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On Hermite-Hadamard Type Integral Inequalities for n-times Differentiable m- and (α, m)-Logarithmically Convex Functions

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Abstract. In this paper, we establish Hermite-Hadamard type inequalities for functions whose nth derivatives are m- and (α, m)-logarithmically convex functions. From our results, several results for classical trapezoidal and classical midpoint inequalities are obtained in terms second derivatives that are m- and (α, m)-logarithmically convex functions as special cases.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I and $a, b \in \mathbb{R}$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

The double inequality (1) was firstly discovered by Ch. Hermite [11] in 1881 in the journal Mathesis but was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result. E. F. Beckenbach [3], a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard [10] in 1893. Later on, in 1974, D. S. Mitrinović [19] found Hermite's note in Mathesis. This is why, the inequality (1) is now commonly referred as the Hermite-Hadamard inequality.

The inequality (1) has been subject of extensive research and has been refined and generalized by a number of mathematicians for over one hundred years see for instance [2], [5]-[1], [12]-[17], [20], [22]-[21], [26]-[29] and the references therein.

In recent years lot of generalizations of classical convexity have been given by a number of mathematicians, some of these are given as follows.

Definition 1.1. *[25] A function $f : [0, b] \to \mathbb{R}$ is said to be m-convex if*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

holds for all $x, y \in [0, b]$, $t \in [0, 1]$ and $m \in (0, 1]$.

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Definition 1.2. *[18] A function $f:[0,b]\to\mathbb{R}$ is said to be m-convex if*

$$f(tx+m(1-t)y)\le t^{\alpha}f(x)+m(1-t^{\alpha})f(y)$$

holds for all $x,y\in[0,b]$, $t\in[0,1]$ and $(\alpha,m)\in(0,1]\times(0,1]$.

Most recently, the above definitions are further generalized in [1] as follows.

Definition 1.3. *[1] A function $f:[0,b]\to(0,\infty)$ is said to be m-logarthmically convex if*

$$f(tx+m(1-t)y)\le[f(x)]^{t}[f(y)]^{m(1-t)}$$

holds for all $x,y\in[0,b]$, $t\in[0,1]$ and $m\in(0,1]$.

Definition 1.4. *[1] A function $f:[0,b]\to(0,\infty)$ is said to be (α,m)-logarithmically convex if*

$$f(tx+m(1-t)y)\le[f(x)]^{t^{\alpha}}[f(y)]^{m(1-t^{\alpha})}$$

holds for all $x,y\in[0,b]$, $t\in[0,1]$ and $(\alpha,m)\in(0,1]\times(0,1]$.

Bai et al. obtained the following Hermite-Hadamard type inequalites for m- and (α,m)-logarithmically convex functions.

Theorem 1.5. *[1] Let $I\subseteq[0,\infty)$ be an open real interval and let $f:I\to(0,\infty)$ be a differentiable function on I such that $f'\in L([a,b])$ for $0\le a<b<\infty$. If $|f'(x)|^{q}$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $q\in[1,\infty)$, $(\alpha,m)\in(0,1]\times(0,1]$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{(b-a)}{2}\left|f'\left(\frac{b}{m}\right)\right|^{m}\left(\frac{1}{2}\right)^{1-1/q}[E_1(\alpha,m,q)]^{1/q}, \tag{2}$$

where

$$\mu=\frac{|f'(a)|}{\left|f'\left(\frac{b}{m}\right)\right|^{m}},\ E_1(\alpha,m,q)=\begin{cases}\frac{1}{2}, & \mu=1\\ F_1(\mu,\alpha q), & 0<\mu<1\\ F_1\left(\mu,\frac{q}{\alpha}\right), & \mu>1\end{cases}$$

and

$$F_1(u,v)=\frac{1}{v^2(\ln u)^2}\left[v(u^{v}-1)\ln u-2\left(u^{v/2}-1\right)^2\right]$$

for $u,v>0$, $u\ne1$.

Corollary 1.6. *[1] Let $I\subseteq[0,\infty)$ be an open real interval and let $f:I\to(0,\infty)$ be a differentiable function on I such that $f'\in L([a,b])$ for $0\le a<b<\infty$. If $|f'(x)|^{q}$ is m-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $q\in[1,\infty)$, $m\in(0,1]$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{(b-a)}{2}\left|f'\left(\frac{b}{m}\right)\right|^{m}\left(\frac{1}{2}\right)^{1-1/q}[E_1(1,m,q)]^{1/q}, \tag{3}$$

where $E_1(\alpha,m,q)$ is defined as in Theorem 1.5.

