Filomat 28:7 (2014), 1333–1343
DOI 10.2298/FIL1407333Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Bahadur Representation of Sample Quantiles for Widely Orthant Dependent Sequences

Wenzhi Yang, Tingting Liu, Xuejun Wang, Shuhe Hu

School of Mathematical Science, Anhui University, Hefei, 230601, P.R. China

Abstract. It can be found that widely orthant dependent (WOD) random variables are weaker than extended negatively orthant dependent (END) random variables, while END random variables are weaker than negatively orthant dependent (NOD) and negatively associated (NA) random variables. In this paper, we investigate the Bahadur representation of sample quantiles based on WOD sequences. Our results extend the corresponding ones of Ling [N.X. Ling, The Bahadur representation for sample quantiles under negatively associated sequence, Statistics and Probability Letters 78(16) (2008), 2660–2663], Xu et al. [S.F. Xu, L. Ge, Y. Miao, On the Bahadur representation of sample quantiles and order statistics for NA sequences, Journal of the Korean Statistical Society 42(1) (2013), 1–7] and Li et al. [X.Q. Li, W.Z. Yang, S.H. Hu, X.J. Wang, The Bahadur representation for sample quantile under NOD sequence, Journal of Nonparametric Statistics 23(1) (2011), 59–65] for the case of NA sequences or NOD sequences.

1. Introduction

Assume that $\{X_n\}_{n\geq 1}$ is a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ with a common marginal distribution function $F(x) = P(X_1 \leq x)$. F is a distribution function (continuous from the right, as usual). For $0 < p < 1$, the pth quantile of F is defined as

$$\xi_p = \inf\{x : F(x) \geq p\}$$

and is alternately denoted by $F^{-1}(p)$. The function $F^{-1}(t)$, $0 < t < 1$, is called the inverse function of F. For a sample $X_1, X_2, \cdots, X_n$, $n \geq 1$, let F_n represent the empirical distribution function based on $X_1, X_2, \ldots, X_n$, which is defined as $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$, $x \in R$, where $I(A)$ denotes the indicator function of a set A and R is the real line. For $0 < p < 1$, we define $F_n^{-1}(p) = \inf\{x : F_n(x) \geq p\}$ as the pth quantile of sample.

Bahadur [3] consider the independent and identically distributed (*i.i.d.*) random variables and get the following results (or see Lemma 2.5.4 E and Theorem 2.5.1 in [15]).

2010 *Mathematics Subject Classification*. Primary 62G30
Keywords. Bahadur representation, Sample quantiles, Widely orthant dependent sequence, Negatively associated sequence
Received: 11 June 2013; Revised: 12 February 2014; Accepted: 12 March 2014
Communicated by Miroslav M. Ristić
Research supported by the NNSF of China (11171001, 11201001, 11326172, 11426032), Anhui Provincial Natural Science Foundation (1208085QA03, 1408085QA02), Higher Education Talent Revitalization Project of Anhui Province (2013SQRL005ZD) and Academic and Technology Leaders to Introduction Projects of Anhui University
Email address: hushuhe@263.net (Shuhe Hu)

Theorem 1.1. *Let $0 < p < 1$ and $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables. Suppose that $F(x)$ is twice differentiable at ξ_p, with $F'(\xi_p) = f(\xi_p) > 0$. Let $\{a_n\}$ be a sequence of positive constants such that*

$$a_n \sim c_0 n^{-1/2}(\log n)^q,\ n \to \infty,$$

for some constants $c_0 > 0$ and $q \geq 1/2$. Put

$$H_{p,n} = \sup_{|x|\leq a_n}\Big|[F_n(\xi_p + x) - F_n(x)] - [F(\xi_p + x) - F(\xi_p)]\Big|.$$

Then with probability 1

$$H_{p,n} = O(n^{-\frac{3}{4}}(\log n)^{\frac{1}{2(q+1)}}),\ n \to \infty.$$

Theorem 1.2. *Let $0 < p < 1$ and $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables. Suppose that $F(x)$ is twice differentiable at ξ_p, with $F'(\xi_p) = f(\xi_p) > 0$. Then with probability* 1

$$\xi_{p,n} = \xi_p - \frac{F_n(\xi_p) - p}{f(\xi_p)} + O(n^{-\frac{3}{4}}(\log n)^{\frac{3}{4}}),\ n \to \infty.$$

At present, many researchers have extended Theorem 1.1 and Theorem 1.2 for *i.i.d.* random variables to the many dependent cases of random variables. For example, Sen [14], Babu and Singh [2], Yoshihara [32], Sun [20], Wang et al. [25] and Zhang et al. [33] studied the Bahadur representation under the cases of φ-mixing sequences or strong mixing (α-mixing) sequences. Wendler [28] investigated the Bahadur representation for U-quantiles of α-mixing sequences and functionals of absolutely regular sequences. Wendler [29] also studied the generalized Bahadur representation for U-quantile processes and generalized linear statistics under dependent data such as α-mixing sequences and β-mixing sequences.

