



Conformal Mappings of Quasi-Einstein Manifolds Admitting Special Vector Fields

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Abstract. As it is known, Einstein manifolds play an important role in geometry as well as in general relativity. Einstein manifolds form a natural subclass of the class of quasi-Einstein manifolds. In this work, we investigate conformal mappings of quasi-Einstein manifolds. Considering this mapping, we examine some properties of these manifolds. After that, we also study some special vector fields under this mapping of these manifolds and some theorems about them are proved.

1. Introduction

A non-flat n -dimensional Riemannian manifold (M, g) ($n > 2$) is said to be an Einstein manifold if the condition

$$S(X, Y) = \frac{r}{n}g(X, Y) \quad (1)$$

holds on M , where S and r denote the Ricci tensor and the scalar curvature of (M, g) , respectively. Einstein manifolds play an important role in Riemannian Geometry, as well as in general relativity. These manifolds form a natural subclass of the class of quasi-Einstein manifolds. A non-flat n -dimensional Riemannian manifold (M, g) ($n > 2$) is defined to be a quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the following condition

$$S(X, Y) = ag(X, Y) + b\phi(X)\phi(Y) \quad (2)$$

where a and b are scalars of which $b \neq 0$ and ϕ is a non-zero 1-form such that

$$g(X, U) = \phi(X), \quad g(U, U) = 1 \quad (3)$$

for all vector fields X on M , U being a unit vector field. Then a and b are called the associated scalars, ϕ is called the associated 1-form and U is called the generator of the manifold, [2]. This manifold is denoted by $(QE)_n$. The notion of quasi-Einstein manifold has been studied by many authors e.g. [3–5]. This manifold arose during the study of exact solutions of the Einstein field equations as well as during

2010 *Mathematics Subject Classification.* Primary 53B20; Secondary 53C25

Keywords. Quasi-Einstein manifold, φ (Ric)-vector field, concircular vector field, Codazzi tensor

Received: 31 August 2014; Accepted: 21 December 2014

Communicated by Ljubica Velimirović and Mića Stanković

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considerations of quasi-umbilical hypersurfaces. For instance, the Robertson-Walker space-times are quasi-Einstein manifolds. Also quasi-Einstein manifold can be taken as a model of the perfect fluid space-time in general relativity.

One of the important concepts of Riemannian Geometry is conformal mapping. Conformal mappings of Riemannian manifolds (or semi-Riemannian manifolds) have been investigated by many authors. In general relativity, conformal mappings are important since they preserve the causal structure up to time orientation and light-like geodesics up to parametrization, [13]. The existence of conformal mappings of Riemannian manifolds onto Einstein manifolds have been studied by Brinkmann [1], Mikeš, Gavrilchenko, Gladysheva [14] and others. Also, conformal mappings between two Einstein manifolds have been examined by Brinkmann. Recently, Kiosak [12] investigated conformal mappings of quasi-Einstein manifolds by considering the generator vector field U as a isotropic vector field, i.e., $g(U, U) = 0$, which is different from the definition of [2] and by taking the associated scalars as $a = \frac{r}{n}$, $b = 1$ in [2]. He also proved that quasi-Einstein manifolds are closed with respect to concircular mappings and he obtained some properties of them. Additionally, Fu, Yang and Zhao [7] studied a class of conformal mappings between two semi-Riemannian manifolds and they found a conformal mapping which transforms a generalized quasi-Einstein manifold into a generalized quasi-Einstein manifold.

In this work, we first examine some special vector fields on quasi-Einstein manifolds satisfying the conditions (2), (3) and we find some properties of these manifolds. After that, considering a conformal mapping between two quasi-Einstein manifolds, we study special vector fields under this conformal transformation and we prove some theorems related to conformal transformation of these manifolds.