Theorem 1.7. *[1] Let $I \subseteq [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $\left|f'(x)\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $q \in [1,\infty)$, $(\alpha,m) \in (0,1]\times(0,1]$, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4}\left|f'\left(\frac{b}{m}\right)\right|^m\left(\frac{1}{2}\right)^{3-1/q}\left[E_2(\alpha,m,q)\right], \tag{4}$$

where

$$\mu=\frac{\left|f'(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^m},\ E_1(\alpha,m,q)=\begin{cases}2\left(\frac{1}{8}\right)^{1/q}, & \mu=1\\ \left[F_2(\mu,\alpha q)\right]^{1/q}+\left[F_3(\mu,\alpha q)\right]^{1/q}, & 0<\mu<1\\ \left[F_2\left(\mu,\frac{q}{\alpha}\right)\right]^{1/q}+\left[F_3\left(\mu,\frac{q}{\alpha}\right)\right]^{1/q}, & \mu>1\end{cases}$$

and

$$F_2(u,v)=\frac{1}{v^2(\ln u)^2}\left[\frac{v}{2}u^{v/2}\ln u-u^{v/2}+1\right]$$
$$F_3(u,v)=\frac{1}{v^2(\ln u)^2}\left[u^v-\frac{v}{2}u^{v/2}\ln u-u^{v/2}\right]$$

for $u, v > 0$, $u \neq 1$.

Corollary 1.8. *Let $I \subseteq [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $\left|f'(x)\right|^q$ is m-logarthmically convex on $\left[0,\frac{b}{m}\right]$ for $q \in [1,\infty)$, $m \in (0,1]$, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4}\left|f'\left(\frac{b}{m}\right)\right|^m\left(\frac{1}{2}\right)^{3-1/q}\left[E_2(1,m,q)\right], \tag{5}$$

where $E_2(\alpha,m,q)$ is defined as in Theorem 1.7.

The main purpose of the present paper to establish new Hermite-Hadamard type inequalities for functions whose nth derivatives in absolute value are m- and (α,m)-logarithmically convex. These results not only generalize the results from [1] but many other interesting results can be obtained for functions whose second derivatives in absolute value are m- and (α,m)-logarithmically convex which may be better than those from [1].

2. Main Results

First we quote and establish some useful lemmas to prove our mains results.

Lemma 2.1. *[12] Suppose $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on I° for $n \in \mathbb{N}$, $n \ge 1$. If $f^{(n)}$ is integrable on $[a,b]$, for $a,b \in I$ with $a > b$, the equality holds*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)$$
$$=\frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)\,f^{(n)}(ta+(1-t)b)dt, \tag{6}$$

where the sum above takes 0 when $n = 1$ and $n = 2$.

Lemma 2.2. *Suppose $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on I° for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, b]$, for $a, b \in I$ with $a > b$, the equality holds*

$$\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$= \frac{(-1)(b-a)^n}{n!}\int_0^1 K_n(t) f^{(n)}(ta + (1-t)b)dt, \tag{7}$$

where

$$K_n(t) := \begin{cases} t^n, & t \in \left[0, \frac{1}{2}\right] \\ (t-1)^n, & t \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

Proof. For $n = 1$, we have

$$(-1)(b-a)\int_0^1 K_1(t) f^{(n)}(ta + (1-t)b)dt$$
$$= -(b-a)\int_0^{\frac{1}{2}} t f'(ta + (1-t)b)dt - (b-a)\int_{\frac{1}{2}}^1 (t-1) f'(ta + (1-t)b)dt$$
$$= f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx,$$

which is the left hand side of (7) for $n = 1$.
Suppose that (7) holds for $n = m - 1$, $m > 2$, that is

$$\sum_{k=0}^{m-2} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$= \frac{(-1)(b-a)^{m-1}}{(m-1)!}\int_0^1 K_{m-1}(t) f^{(m-1)}(ta + (1-t)b)dt. \tag{8}$$

Now for $n = m$, by integration by parts and using (8), we have

$$\frac{(-1)(b-a)^m}{m!}\int_0^1 K_m(t) f^{(m)}(ta + (1-t)b)dt$$
$$= \frac{\left[(-1)^{m-1} + 1\right](b-a)^{m-1}}{2^m m!} f^{(m-1)}\left(\frac{a+b}{2}\right)$$
$$+ \frac{(-1)(b-a)^{m-1}}{(m-1)!}\int_0^1 K_{m-1}(t) f^{(m-1)}(ta + (1-t)b)dt$$
$$= \frac{(b-a)^{m-1}\left[(-1)^m + 1\right]}{2^m m!} f^{(m-1)}\left(\frac{a+b}{2}\right)$$
$$+ \sum_{k=0}^{m-2} \frac{\left[(-1)^{k+1} - 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx, \tag{9}$$

which is the required identity (7). This completes the proof of the Lemma. □

The following useful result will also help us establishing our results:

Lemma 2.3. *If $\mu > 0$, $\mu \neq 1$ and $\mu \in N \cup \{0\}$, then*

$$\int_0^1 t^n \mu^t dt = \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,(\ln \mu)^{k+1}}. \tag{10}$$

Proof. For $n = 0$, we have

$$\int_0^1 \mu^t dt = \frac{\mu - 1}{\ln \mu},$$

which coincides with the right hand side of (10) for $n = 0$.

