $\underline{1_X}$ defined with the identity function $1_X$. By Definition 3.3 the following identities are true:

$$\left[\underline{f}\right]_{x_0} \circ \left[\underline{1_X}\right]_{x_0} = \left[\underline{f} \circ \underline{1_X}\right]_{x_0} = \left[\underline{f}\right]_{x_0}$$

So, an identity morphism in the category of pointed intrinsic shape is the pointed homotopy class of $\left[\underline{1_X}\right]_{x_0}$ pointed proximate net $\underline{1_X}$ defined with the identity function $1_X$ in the topological space $X$.

The induced function $1_{prox} : prox\pi_1(X, x_0) \to prox\pi_1(X, x_0)$ associated to the identity morphism is defined in the following way: $1_{prox}([\underline{k}]_{x_0}) = [\underline{1_X} \circ \underline{k}]_{x_0}$, where $[\underline{k}]_{x_0} \in prox\pi_1\,(X, x_0)$ is the homotopy class of the proximate loop $\underline{k} = (k_{\mathcal{U}} \,|\mathcal{U} \in CovX)$ in $x_0$. Since $1_{prox}([\underline{k}]_{x_0}) = [\underline{1_X} \circ \underline{k}]_{x_0} = [\underline{k}]_{x_0} = 1_{prox\pi_1(X,x_0)}\left([\underline{k}]_{x_0}\right)$ we conclude that the function from the category of pointed intrinsic shape to category of groups preserves the identity morphisms. □

By this theorem we proved that $prox\pi_1\,(X, x_0)$ is an invariant of pointed intrinsic shape of a pointed space $(X, x_0)$. If $(X, x_0)$ and $(Y, y_0)$ have same pointed intrinsic shape then their proximate fundamental groups are isomorphic.

**Example 5.1.** *The proximate fundamental group of a circle and Warsaw circle are isomorphic to additive group of integers.*

*Proof.* Notions of shape and homotopy for finite polyhedra coincide. So, there is $1-1$ corespondence between homotopy classes of pointed maps $(S^1, 1) \to (S^1, 1)$ and homotopy classes of pointed proximate nets $(S^1, 1) \to (S^1, 1)$.

We consider the unit circle $S^1$ in the complex plain and define maps $f^n : (S^1, 1) \to (S^1, 1)$ by $f^n(z) = z^n$, $n \in \mathbb{Z}$.

Then, the only classes of pointed homotopy of maps $(S^1, 1) \to (S^1, 1)$ are $[f^n]$, $n \in \mathbb{Z}$, and these are exactly the elements of the fundamental group of the circle, i.e., $\pi_1(S^1) = \{[f^n] \mid n \in \mathbb{Z}\}$.

Since there is $1-1$ corespondence between homotopy classes of pointed maps $(S^1, 1) \to (S^1, 1)$ and homotopy classes of pointed proximate nets $(S^1, 1) \to (S^1, 1)$, the only pointed homotopy classes of pointed proximate nets $(S^1, 1) \to (S^1, 1)$ are $[(f^n_{\mathcal{V}})]$, $n \in \mathbb{Z}$, where the proximate net $(f^n_{\mathcal{V}})$ is defined by $f^n_{\mathcal{V}} = f^n$ for all coverings $\mathcal{V}$. The pointed homotopy classes of pointed proximate nets $[(f^n_{\mathcal{V}})]$, $n \in \mathbb{Z}$, are exactly the elements of the **proximate** fundamental group of the circle, i.e., $prox\pi_1(S^1) = \{[(f^n_{\mathcal{V}})] \mid n \in \mathbb{Z}\}$.

The operation "$*$" in fundamental group of a circle is defined by concatenation of paths. It is well known that the definition leads to $[f^n] * [f^m] = [f^{n+m}]$, i.e., the fundamental group of a circle is isomorphic to additive group of integers ( for example, see [10] ).

Since the operation $*$ in proximate fundamental group of a circle is defined also by concatenation of paths then the operation in $prox\pi_1(S^1)$ is given by

$$[(f^n_{\mathcal{V}})] * [(f^m_{\mathcal{V}})] = [(f^{n+m}_{\mathcal{V}})].$$

Then, with $[f^n] \to [(f^n_{\mathcal{V}})]$ is defined a natural isomorphism $\pi_1(S^1) \to prox\pi_1(S^1)$, between fundamental group and **proximate** fundamental group of the circle.

Finally, by Theorem 5.5 the **proximate** fundamental group is an invariant of pointed intrinsic shape. Since a circle and Warsaw circle have the same shape, they also have the same intrinsic shape and isomorphic proximate fundamental groups. □

## References


[1] K. Borsuk, Theory of Shape, Polish Scientific Publisher, Warszawa, 1975.
[2] R. Engelking, General Topology, Heldermann Verla Berlin, 1989.
[3] James E. Felt, $\varepsilon$ - continuity and shape, Proc. Amer. Math. Soc. 46 (1974), 426–430.
[4] R. W. Kieboom, An intrinsic characterization of the shape of paracompacta by means of non-continuous single-valued maps, Bull. Belg. Math. Soc. 1 Volume 1, Number 5 (1994) 701–711.
[5] S. Mardešić, J. Segal, Shape Theory, North-Holland Mathematical Library, vol. 26, North-Holland Publishing Co., Amsterdam-New York, 1982.
[6] J.M.R.Sanjurjo, A non-continuous description of the shape category of compacta, Quart.J.Math. Oxford Series 40 (1989), 351–359.
[7] N. Shekutkovski, Intrinsic definition of strong shape for compact metric spaces, Topology Proceedings, (2012) Vol. 39, 27–39.
[8] N. Shekutkovski, B. Andonovik, Intrinsic definition of strong shape of strong paracompacta, Proceedings of IV Congress of the mathematicians of R. Macedonia, Struga, October 18–22, 2008 (2010), 287–299.
[9] N. Shekutkovski, T.A. Pachemska, G. Markoski, (2012) Maps of quasicomponents induced by a shape morphism, Glasnik Matematicki, Vol. 47(67), 431–439.
[10] N. Shekutkovski, Topologija, Prirodno matematicki fakultet, Univerzitet Sv. Kiril I Metodij, Skopje, 2002.
[11] N. Shekutkovski, A. Velkoska, Proximate Fundamental Group, Proceedings of the Fifth International Scientific Conference–FMNS2013, June 12–16, 2013 (2013), 80–86.
[12] N. Shekutkovski, Z. Misajleski, Intrinsic shape based on $\varepsilon$ - continuity and on continuity up to a covering are equivalent (II), Proceedings of FMNS 2013, Volume 1 (2013), 87–94.
[13] N. Shekutkovski, Z. Misajleski, Equivalence of intrinsic shape and shape, God. Zb. Inst. Mat. 42 (2013), 69–80.