2. Special vector fields on quasi-Einstein manifolds

This section provides an investigation of some special vector fields on a $(QE)_n$. Firstly, we mention about the known properties of this manifold and the definitions of special vector fields are given. After that, we examine some properties of $(QE)_n$ admitting special vector fields.

Let $\{e_i : i = 1, 2, \dots, n\}$ be an orthonormal frame field at any point of $(QE)_n$. Then putting $X = Y = e_i$ in (2) and taking summation over i , we get

$$r = na + b \quad (4)$$

where r is the scalar curvature of the manifold. Since U is an unit vector field, putting $X = Y = U$ in (2), we obtain

$$S(U, U) = a + b. \quad (5)$$

Definition 2.1. A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the condition

$$\nabla_X \xi = \rho X + \lambda(X)\xi \quad (6)$$

where $X \in TM$, λ is a linear form and ρ is a function, [20]. In the local transcription, this reads

$$\nabla_i \xi^h = \rho \delta_i^h + \xi^h \lambda_i \quad (7)$$

where ξ^h and λ_i are the components of ξ and λ , and δ_i^h is the Kronecker symbol.

A torse-forming vector field ξ is called recurrent if $\rho = 0$; concircular if the form λ_i is a gradient covector, i.e., there is a function $v(x)$ such that $\lambda = dv(x)$.

Thus, for a recurrent vector field, we have from (7)

$$\nabla_i \xi_j = \lambda_i \xi_j. \quad (8)$$

Also, for a concircular vector field ξ , we get

$$\nabla_i \xi_j = \rho g_{ij}. \quad (9)$$

A Riemannian manifold with a concircular vector field is called equidistant, [18, 19]. Recently, torse-forming and concircular vector fields have been studied by many authors e.g. [15–17].

Now, we deal with $\varphi(Ric)$ -vector fields introduced by Hinterleitner and Kiosak [9]. As it has been mentioned before, Einstein manifolds are characterized by the proportionality of the Ricci tensor to the metric tensor. So, in these manifolds, concircular vector fields could equally be defined by $\nabla\xi = \rho Ric$. This inspires us to a general investigation of vector fields satisfying the latter relation and the conditions for their existence in general (i.e. non-Einstein) Riemannian manifolds, with the specialization $\rho = \mu = \text{const}$, [9]. It is also indicated in [10] that $\varphi(Ric)$ -vector fields are closely related to Ricci flows introduced by Hamilton, [8].

Definition 2.2. A $\varphi(Ric)$ -vector field is a vector field on an n -dimensional Riemannian manifold (M, g) and Levi-Civita connection ∇ , which satisfies the condition

$$\nabla\varphi = \mu Ric \tag{10}$$

where μ is a constant and Ric is the Ricci tensor, [9]. When (M, g) is an Einstein manifold, the vector field φ is concircular. If $\mu \neq 0$, then we call that the vector field φ is proper $\varphi(Ric)$ -vector field. Moreover, when $\mu = 0$, the vector field φ is covariantly constant.

In [9], it was shown that Riemannian manifolds with a $\varphi(Ric)$ -vector field of constant length have constant scalar curvature. Now, we show that the converse of this theorem is also true. Therefore, we can state and prove the following theorem.

Theorem 2.3. Let V_n be a Riemannian manifold with constant scalar curvature. If V_n admits a $\varphi(Ric)$ -vector field, then the length of φ is constant.

Proof. Suppose that V_n is a Riemannian manifold with constant scalar curvature admitting a $\varphi(Ric)$ -vector field. Using the Ricci identity and the equation (10), we obtain

$$\varphi_\alpha R_{ijk}^\alpha = \mu(\nabla_k S_{ij} - \nabla_j S_{ik}) \tag{11}$$

where $\varphi_\alpha, R_{ijk}^\alpha, S_{ij}$ denote the components of the vector field φ , the curvature tensor, the Ricci tensor, respectively and μ is a constant.