For $n = 1$, we have

$$\int_0^1 t\mu^t dt = \frac{\mu}{\ln \mu} - \frac{\mu}{(\ln \mu)^2} + \frac{1}{(\ln \mu)^2},$$

and it coincides with the right hand side of (10) for $n = 1$.

Suppose (10) is true for $n - 1$, i.e.

$$\int_0^1 t^{n-1} \mu^t dt = \frac{(-1)^n (n-1)!}{(\ln \mu)^n} + (n-1)!\mu \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)!\,(\ln \mu)^{k+1}}. \tag{11}$$

Now by integration by parts and using (11), we have

$$\begin{aligned}
\int_0^1 t^n \mu^t dt &= \frac{\mu}{\ln \mu} - \frac{n}{\ln \mu} \int_0^1 t^{n-1} \mu^t dt \\
&= \frac{\mu}{\ln \mu} - \frac{n}{\ln \mu} \left[\frac{(-1)^n (n-1)!}{(\ln \mu)^n} + (n-1)!\mu \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)!\,(\ln \mu)^{k+1}} \right] \\
&= \frac{\mu}{\ln \mu} + \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(n-1-k)!\,(\ln \mu)^{k++2}} \\
&= \frac{n!\mu}{n! \ln \mu} + \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=1}^{n} \frac{(-1)^k}{(n-k)!\,(\ln \mu)^{k+1}} \\
&= \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,(\ln \mu)^{k+1}}.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.4. *If $\mu > 0$, $\mu \neq 1$ and $\mu \in N \cup \{0\}$, then*

$$\int_0^{\frac{1}{2}} t^n \mu^t dt = \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu^{1/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k} (n-k)!\,(\ln \mu)^{k+1}}. \tag{12}$$

Proof. It follows from Lemma 2.3 after making use of the substitution $t = \frac{u}{2}$. □

Lemma 2.5. *If $\mu > 0$, $\mu \neq 1$ and $\mu \in N \cup \{0\}$, then*

$$\int_{\frac{1}{2}}^1 (1-t)^n \mu^t dt = \frac{n!\mu}{(\ln \mu)^{n+1}} - n!\mu^{1/2} \sum_{k=0}^{n} \frac{1}{2^{n-k} (n-k)!\,(\ln \mu)^{k+1}}. \tag{13}$$

Proof. It follows from Lemma 2.4 after making the substitution $1 - t = u$. □

Lemma 2.6. *[26] For $\alpha > 0$ and $\mu > 0$, we have*

$$I(\alpha,\mu) := \int_0^1 t^{\alpha-1}\mu^t dt = \mu \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\ln\mu)^{k-1}}{(\alpha)_k} < \infty,$$

where

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+k-1).$$

Moreover, it holds

$$\left| I(\alpha,\mu) - \mu \sum_{k=1}^{m} (-1)^{k-1} \frac{(\ln\mu)^{k-1}}{(\alpha)_k} \right| \le \frac{|\ln\mu|}{\alpha\sqrt{2\pi(m-1)}} \left(\frac{|\ln\mu| e}{m-1} \right)^{m-1}.$$

We are now ready to set off our first result.

Theorem 2.7. *Let $I \subseteq [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I and $f^{(n)}$ is integrable on $[a,b]$ for $n \in \mathbb{N}$, $n \ge 2$, $0 \le a < b < \infty$. If $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $q \in [1,\infty)$, $(\alpha,m) \in (0,1] \times (0,1]$, we have*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \le \frac{(b-a)^n}{2n!} \left| f^{(n)}\left(\frac{b}{m}\right) \right|^m \left(\frac{n-1}{n+1}\right)^{1-1/q} \left[E_1(\alpha,\mu,n,q)\right]^{1/q}, \quad (14)$$

where

$$\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m},\ E_1(\alpha,m,n,q) = \begin{cases} \frac{n-1}{n+1}, & \mu = 1 \\ F_1(\mu,\alpha q,n), & 0<\mu<1 \\ \mu^{q(1-\alpha)}F_1(\mu,\alpha q,n), & \mu > 1 \end{cases}$$

and

$$F_1(u,v,n) = \frac{(-1)^n n!\,[v\ln u + 2]}{v^{n+1}(\ln u)^{n+1}} - \frac{2u^v}{\ln u} - n!u^v \sum_{k=1}^{n} \frac{(-1)^k [v\ln u + 2]}{v^{k+1}(n-k)!(\ln u)^{k+1}}$$

for $u, v > 0$, $u \ne 1$.