Meanwhile, Ling [10] and Xu et al. [31] investigated the Bahadur representation under the case of negatively associated (NA) sequences, Li et al. [9] extended and improved the results of Ling [10] to the case of negatively orthant dependent (NOD) random variables, which are weaker than NA random variables. For the other works on Bahadur representation and related works, one can refer to [4], [6], [7], [17], [30] and the references therein. In this paper, we study the Bahadur representation of sample quantiles based on widely orthant dependent (WOD) sequences, which are weaker than extended negatively orthant dependent (END) random variables. It is pointed out that END random variables are weaker than NOD and NA random variables. The concept of WOD random variables can be found in many paper such as in Wang et al. [27].

Definition 1.3. *For the random variables $\{X_n\}_{n\geq 1}$, if there exists a finite sequence of real numbers $\{g_u(n)\}_{n\geq 1}$ such that for each $n \geq 1$ and for all $x_i \in (-\infty, \infty), 1 \leq i \leq n$,*

$$P\Big(\bigcap_{i=1}^{n}(X_i > x_i)\Big) \leq g_u(n)\prod_{i=1}^{n} P(X_i > x_i),$$

then we say that the random variables $\{X_n\}_{n\geq 1}$ are widely upper orthant dependent (WUOD), if there exists a finite sequence of real numbers $\{g_l(n)\}_{n\geq 1}$ such that for each $n \geq 1$ and for all $x_i \in (-\infty, \infty), 1 \leq i \leq n$,

$$P\Big(\bigcap_{i=1}^{n}(X_i \leq x_i)\Big) \leq g_l(n)\prod_{i=1}^{n} P(X_i \leq x_i),$$

then we say that the random variables $\{X_n\}_{n\geq 1}$ are widely lower orthant dependent (WLOD). If the random variables $\{X_n\}_{n\geq 1}$ are both WUOD and WLOD, then we say that the random variables $\{X_n\}_{n\geq 1}$ are widely orthant dependent (WOD).

It can be found that $g_u(n) \geq 0$ and $g_l(n) \geq 0$ for all $n \geq 1$. Sometimes, we can take

$$g_u(n) = \sup_{x_i \in (-\infty,\infty), 1 \leq i \leq n} \frac{P\Big(\bigcap_{i=1}^n (X_i > x_i)\Big)}{\prod_{i=1}^n P(X_i > x_i)}, \ n \geq 1,$$

and

$$g_l(n) = \sup_{x_i \in (-\infty,\infty), 1 \leq i \leq n} \frac{P\Big(\bigcap_{i=1}^n (X_i \leq x_i)\Big)}{\prod_{i=1}^n P(X_i \leq x_i)}, \ n \geq 1,$$

if $g_u(n) < \infty$ and $g_l(n) < \infty$ for all $n \geq 1$.

Wang et al. [27] studied the uniform asymptotics for the finite-time ruin probability of risk model with a constant interest rate under the case of WOD random variables. They also gave some examples to illustrate WLOD and WUOD structures (see Section 3 of Wang et al. [27]). For more results of risk model under the case of WOD random variables, one can refer to Wang and Cheng [22], Liu et al. [12], and Wang et al. [23], etc.

If $g_u(n) = g_l(n) = 1$, the WOD random variables are NOD random variables. The concepts of NOD and NA sequences were introduced by Joag-Dev and Proschan [8]. They pointed out that NA random variables are NOD random variables, but the converse statement cannot always be true. Various results and examples of NOD and NA random variables can be found in [1], [11], [13], [16], [19], [21], [24], etc.

On the other hand, if $g_u(n) = g_l(n) = M > 0$, then WOD random variables form END random variables. The concept of END sequence was introduced by Liu [11]. Obviously, END random variables extend the corresponding one of NOD random variables. For the works on the END random variables, one can refer to [5], [18], [26] and the references therein.