Considering the second Bianchi identity, (11) reduces to

$$\varphi_\alpha R_{ijk}^\alpha = \mu \nabla_\alpha R_{ijk}^\alpha. \tag{12}$$

From the contracted second Bianchi identity, we also have

$$\varphi_\alpha S_k^\alpha = \frac{\mu}{2} \nabla_k r \tag{13}$$

where $S_k^\alpha = g^{i\alpha} S_{ik}$ ($i = 1, 2, \dots, n$) and r is the scalar curvature of V_n .

Since the scalar curvature of V_n is constant, then we get from (13)

$$\varphi_\alpha S_k^\alpha = 0. \tag{14}$$

On the other hand, taking the covariant derivative of the length of φ , using (10) and (14), it is obtained that

$$\begin{aligned} \nabla_k(g^{ij}\varphi_i\varphi_j) &= g^{ij}(\nabla_k\varphi_i)\varphi_j + g^{ij}\varphi_i\nabla_k\varphi_j \\ &= \mu(g^{ij}S_{ik}\varphi_j + g^{ij}S_{jk}\varphi_i) \\ &= 2\mu \varphi_j S_k^j \\ &= 0. \end{aligned} \tag{15}$$

From (15), it can be seen that the length of the vector field φ is constant. Thus, the proof is completed. \square

Now, we consider a $(QE)_n$ admitting the generator vector field U as a $\phi(Ric)$ -vector field. Then we have from (10)

$$\nabla_j \phi_i = \mu S_{ij} \tag{16}$$

where μ is a constant. Then, we give the following theorem.

Theorem 2.4. *In a $(QE)_n$, if the vector field U corresponding to the 1-form ϕ is a $\phi(Ric)$ -vector field, then U is covariantly constant.*

Proof. We consider a $(QE)_n$ whose generator vector field is a $\phi(Ric)$ -vector field. Putting (2) in (16), we obtain

$$\nabla_j \phi_i = \mu(ag_{ij} + b\phi_i\phi_j). \tag{17}$$

Multiplying (17) by ϕ^i and using the condition $g(U, U) = 1$, it can be seen that

$$\mu(a + b)\phi_j = 0. \tag{18}$$

Suppose that μ is a non-zero constant. Then, we get from (18)

$$a = -b. \tag{19}$$

By the aid of (2) and (19), we obtain

$$S_{ij} = a(g_{ij} - \phi_i\phi_j). \tag{20}$$

Thus, we have from (20)

$$S_{ij}\phi^i = 0. \tag{21}$$

Taking the covariant derivative of the condition (21) and using (16), we get

$$(\nabla_k S_{ij})\phi^i + \mu S_{ij}S_k^i = 0. \tag{22}$$

Multiplying (22) by g^{jk} , we obtain

$$(\nabla_k S_i^k)\phi^i + \mu S_{ij}S^{ij} = 0 \tag{23}$$

where $S^{ij} = g^{jk}S_k^i$.

It was shown, [9], that Riemannian manifolds with a $\phi(Ric)$ vector field of constant length have constant scalar curvature. Since the generator U is a unit vector field and it is also a $\phi(Ric)$ vector field, the scalar curvature of the manifold is constant. In this case, using the contracted second Bianchi identity and considering that the scalar curvature of the manifold is constant, it is obtained that

$$\nabla_k S_i^k = \frac{1}{2}\nabla_i r = 0. \tag{24}$$

Using (23), (24) and assuming that μ is a non-zero constant, we obtain

$$S_{ij}S^{ij} = 0. \tag{25}$$

By the aid of (20) and (25) it follows that

$$(n - 1)a^2 = 0. \tag{26}$$

Since $n > 2$, from (26) it is obtained that $a = 0$. In this case, it is seen from (20) that the Ricci tensor vanishes which is a contradiction. Therefore, the constant μ must be zero and so, the generator vector field U is covariantly constant. This completes the proof. \square