Proof. Suppose $n \ge 2$ and $a,b \in I$, $0 \le a < b < \infty$. By (α,m)-logarithmically convexity of $\left|f^{(n)}\right|^q$ on $\left[0,\frac{b}{m}\right]$, Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\ &\le \frac{(b-a)^n \left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2n!} \left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-1/q} \left(\int_0^1 t^{n-1}(n-2t)\,\mu^{qt^\alpha}dt\right)^{1/q} \\ &= \frac{(b-a)^n \left|f^{(n)}(\frac{b}{m})\right|^m}{2n!} \left(\frac{n-1}{n+1}\right)^{1-1/q} \left(n\int_0^1 t^{n-1}\mu^{qt^\alpha}dt - 2\int_0^1 t^n \mu^{qt^\alpha}dt\right)^{1/q}, \end{aligned} \quad (15)$$

where $\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.
For $\mu = 1$, we have

$$n \int_0^1 t^{n-1} \mu^{qt^\alpha} dt - 2 \int_0^1 t^n \mu^{qt^\alpha} dt = n \int_0^1 t^{n-1} dt - 2 \int_0^1 t^n dt = \frac{n-1}{n+1}. \tag{16}$$

For $0 < \mu < 1$, $\mu^{qt^\alpha} \leq \mu^{\alpha qt}$ and hence by using Lemma 2.3

$$\begin{aligned} & n \int_0^1 t^{n-1} \mu^{qt^\alpha} dt - 2 \int_0^1 t^n \mu^{qt^\alpha} dt \leq n \int_0^1 t^{n-1} \mu^{\alpha qt} dt - 2 \int_0^1 t^n \mu^{\alpha qt} dt \\ & = \frac{(-1)^n n!}{(\alpha q)^n (\ln \mu)^n} - n! \mu^{\alpha q} \sum_{k=1}^n \frac{(-1)^k}{(\alpha q)^k (n-k)! (\ln \mu)^k} \\ & - \frac{2(-1)^{n+1} n!}{(\alpha q)^{n+1} (\ln \mu)^{n+1}} - 2n! \mu^{\alpha q} \sum_{k=0}^n \frac{(-1)^k}{(\alpha q)^k (n-k)! (\ln \mu)^{k+1}} \\ & = \frac{(-1)^n n! \left[\alpha q \ln \mu + 2\right]}{(\alpha q)^{n+1} (\ln \mu)^{n+1}} - \frac{2\mu^{\alpha q}}{\ln \mu} - n! \mu^{\alpha q} \sum_{k=1}^n \frac{(-1)^k \left[\alpha q \ln \mu + 2\right]}{(\alpha q)^{k+1} (n-k)! (\ln \mu)^{k+1}}. \end{aligned} \tag{17}$$

For $\mu > 1$, $\mu^{qt^\alpha} \leq \mu^{q\alpha t + q - q\alpha}$, and hence by Lemma 2.3

$$\begin{aligned} & n \int_0^1 t^{n-1} \mu^{qt^\alpha} dt - 2 \int_0^1 t^n \mu^{qt^\alpha} dt \leq n \int_0^1 t^{n-1} \mu^{q\alpha t + q(1-\alpha)} dt - 2 \int_0^1 t^n \mu^{q\alpha t + q(1-\alpha)} dt \\ & = \mu^{q(1-\alpha)} \left[\frac{(-1)^n n! \left[(q\alpha) \ln \mu + 2\right]}{(q\alpha)^{n+1} (\ln \mu)^{n+1}} - \frac{2\mu^{q\alpha}}{\ln \mu} - n! \mu^{q\alpha} \sum_{k=1}^n \frac{(-1)^k \left[(q\alpha) \ln \mu + 2\right]}{(q\alpha)^{k+1} (n-k)! (\ln \mu)^{k+1}}\right]. \end{aligned} \tag{18}$$

Combining (16), (17) and (18), we obtain the required result. This completes the proof of the theorem. □

Corollary 2.8. *Suppose the assumptions of Theorem 2.7 are satisfied and if $q = 1$, we have*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a)\right| \leq \frac{(b-a)^n}{2n!} \left|f^{(n)}\left(\frac{b}{m}\right)\right|^m E_1(\alpha, \mu, n, 1), \tag{19}$$

where $E_1(\alpha, \mu, n, q)$ is as defined in Theorem 2.7.