In this paper, by using an exponential inequality for WOD sequences (see Lemma 2.3 in Section 2), we investigate the Bahadur representation of sample quantiles based on this stochastic processes. Our results extend the corresponding ones of Ling [10], Xu et al. [31] and Li et al. [9]. For the details, please see the main results in Section 3. Some lemmas are presented in Section 2.

Though out the paper, for a fixed $p \in (0, 1)$, let $\xi_p = F^{-1}(p)$, $\xi_{p,n} = F_n^{-1}(p)$. Meanwhile, let $\lceil x \rceil$ denote the largest integer not exceeding x, $C, C_1, C_2, \cdots$ denote positive constants whose values do not depend on n and may vary at each occurrence.

2. Some Lemmas

Lemma 2.1 (Wang et al. [23], Proposition 1.1). (1) *Let $\{X_n\}_{n\geq1}$ be WUOD (WLOD) with dominating coefficients $g_u(n)$, $n \geq 1$ $(g_l(n), n \geq 1)$. If $\{f_n(\cdot)\}_{n\geq1}$ are nondecreasing, then $\{f_n(X_n)\}_{n\geq1}$ are still WUOD (WLOD) with dominating coefficients $g_u(n)$, $n \geq 1$ $(g_l(n), n \geq 1)$; if $\{f_n(\cdot)\}_{n\geq1}$ are nonincreasing, then $\{f_n(X_n)\}_{n\geq1}$ are WLOD (WUOD) with dominating coefficients $g_l(n)$, $n \geq 1$ $(g_u(n), n \geq 1)$.*

(2) *If $\{X_n\}_{n\geq1}$ are non-negative and WUOD, then*

$$E\left(\prod_{i=1}^n X_i\right) \leq g_u(n) \prod_{i=1}^n E(X_i), \ n \geq 1.$$

In particular, if $\{X_n\}_{n\geq1}$ are WUOD, then for any $s > 0$,

$$E\left(\exp\left\{s\sum_{i=1}^n X_i\right\}\right) \leq g_u(n) \prod_{i=1}^n E\left(\exp\{sX_i\}\right), \ n \geq 1.$$

Lemma 2.2. *Let X be a random variable with $E(X) = 0$ and $|X| \leq b$, where b is a positive constant. Then for $\lambda > 0$,*

$$E\left(\exp(\lambda X)\right) \leq \exp\left\{\frac{\lambda^2 b^2}{2(1 - C)}\right\},$$

where $\lambda b \leq C < 1$.

Proof. Let $\delta^2 = E(X^2)$. For $|X| \le b$, it can be seen that

$$E|X|^n \le \delta^2 b^{n-2}, \quad n \ge 2.$$

By Taylor's expansion, $E(X) = 0$ and the fact $1 + x \le e^x$, for any $\lambda b \le C < 1$, we can get that

$$\begin{aligned} E\left(\exp\{\lambda(X)\}\right) &= 1 + \sum_{j=2}^{\infty} \frac{E(\lambda X)^j}{j!} \\ &\le 1 + \frac{\lambda^2}{2}\delta^2 + \frac{\lambda^3}{3!} b\delta^2 + \frac{\lambda^4}{4!} b^2\delta^2 \cdots + \frac{\lambda^j b^{j-2}\delta^2}{j!} + \cdots \\ &\le 1 + \frac{\lambda^2\delta^2}{2}\left(1 + \lambda b + (\lambda b)^2 + \cdots + (\lambda b)^{j-2} + \cdots\right) \\ &\le 1 + \frac{\lambda^2\delta^2}{2(1-\lambda b)} \le \exp\left\{\frac{\lambda^2 b^2}{2(1-C)}\right\}. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.3. *Let $\{X_n\}_{n\ge1}$ be a sequence of WOD random variables with dominating coefficients $g(n) \doteq \max\{g_u(n), g_l(n)\}$. Assume that $EX_n = 0$ and $|X_n| \le b$ for each $n \ge 1$, where b is a positive constant. Then for any $0 < C < 1$ and $0 < \epsilon < \frac{bC}{1-C}$, we have that*

$$P\left(\left|\sum_{i=1}^{n} X_i\right| > n\epsilon\right) \le 2g(n)\exp\left(-\frac{n\epsilon^2}{4K}\right), \tag{1}$$

where $K = \frac{b^2}{2(1-C)}$.