Theorem 2.5. *If a $(QE)_n$ admits a $\varphi(\text{Ric})$ -vector field with constant length, then either ϕ_i and φ_i are collinear or the Ricci tensor of the manifold reduces to the following form*

$$S_{ij} = b \phi_i \phi_j.$$

Proof. We assume that $(QE)_n$ admits a $\varphi(\text{Ric})$ -vector field with constant length. Then, we have

$$\varphi_i \varphi^i = c \tag{27}$$

where c is a constant. Taking the covariant derivative of the condition (27), using the equation (10) and considering μ as a non-zero constant (that is φ is proper $\varphi(\text{Ric})$ vector field), it follows that

$$S_{ik} \varphi^i = 0. \tag{28}$$

By the aid of (2) and (28), we get

$$a \varphi_k + b (\varphi^i \phi_i) \phi_k = 0. \tag{29}$$

Multiplying (29) by ϕ^k and using (3), it is obtained that

$$(a + b) \varphi_k \phi^k = 0. \tag{30}$$

So either $\varphi_k \phi^k = 0$ which gives from (29) that $a = 0$ and so, the Ricci tensor of the manifold reduces to the form

$$S_{ij} = b \phi_i \phi_j \tag{31}$$

or $\varphi_k \phi^k \neq 0$ which gives from (30) that $a = -b$. Since $b \neq 0$ then $a \neq 0$ and from (29) we obtain that

$$\varphi_k = (\varphi^i \phi_i) \phi_k \tag{32}$$

so, φ_k and ϕ_k are collinear. This completes the proof. \square

From the previous theorem we have the following corollary.

Corollary 2.6. *If a $(QE)_n$ admits a $\varphi(\text{Ric})$ -vector field with constant length which is not orthogonal to the generator, then the associated scalars of the manifold must be constants and the vector field φ is covariantly constant.*

Proof. As it has been mentioned before, a Riemannian manifold admitting a $\varphi(\text{Ric})$ -vector field with constant length has constant scalar curvature. Moreover, under the assumptions and from Theorem 2.5., we obtain that the associated scalars of $(QE)_n$ are related by $a = -b$, and from (4), we get

$$r = (n - 1)a. \tag{33}$$

Since the scalar curvature of the manifold is constant, in this case, from (4) and (33), we see that the associated scalars of the manifold are constants.

For the second part, multiplying (32) by φ^k and using (27), it can be seen that $\varphi^i \phi_i$ is a constant. So, (32) shows that the generator vector field U is also a $\varphi(\text{Ric})$ -vector field. In this case, U must be covariantly constant by Theorem 2.4., and due to the collinearity of φ and U , φ is also covariantly constant. Hence, the proof is completed. \square

3. An example of quasi-Einstein manifold

We define a Riemannian metric g on the 4-dimensional real number space \mathbb{R}^4 by the formula

$$ds^2 = g_{ij}dx^i dx^j = (1 + e^{x^1})[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2] \tag{34}$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of \mathbb{R}^4 . Then the only non-vanishing components of the Christoffel symbols, curvature tensor and the Ricci tensor are

$$\Gamma_{22}^1 = \Gamma_{33}^1 = \Gamma_{44}^1 = -e^{x^1}/2(1 + e^{x^1}), \quad \Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{13}^3 = \Gamma_{14}^4 = e^{x^1}/2(1 + e^{x^1}),$$

$$R_{1221} = R_{1331} = R_{1441} = e^{x^1}/2(1 + e^{x^1}),$$

$$R_{2332} = R_{2442} = R_{3443} = e^{2x^1}/4(1 + e^{x^1}),$$

$$S_{11} = 3e^{x^1}/2(1 + e^{x^1})^2,$$

$$S_{22} = S_{33} = S_{44} = e^{x^1}/2(1 + e^{x^1})$$

and the components which can be obtained from these by the symmetry properties.