Corollary 2.9. *Under the assumptions of Theorem 2.7, if $n = 2$, we have the inequality*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx\right| \leq \frac{(b-a)^2}{4} \left|f''\left(\frac{b}{m}\right)\right|^m \left(\frac{1}{3}\right)^{1-1/q} \left[E_1(\alpha, \mu, 2, q)\right]^{1/q}, \tag{20}$$

where

$$\mu = \frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m}, E_1(\alpha, m, 2, q) = \begin{cases} \frac{1}{3}, & \mu = 1 \\ F_1(\mu, \alpha q, 2), & 0 < \mu < 1 \\ \mu^{q(1-\alpha)} F_1(\mu, \alpha q, 2), & \mu > 1 \end{cases}$$

and

$$F_1(u, v, 2) = \frac{2v(1 + u^v) \ln u + 4(1 - u^v)}{v^3 (\ln u)^3}$$

for $u, v > 0$, $u \neq 1$.

Corollary 2.10. *Under the assumptions of Theorem 2.7, if $n = 2$ and $\alpha = 1$, then following inequality holds when the absolute value of the second derivative of f is an m-logarthmically convex function*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{4}\left|f''\left(\frac{b}{m}\right)\right|^m \left(\frac{1}{3}\right)^{1-1/q}\left[E_1(1,\mu,2,q)\right]^{1/q}, \quad (21)$$

where

$$\mu = \frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m},\; E_1(1,m,2,q) = \begin{cases} \frac{1}{3}, & \mu = 1 \\ F_1(\mu,q,2), & \mu > 0, \mu \neq 1 \end{cases}$$

and

$$F_1(u,v,2) = \frac{2v(1+u^v)\ln u + 4(1-u^v)}{v^3(\ln u)^3}$$

for $u, v > 0$, $u \neq 1$.

Remark 2.11. *The inequalities (20) and (21) may be better than those given in Theorem 1.5 and Corollary 1.6.*

Theorem 2.12. *Let $I \subseteq [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I and $f^{(n)}$ is integrable on $[a,b]$ for $n \in \mathbb{N}$, $n \geq 2$, $0 \leq a < b < \infty$. If $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $q \in (1,\infty)$, $(\alpha,m) \in (0,1]\times(0,1]$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\le \frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left[E_2(\alpha,m,n,q)\right]^{1/q}, \quad (22)$$

where

$$\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m},\; E_2(\alpha,m,n,q) = \begin{cases} \frac{1}{nq-q+1}, & \mu = 1 \\ F_2(\mu,\alpha q,n), & 0<\mu<1 \\ \mu^{q(1-\alpha)}F_2(\mu,\alpha q,n), & \mu > 1, \end{cases}$$

$$F_2(u,v,n) = u^v\sum_{k=1}^{\infty}\frac{(-1)^{k-1}v^{k-1}(\ln u)^{k-1}}{(nq-q+1)_k} < \infty$$

for $u, v > 0$, $u \neq 1$ and $(nq-q+1)_k = (nq-q+1)(nq-q+2)\cdots(nq-q+k)$.

Proof. Since $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $q \in (1,\infty)$, $(\alpha,m) \in (0,1]\times(0,1]$, Lemma 2.1 and Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\le \frac{(b-a)^n}{2n!}\left(\int_0^1 (n-2t)^{q/(q-1)}\,dt\right)^{1-1/q}\left(\int_0^1 t^{q(n-1)}\left|f^{(n)}(ta+(1-t)b\right|^q dt\right)^{1/q}$$
$$\le \frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left(\int_0^1 t^{q(n-1)}\mu^{qt^\alpha}dt\right)^{1/q}, \quad (23)$$

where $\mu = \frac{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{\left|f^{(n)}(a)\right|}$.