Proof. By Markov's inequality, Lemma 2.1(2) and Lemma 2.2, for $0 < \lambda b \le C < 1$, we have that

$$\begin{aligned} P\left(\sum_{i=1}^{n} X_i > n\epsilon\right) &\le \exp(-\lambda n\epsilon) E\left(\exp\left(\lambda\sum_{i=1}^{n} X_i\right)\right) \\ &\le \exp(-\lambda n\epsilon) g(n) \prod_{i=1}^{n} E\left(\exp(\lambda X_i)\right) \\ &\le \exp(-\lambda n\epsilon) g(n) \prod_{i=1}^{n} \exp\left(\frac{\lambda^2 b^2}{2(1-C)}\right) \\ &= g(n)\exp(-\lambda n\epsilon + K\lambda^2 n). \end{aligned}$$

Optimizing the exponent in the term of this upper bound, we find $\lambda = \epsilon/(2K)$. Taking $\lambda = \frac{\epsilon}{2K} = \frac{\epsilon(1-C)}{b^2}$, it is easily seen that $\lambda b = \frac{\epsilon(1-C)}{b} < C$. Then

$$P\left(\sum_{i=1}^{n} X_i > n\epsilon\right) \le g(n)\exp\left(-\frac{n\epsilon^2}{4K}\right). \tag{2}$$

Since $\{-X_n\}_{n\ge1}$ are also WOD, we can replace X_i by $-X_i$ in the above statement and get that

$$P\left(\sum_{i=1}^{n} X_i < -n\epsilon\right) \le g(n)\exp\left(-\frac{n\epsilon^2}{4K}\right). \tag{3}$$

By combining (2) with (3), (1) holds true. □

Lemma 2.4 (Serfling [15], Lemma 1.1.4). *Let $F(x)$ be a right-continuous distribution function. The inverse function $F^{-1}(t)$, $0 < t < 1$, is nondecreasing and left-continuous, and satisfies*

(i) $F^{-1}(F(x)) \le x, \ -\infty < x < \infty;$

(ii) $F(F^{-1}(t)) \ge t, \ 0 < t < 1;$

(iii) $F(x) \ge t$ *if and only if* $x \ge F^{-1}(t)$.

Lemma 2.5. *Let $0 < p < 1$ and $\{X_n\}_{n\ge1}$ be a sequence of WOD random variables with dominating coefficient $g(n) \doteq \max\{g_u(n), g_l(n)\}$, which satisfy that $\frac{\log(g(n)+1)}{n} \to 0$ as $n \to \infty$. Assume that the common marginal distribution function $F(x)$ is differentiable at ξ_p with $F'(\xi_p) = f(\xi_p) > 0$. Suppose that $f'(x)$ is bounded in a neighborhood of ξ_p, say $\aleph_p$. Then there exists some positive constant λ, with probability* 1

$$|\xi_{p,n} - \xi_p| \le \sqrt{\lambda + 1}\, \frac{[\log(g(n) + 1)) + \log n)]^{1/2}}{f(\xi_p)n^{1/2}}, \tag{4}$$

for all n sufficiently large.

Proof. Let $\lambda > 0$, whose value will be given later, and denote

$$\varepsilon_n = \sqrt{\lambda + 1}\frac{[\log(g(n) + 1) + \log n]^{1/2}}{f(\xi_p)n^{1/2}}, \ n \ge 1.$$

Write

$$P(|\xi_{p,n} - \xi_p| > \varepsilon_n) = P(\xi_{p,n} > \varepsilon_n + \xi_p) + P(\xi_{p,n} < \varepsilon_n - \xi_p).$$

By Lemma 2.4(*iii*),

$$\begin{aligned} P(\xi_{p,n} > \xi_p + \varepsilon_n) &= P(p > F_n(\xi_p + \varepsilon_n)) = P(1 - F_n(\xi_p + \varepsilon_n) > 1 - p) \\ &= P\left(\sum_{i=1}^{n} I(X_i > \xi_p + \varepsilon_n) > n(1-p)\right) \\ &= P\left(\sum_{i=1}^{n} (V_i - E(V_i)) > n\delta_{n1}\right), \end{aligned}$$

where $V_i = I(X_i > \xi_p + \varepsilon_n)$ and $\delta_{n1} = F(\xi_p + \varepsilon_n) - p$. Similarly,

$$P(\xi_{p,n} < \xi_p - \varepsilon_n) \le P(p \le F_n(\xi_p - \varepsilon_n)) = P\left(\sum_{i=1}^{n} (W_i - E(W_i)) \ge n\delta_{n2}\right),$$

where $W_i = I(X_i \le \xi_p - \varepsilon_n)$ and $\delta_{n2} = p - F(\xi_p - \varepsilon_n)$.