Moreover, it can be shown that the scalar curvature of the manifold is $r = \frac{3e^{x^1}(2+e^{x^1})}{2(1+e^{x^1})^3}$ which is non-vanishing and non-constant. Therefore \mathbb{R}^4 with the considered metric is a Riemannian manifold (M_4, g) of non-vanishing scalar curvature. We shall now show that M_4 is a $(QE)_4$. Let us now consider the associated scalars and the components of the associated 1-form ϕ as follows:

$$a = \frac{e^{x^1}}{2(1 + e^{x^1})^2}, \quad b = \frac{2e^{x^1} - e^{2x^1}}{2(1 + e^{x^1})^3}$$

$$\phi_i(x) = \begin{cases} \sqrt{1 + e^{x^1}}, & \text{if } i = 1 \\ 0, & \text{if } i = 2, 3, 4 \end{cases} \tag{35}$$

at any point $x \in M_4$. In our M_4 , it is seen that the following equations are satisfied:

(i) $S_{11} = ag_{11} + b\phi_1\phi_1,$

(ii) $S_{22} = ag_{22} + b\phi_2\phi_2,$

(iii) $S_{33} = ag_{33} + b\phi_3\phi_3,$

(iv) $S_{44} = ag_{44} + b\phi_4\phi_4,$

(v) $g^{ij}\phi_i\phi_j = 1$

since for the cases other than (i) – (v) the components of each term of (2) vanishes identically and the relation (2) holds trivially. Therefore, (M_4, g) endowed with the metric (34) is a $(QE)_4$ satisfying the conditions (2) and (3).

4. Conformal mappings of $(QE)_n$ admitting special vector fields

In this section, we consider a conformal mapping between two quasi-Einstein manifolds denoted by V_n and \bar{V}_n with metrics g and \bar{g} , respectively, and we examine some special vector fields under this mapping.

Definition 4.1. A conformal mapping is a diffeomorphism of V_n onto \bar{V}_n such that

$$\bar{g} = e^{2\sigma}g \tag{36}$$

where σ is a function on V_n . If σ is constant, then it is called homothetic mapping. In local coordinates, (36) is written as

$$\bar{g}_{ij}(x) = e^{2\sigma(x)}g_{ij}(x) \tag{37}$$

From (37), we obtain

$$\bar{g}^{ij} = e^{-2\sigma} g^{ij} \tag{38}$$

where g^{ij} and \bar{g}^{ij} are the inverse matrices of the metric tensor on V_n and \bar{V}_n , respectively.

Besides those equations, under this conformal mapping, the Christoffel symbols, the components of the curvature tensor, the Ricci tensor and the scalar curvature are, respectively, [6]

$$\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h + \delta_i^h \sigma_j + \delta_j^h \sigma_i - \sigma^h g_{ij}, \tag{39}$$

$$\begin{aligned} \bar{R}_{ijk}^h &= R_{ijk}^h + \delta_k^h \sigma_{ij} - \delta_j^h \sigma_{ik} + g^{h\alpha} (\sigma_{\alpha k} g_{ij} - \sigma_{\alpha j} g_{ik}) \\ &+ \Delta_1 \sigma (\delta_k^h g_{ij} - \delta_j^h g_{ik}), \end{aligned} \tag{40}$$

$$\bar{S}_{ij} = S_{ij} + (n - 2)\sigma_{ij} + (\Delta_2 \sigma + (n - 2)\Delta_1 \sigma)g_{ij}, \tag{41}$$

$$\bar{r} = e^{-2\sigma} (r + 2(n - 1)\Delta_2 \sigma + (n - 1)(n - 2)\Delta_1 \sigma), \tag{42}$$

where $S_{ij} = R_{ij\alpha}^\alpha$, $r = S_{\alpha\beta} g^{\alpha\beta}$, $\sigma_i = \frac{\partial \sigma}{\partial x^i} = \nabla_i \sigma$, $\sigma^h = \sigma_\alpha g^{\alpha h}$ and

$$\sigma_{ij} = \nabla_j \nabla_i \sigma - \nabla_i \sigma \nabla_j \sigma, \tag{43}$$