For $\mu = 1$, we have

$$\int_0^1 t^{q(n-1)}\mu^{qt^\alpha}dt = \int_0^1 t^{q(n-1)}dt = \frac{1}{nq-q+1}.$$

For $0 < \mu < 1$, $\mu^{qt^\alpha} \le \mu^{\alpha qt}$ and hence by using Lemma 2.6, we have

$$\begin{aligned}\int_0^1 t^{q(n-1)}\mu^{qt^\alpha}dt &\le \int_0^1 t^{q(n-1)}\mu^{q\alpha t}dt \\ &\le \mu^{\alpha q}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\alpha q)^{k-1}(\ln\mu)^{k-1}}{(nq-q+1)_k} < \infty.\end{aligned}$$

For $\mu > 1$, $\mu^{qt^\alpha} \le \mu^{q\alpha t+q(1-\alpha)}$, and hence by Lemma 2.6, we obtain

$$\begin{aligned}\int_0^1 t^{q(n-1)}\mu^{qt^\alpha}dt &\le \int_0^1 t^{q(n-1)}\mu^{q\alpha t+q(1-\alpha)}dt \\ &\le \mu^{q(1-\alpha)}\left(\mu^{q\alpha}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(q\alpha)^{k-1}(\ln\mu)^{k-1}}{(nq-q+1)_k}\right) < \infty.\end{aligned}$$

Thus the inequality (22) follows. This completes the proof of the theorem. □

Corollary 2.13. *Suppose the assumptions of Theorem 2.12 are satisfied and $n = 2$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f''\left(\frac{b}{m}\right)\right|^m \left[E_2(\alpha,m,2,q)\right]^{1/q}, \quad (24)$$

where

$$\mu = \frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m},\ E_2(\alpha,m,2,q) = \begin{cases} \frac{1}{q+1}, & \mu = 1 \\ F_2(\mu,\alpha q,2), & 0<\mu<1 \\ \mu^{q(1-\alpha)}F_2(\mu,\alpha q,2), & \mu>1, \end{cases}$$

$$F_2(u,v,2) = u^v\sum_{k=1}^{\infty}\frac{(-1)^{k-1}v^{k-1}(\ln u)^{k-1}}{(q+1)_k} < \infty$$

for $u, v > 0$, $u \ne 1$ and $(q+1)_k = (q+1)(q+2)\cdots(q+k)$.

Corollary 2.14. *Suppose the assumptions of Theorem 2.12 are satisfied and $n = 2$, $\alpha = 1$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f''\left(\frac{b}{m}\right)\right|^m \left[E_2(1,m,2,q)\right]^{1/q}, \quad (25)$$

where

$$\mu = \frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m},\ E_2(1,m,2,q) = \begin{cases} \frac{1}{q+1}, & \mu = 1 \\ F_2(\mu,q,2), & \mu>1, \mu\ne 1 \end{cases}$$

$$F_2(u,v,2) = u^v\sum_{k=1}^{\infty}\frac{(-1)^{k-1}v^{k-1}(\ln u)^{k-1}}{(q+1)_k} < \infty$$

for $u, v > 0$, $u \ne 1$ and $(q+1)_k = (q+1)(q+2)\cdots(q+k)$.

Now we give some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable log-preinvex functions.

Theorem 2.15. *Let $I \subseteq [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I and $f^{(n)}$ is integrable on $[a,b]$ for $n \in \mathbb{N}$, $0 \le a < b < \infty$. If $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $q \in [1,\infty)$, $(\alpha,m) \in (0,1]\times(0,1]$, we have*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2^{(n+1)(q-1)/q}(n+1)^{1-1/q}n!}E_3(\alpha,m,n,q), \quad (26)$$

where

$$\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m},\ E_3(\alpha,m,n,q)=\begin{cases}\frac{2}{2^{(n+1)/q}(n+1)^{1/q}}, & \mu=1\\ \begin{array}{l}\left[F_3(\mu,\alpha q,n)\right]^{1/q}\\ +\left[F_4(\mu,\alpha q,n)\right]^{1/q},\end{array} & 0<\mu<1\\ \mu^{1-\alpha}\left\{\begin{array}{c}\left[F_3(\mu,\alpha q,n)\right]^{1/q}\\ +\left[F_4(\mu,\alpha q,n)\right]^{1/q}\end{array}\right\}, & \mu>1,\end{cases}$$

and

$$F_3(u,v,n)=\frac{(-1)^{n+1}n!}{v^{n+1}(\ln u)^{n+1}}+n!u^{v/2}\sum_{k=0}^{n}\frac{(-1)^k}{2^{n-k}v^{k+1}(n-k)!(\ln u)^{k+1}},$$
$$F_4(u,v,n)=\frac{n!u}{v^{n+1}(\ln u)^{n+1}}-n!u^{v/2}\sum_{k=0}^{n}\frac{1}{2^{n-k}v^{k+1}(n-k)!(\ln u)^{k+1}}$$

for $u,v>0$, $u\neq 1$.

Proof. Suppose $n \ge 1$. By using Lemma 2.2, the (α,m)-logarithmically convexity of $\left|f^{(n)}\right|$ and the Hölder inequality, we have

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
&\le\frac{(b-a)^n}{n!}\left[\int_0^{\frac12}t^n\left|f^{(n)}(ta+(1-t)b)\right|dt+\int_{\frac12}^1(1-t)^n\left|f^{(n)}(ta+(1-t)b)\right|dt\right]\\
&\le\frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n!}\left[\left(\int_0^{\frac12}t^n dt\right)^{1-1/q}\left(\int_0^{\frac12}t^n\mu^{qt^\alpha}dt\right)^{1/q}+\left(\int_{\frac12}^1(1-t)^n dt\right)^{1-1/q}\left(\int_{\frac12}^1(1-t)^n\mu^{qt^\alpha}dt\right)^{1/q}\right]\\
&=\frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n!2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}\left[\left(\int_0^{\frac12}t^n\mu^{qt^\alpha}dt\right)^{1/q}+\left(\int_{\frac12}^1(1-t)^n\mu^{qt^\alpha}dt\right)^{1/q}\right],\end{aligned} \quad (27)$$

where $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.

For $\mu = 1$, we have

$$\left(\int_0^{\frac{1}{2}} t^n \mu^{qt^\alpha} dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^{qt^\alpha} dt\right)^{1/q}$$
$$= \left(\int_0^{\frac{1}{2}} t^n dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1/q} = \frac{2}{2^{(n+1)/q} (n+1)^{1/q}}.$$

For $0 < \mu < 1$, $\mu^{qt^\alpha} \leq \mu^{\alpha q t}$ and hence by using Lemma 2.4 and Lemma 2.5, we have

$$\left(\int_0^{\frac{1}{2}} t^n \mu^{qt^\alpha} dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^{qt^\alpha} dt\right)^{1/q}$$
$$\leq \left(\frac{(-1)^{n+1} n!}{(\alpha q)^{n+1} (\ln \mu)^{n+1}} + n! \mu^{\alpha q/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k} (\alpha q)^{k+1} (n-k)! (\ln \mu)^{k+1}}\right)^{1/q}$$
$$+ \left(\frac{n! \mu}{(\alpha q)^{n+1} (\ln \mu)^{n+1}} - n! \mu^{\alpha q/2} \sum_{k=0}^{n} \frac{1}{2^{n-k} (\alpha q)^{k+1} (n-k)! (\ln \mu)^{k+1}}\right)^{1/q}.$$

For $\mu > 1$, $\mu^{qt^\alpha} \leq \mu^{q\alpha t + q(1-\alpha)}$ and hence by using Lemma 2.4 and Lemma 2.5, we have

$$\left(\int_0^{\frac{1}{2}} t^n \mu^{qt^\alpha} dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^{qt^\alpha} dt\right)^{1/q}$$
$$\leq \mu^{1-\alpha} \left(\frac{(-1)^{n+1} n!}{(q\alpha)^{n+1} (\ln \mu)^{n+1}} + n! \mu^{\alpha q/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k} (q\alpha)^{k+1} (n-k)! (\ln \mu)^{k+1}}\right)^{1/q}$$
$$+ \mu^{1-\alpha} \left(\frac{n! \mu}{(q\alpha)^{n+1} (\ln \mu)^{n+1}} - n! \mu^{\alpha q/2} \sum_{k=0}^{n} \frac{1}{2^{n-k} (q\alpha)^{k+1} (n-k)! (\ln \mu)^{k+1}}\right)^{1/q}.$$

Hence the inequality (26) follows from the above facts. This completes the proof of the theorem. □

Corollary 2.16. *Suppose the assumptions of Theorem 2.15 are fulfilled and if $q = 1$, we have*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1} (k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\, dx\right| \leq \frac{(b-a)^n \left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n!} E_3(\alpha, m, n, 1), \quad (28)$$

where $E_3(\alpha, m, n, q)$ is as defined in Theorem 2.15.

Remark 2.17. *Suppose the conditions of Theorem 2.15 are satisfied and if $n = 1$, we get the corrected inequality given in Theorem 1.7.*

Remark 2.18. *Suppose the conditions of Theorem 2.15 are satisfied and if $\alpha = 1$, we get the corrected inequality given in Corollary 1.7.*