$\Delta_1 \sigma$ and $\Delta_2 \sigma$ are the first and the second Beltrami's symbols which are determined by

$$\Delta_1 \sigma = g^{\alpha\beta} \nabla_\alpha \sigma \nabla_\beta \sigma, \quad \Delta_2 \sigma = g^{\alpha\beta} \nabla_\beta \nabla_\alpha \sigma \tag{44}$$

where ∇ is the covariant derivative according to the Riemannian connection in V_n . We denote the objects of space conformally corresponding to V_n by bar, i.e., \bar{V}_n .

Taking the covariant derivative of \bar{S}_{ij} and using (41), it can be obtained that

$$\begin{aligned} \bar{\nabla}_k \bar{S}_{ij} &= \nabla_k S_{ij} + (n - 2)\nabla_k \sigma_{ij} + \partial_k (\Delta_2 \sigma + (n - 2)\Delta_1 \sigma)g_{ij} - 2\sigma_k S_{ij} \\ &- \sigma_i S_{jk} - \sigma_j S_{ik} - 2(\Delta_2 \sigma + (n - 2)\Delta_1 \sigma)g_{ij} \sigma_k + \sigma^h (S_{ih} g_{jk} + S_{hj} g_{ik}) \\ &+ (n - 2)(\sigma^h \sigma_{hj} g_{ik} + \sigma^h \sigma_{ih} g_{jk} - 2\sigma_k \sigma_{ij} - \sigma_i \sigma_{kj} - \sigma_j \sigma_{ik}) \end{aligned} \tag{45}$$

where $\bar{\nabla}$ and ∇ denote the Levi-Civita connections and ∂_k is the partial derivative with respect x^k .

Definition 4.2. A symmetric tensor field T of type $(0,2)$ on a Riemannian manifold (M, g) is said to be a Codazzi tensor if it satisfies the following condition

$$(\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z) \tag{46}$$

for arbitrary vector fields X, Y and Z .

Now, we assume that the Ricci tensors \bar{S} and S of the quasi-Einstein manifolds are Codazzi tensors with respect to the Levi-Civita connections $\bar{\nabla}$ and ∇ , respectively. Then, from (46), we have the following relations

$$\bar{\nabla}_k \bar{S}_{ij} = \bar{\nabla}_j \bar{S}_{ik}, \tag{47}$$

and

$$\nabla_k S_{ij} = \nabla_j S_{ik}. \tag{48}$$

On the other hand, if the Ricci tensor of the manifold is a Codazzi tensor, then from the second Bianchi identity, it can be seen that the scalar curvature is constant. According to our assumptions, the scalar curvatures \bar{r} and r of the quasi-Einstein manifolds are constants. So, we state and prove the following theorems.

Theorem 4.3. *Let us consider a conformal mapping $\bar{g} = e^{2\sigma}g$ of quasi-Einstein manifolds whose Ricci tensors are Codazzi type. If the vector field generated by the 1-form σ is a $\sigma(\text{Ric})$ -vector field, then either this conformal mapping is homothetic or the relation*

$$\mu = \frac{(2-n)(n-1)c-r}{2(n-1)r} \tag{49}$$

is satisfied where c is the square of the length of $\sigma_i = \frac{\partial\sigma}{\partial x^i} = \partial_i\sigma$ and μ denotes the constant corresponding to the $\sigma(\text{Ric})$ -vector field and $r \neq 0$.