Corollary 2.19. *Let $I \subseteq [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a function such that $f^{(n)}$ exists on I and $f^{(n)}$ is integrable on $[a, b]$ for $n \in \mathbb{N}$, $n \geq 1$, $0 \leq a < b < \infty$. If $\left|f^{(n)}\right|^q$ is m-logarthmically convex on $\left[0, \frac{b}{m}\right]$ for $q \in [1, \infty)$, $m \in (0, 1]$, we have*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1} (k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\, dx\right| \leq \frac{(b-a)^n \left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2^{(n+1)(q-1)/q} (n+1)^{1-1/q}\, n!} E_3(1, m, n, q), \quad (29)$$

where

$$\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^{m}},\ E_3(1,m,n,q)=\begin{cases}\frac{2}{2^{(n+1)/q}(n+1)^{1/q}}, & \mu=1\\ \left[F_3(\mu,q,n)\right]^{1/q}+\left[F_4(\mu,q,n)\right]^{1/q}, & \mu>0,\mu\neq 1\end{cases}$$

and

$$F_3(u,v,n)=\frac{(-1)^{n+1}n!}{v^{n+1}(\ln u)^{n+1}}+n!u^{v/2}\sum_{k=0}^{n}\frac{(-1)^k}{2^{n-k}v^{k+1}(n-k)!(\ln u)^{k+1}},$$

$$F_4(u,v,n)=\frac{n!u}{v^{n+1}(\ln u)^{n+1}}-n!u^{v/2}\sum_{k=0}^{n}\frac{1}{2^{n-k}v^{k+1}(n-k)!(\ln u)^{k+1}}$$

for $u,v>0$, $u\neq 1$.

Theorem 2.20. *Let* $I\subseteq[0,\infty)$ *be an open real interval and let* $f:I\to(0,\infty)$ *be a function such that* $f^{(n)}$ *exists on* I *and* $f^{(n)}$ *is integrable on* $[a,b]$ *for* $n\in\mathbb{N}$, $n\geq 1$, $0\leq a<b<\infty$. *If* $\left|f^{(n)}\right|^q$ *is* (α,m)*-logarithmically convex on* $\left[0,\frac{b}{m}\right]$ *for* $q\in(1,\infty)$, $(\alpha,m)\in(0,1]\times(0,1]$, *we have*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{(b-a)^n}{2^{n+1/p}(np+1)^{1/p}n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left\{\left[\frac{\alpha q\left(\frac{1}{2}\right)^{\alpha-1}\mu^{q\left(\frac{1}{2}\right)^{\alpha}}}{\ln\mu}\right]^{1/q}+\left[\frac{\alpha q\mu^q-\alpha q\left(\frac{1}{2}\right)^{\alpha-1}\mu^{q\left(\frac{1}{2}\right)^{\alpha}}}{\ln\mu}\right]^{1/q}\right\},\quad(30)$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.

Proof. From Lemma 2.2, the Hölder integral inequality and (α,m)-logarithmically convexity of $\left|f^{(n)}\right|^q$, we have

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n!}\left[\left(\int_0^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}\right)^{qt^{\alpha}}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}\right)^{qt^{\alpha}}dt\right)^{\frac{1}{q}}\right]\quad(31)$$

from which the required inequality follows. This completes the proof of the theorem. □

Corollary 2.21. *Under the assumptions of Theorem 2.20, if* $n=1$, *we have the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{(b-a)}{2^{1+1/p}(p+1)^{1/p}}\left|f'\left(\frac{b}{m}\right)\right|^m\left\{\left[\frac{\alpha q\left(\frac{1}{2}\right)^{\alpha-1}\mu^{q\left(\frac{1}{2}\right)^{\alpha}}}{\ln\mu}\right]^{1/q}+\left[\frac{\alpha q\mu^q-\alpha q\left(\frac{1}{2}\right)^{\alpha-1}\mu^{q\left(\frac{1}{2}\right)^{\alpha}}}{\ln\mu}\right]^{1/q}\right\},\quad(32)$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mu=\frac{|f'(a)|}{|f'(\frac{b}{m})|^m}$.

Corollary 2.22. *Let* $I\subseteq[0,\infty)$ *be an open real interval and let* $f:I\to(0,\infty)$ *be a functions such that* $f^{(n)}$ *exists on* I *and* $f^{(n)}$ *is integrable on* $[a,b]$ *for* $n\in\mathbb{N}$, $n\geq 1$, $0\leq a<b<\infty$. *If* $\left|f^{(n)}\right|^q$ *is m-logarthmically convex on* $\left[0,\frac{b}{m}\right]$ *for* $q\in(1,\infty)$, $m\in(0,1]$, *we have*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2^{n+1/p}(np+1)^{1/p}n!}\left(\frac{q}{\ln\mu}\right)^{1/q}\mu^{1/2}\left\{1+\left[\mu^{q/2}-1\right]^{1/q}\right\}, \quad (33)$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mu=\frac{|f^{(n)}(a)|}{|f^{(n)}(\frac{b}{m})|^m}$.

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