Proof. Suppose that the Ricci tensors of V_n and \bar{V}_n are Codazzi tensors and suppose that $\bar{g} = e^{2\sigma}g$ is a conformal mapping with a $\sigma(\text{Ric})$ -vector field. By using the second Bianchi identity, it can be seen that the scalar curvatures r and \bar{r} are constants. Since r is constant, then the length of σ_i is constant by Theorem 2.3., (and $r \neq 0$ which can be seen from Theorem 2.5. and Corollary 2.6.) and so we have the condition

$$\sigma_i\sigma^i = c \tag{50}$$

where c is a constant. If we assume that the vector field generated by the 1-form σ in the conformal mapping (36) is a $\sigma(\text{Ric})$ -vector field, we get

$$\nabla_j\sigma_i = \mu S_{ij} \tag{51}$$

where μ is a constant. Using (44), (50) and (51), we have the following relations

$$\Delta_2\sigma = \mu r, \quad \Delta_1\sigma = c \tag{52}$$

and so, $\Delta_1\sigma$ and $\Delta_2\sigma$ are constants.

Using the relations (52) in (42), we find

$$\bar{r} = e^{-2\sigma}B \tag{53}$$

where r, \bar{r} and $B = [r + 2(n-1)\mu r + (n-1)(n-2)c]$ are constants. In this case, if \bar{r} is non-zero then we get from (53) that B is non-zero and so, $e^{-2\sigma}$ is constant. Thus, σ is constant. Therefore, this mapping is homothetic. If \bar{r} is zero then B must be zero. So we obtain

$$\mu = \frac{(2-n)(n-1)c-r}{2(n-1)r} \tag{54}$$

where $r \neq 0$. From (54), it can be seen that if $r \neq (2-n)(n-1)c$, that is, $\mu \neq 0$ then this conformal mapping admits a proper $\sigma(\text{Ric})$ -vector field. In case of $r = (2-n)(n-1)c$, then σ_i is covariantly constant. This completes the proof. \square

Next we consider a conformal mapping between two quasi-Einstein manifolds admitting a concircular vector field σ_i .

Theorem 4.4. *Let us consider a conformal mapping $\bar{g} = e^{2\sigma}g$ of quasi-Einstein manifolds whose Ricci tensors are Codazzi type. If σ_i is a concircular vector field, then either*

- i. ϕ_i and σ_i are orthogonal or
- ii. the function ρ is found as

$$\rho = \frac{b - (n-2)\Delta_1\sigma}{n+2}$$

where ϕ_i denote the components of the vector field associated 1-form ϕ , $\sigma_i = \frac{\partial\sigma}{\partial x^i} = \partial_i\sigma$, b is the associated scalar of V_n and ρ denotes the function corresponding to the concircular vector field.

Proof. Let the Ricci tensors of V_n and \bar{V}_n be Codazzi tensors and σ_i be a concircular vector field. In this case, we have from (9)

$$\nabla_j \sigma_i = \rho g_{ij} \tag{55}$$

where ρ is a function.

Changing the indices j and k in (45) and subtracting the last equation from (45) and using (43), (47), (48) and (55), it can be seen that

$$2(n-1)(\rho_k g_{ij} - \rho_j g_{ik}) + [(n-2)\Delta_1\sigma + (n+2)\rho](\sigma_j g_{ik} - \sigma_k g_{ij}) + \sigma_j S_{ik} - \sigma_k S_{ij} + \sigma^h S_{hj} g_{ik} - \sigma^h S_{hk} g_{ij} = 0 \tag{56}$$

where $\rho_k = \partial_k \rho$ and $\Delta_1\sigma = \sigma^h \sigma_h$.

Multiplying (56) by g^{ij} , it is obtained that

$$2(n-1)^2 \rho_k + [(n-2)(1-n)\Delta_1\sigma + (n+2)(1-n)\rho - r]\sigma_k + (2-n)\sigma^h S_{hk} = 0. \tag{57}$$

On the other hand, we have from the Ricci identity and the equation (55)

$$\sigma_\alpha R_{ijk}^\alpha = \rho_k g_{ij} - \rho_j g_{ik} \tag{58}$$

where R_{ijk}^α denote the components of the curvature tensor.

Multiplying (58) by g^{ij} , we get

$$\sigma_\alpha S_k^\alpha = (n-1)\rho_k. \tag{59}$$

Substituting ρ_k obtained from (59) in (57), it can be obtained that

$$n\sigma^h S_{hk} + [(n-2)(1-n)\Delta_1\sigma + (n+2)(1-n)\rho - r]\sigma_k = 0. \tag{60}$$

Considering (2) in (60) and using (4), we get

$$nb\sigma^h \phi_h \phi_k + [(n-2)(1-n)\Delta_1\sigma + (n+2)(1-n)\rho - b]\sigma_k = 0. \tag{61}$$

Multiplying (61) by ϕ^k and using (3), we obtain

$$[(n-1)b + (n-2)(1-n)\Delta_1\sigma + (n+2)(1-n)\rho]\sigma^k \phi_k = 0. \tag{62}$$

From (62), we see that either

$$\sigma^k \phi_k = 0$$

or

$$(n-1)b + (n-2)(1-n)\Delta_1\sigma + (n+2)(1-n)\rho = 0.$$

Thus, we obtain that either σ_k is orthogonal to ϕ_k or the function ρ is found as

$$\rho = \frac{b - (n-2)\Delta_1\sigma}{n+2}. \tag{63}$$

Hence, the proof is completed. \square

Now, we consider a conharmonic transformation between two quasi-Einstein manifolds V_n and \bar{V}_n . A harmonic function is not transformed into a harmonic function by this conformal transformation in general. The conharmonic transformation is a conformal transformation preserving the harmonicity of a certain function. If the conformal mapping is also conharmonic, then we have, [11]

$$\nabla_i \sigma^i + \frac{1}{2}(n-2)\sigma^i \sigma_i = 0. \tag{64}$$

Using the above relation, we can state and prove the following theorem.

Theorem 4.5. *Let us consider the conformal mapping of two quasi-Einstein manifolds V_n and \bar{V}_n . A necessary and sufficient condition for this conformal mapping to be conharmonic is that the associated scalars \bar{a} and \bar{b} be transformed by $\bar{a} = e^{-2\sigma}a$ and $\bar{b} = e^{-2\sigma}b$.*

Proof. We consider a conformal mapping of quasi-Einstein manifolds V_n and \bar{V}_n . Then, we have from (2) and (41)

$$\bar{a}\bar{g}_{ij} + \bar{b}\bar{\phi}_i\bar{\phi}_j = ag_{ij} + b\phi_i\phi_j + (n-2)\sigma_{ij} + (\Delta_2\sigma + (n-2)\Delta_1\sigma)g_{ij}. \quad (65)$$

Multiplying (65) by \bar{g}^{ij} and using (3), (38), (43) and (44), it can be seen that the following relation is satisfied

$$n\bar{a} + \bar{b} = e^{-2\sigma}[na + b + 2(n-1)\Delta_2\sigma + (n-1)(n-2)\Delta_1\sigma]. \quad (66)$$

If the conformal mapping is also conharmonic, then we have from (44) and (64)

$$2\Delta_2\sigma + (n-2)\Delta_1\sigma = 0. \quad (67)$$

Considering (67) in (66), it is found that

$$n\bar{a} + \bar{b} = nae^{-2\sigma} + be^{-2\sigma}. \quad (68)$$

From the equation (68), it can be seen that the associated scalars are transformed by

$$\bar{a} = e^{-2\sigma}a \quad \text{and} \quad \bar{b} = e^{-2\sigma}b. \quad (69)$$

Conversely, if the associated scalars of the manifolds are transformed by (69), then we have from (66)

$$2(n-1)\Delta_2\sigma + (n-1)(n-2)\Delta_1\sigma = 0 \quad (70)$$

and so, we get the relation (64). Thus, the conformal mapping is also conharmonic. This completes the proof. \square

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