Since $F(x)$ is continuous at ξ_p with $F'(\xi_p) > 0$, ξ_p is the unique solution of $F(x-) \le p \le F(x)$ and $F(\xi_p) = p$. By the assumption on $f'(x)$ and Taylor's expansion, it follows that

$$\delta_{n1} = F(\xi_p + \varepsilon_n) - p = F(\xi_p + \varepsilon_n) - F(\xi_p) = f(\xi_p)\varepsilon_n + o(\varepsilon_n),$$

$$\delta_{n2} = p - F(\xi_p - \varepsilon_n) = F(\xi_p) - F(\xi_p - \varepsilon_n) = f(\xi_p)\varepsilon_n + o(\varepsilon_n).$$

Therefore, we can get that

$$\sqrt{\lambda + 1/2}[\log(g(n) + 1) + \log n]^{1/2}/n^{1/2} \le F(\xi_p + \varepsilon_n) - p,$$

for all n sufficiently large. Similarly, $p - F(\xi_p - \epsilon_n)$ satisfies a similar relation. Consequently, it has that

$$n[\min(\delta_{n1}, \delta_{n2})]^2 \ge (\lambda + 1/2)[\log(g(n) + 1) + \log n], \tag{5}$$

for all n sufficiently large.

By Lemma 2.1(1), $\{V_i - E(V_i)\}_{1\le i\le n}$ and $\{W_i - E(W_i)\}_{1\le i\le n}$ are also WOD random variables with dominating coefficients $g(n) = \max\{g_u(n), g_l(n)\}$. Obviously, by the fact $|V_i - E(V_i)| \le 1$, $|W_i - E(W_i)| \le 1$ and $\varepsilon_n \to 0$ as $n \to \infty$, there exists a constant $0 < C_1 < 1$ such that $0 < \delta_{n1} < \frac{C_1}{1-C_1}$ and $0 < \delta_{n2} < \frac{C_1}{1-C_1}$ for all n sufficiently large. So by (2) and (3), for all n sufficiently large, we obtain that

$$P(\xi_{p,n} > \xi_p + \varepsilon_n) \quad \le \quad g(n)\exp\left\{-\frac{n\delta_{n1}^2}{4K}\right\}$$

and

$$P(\xi_{p,n} < \xi_p - \varepsilon_n) \quad \le \quad g(n)\exp\left\{-\frac{n\delta_{n2}^2}{4K}\right\},$$

where $K = \frac{1}{2(1-C_1)}$. Consequently,

$$P(|\xi_{p,n} - \xi_p| > \varepsilon_n) \le 2g(n)\exp\left\{-\frac{n[\min(\delta_{n1},\delta_{n2})]^2}{4K}\right\} \tag{6}$$

for all n sufficiently large. By (5) and (6), we take $\lambda = 4K$ and get that

$$\begin{aligned}\sum_{n=1}^{\infty} P(|\xi_{p,n} - \xi_p| > \varepsilon_n) &\le C\sum_{n=1}^{\infty}\frac{g(n)}{(n[g(n)+1])^{1+1/(8K)}} \\ &\le C\sum_{n=1}^{\infty}\frac{1}{n^{1+1/(8K)}(g(n)+1)^{1/(8K)}} \\ &\le C\sum_{n=1}^{\infty}\frac{1}{n^{1+1/(8K)}} < \infty,\end{aligned}$$

which implies that with probability 1, the relations $|\xi_{p,n} - \xi_p| > \varepsilon_n$ hold for only finitely many n by Borel-Cantelli Lemma. Thus (4) holds true. □

3. Main Results and their Proofs

Theorem 3.1. *Let $0 < p < 1$ and $\{X_n\}_{n\ge1}$ be a sequence of WOD random variables with dominating coefficients $g(n) \doteq \max\{g_u(n), g_l(n)\}$, which satisfy that $\frac{\log(g(n)+1)}{n} \to 0$ as $n \to \infty$. Assume that the common marginal distribution function $F(x)$ is differentiable with derivative function $f(x)$ in a neighborhood $\aleph_p$ of ξ_p such that*

$$0 < d \doteq \sup\{f(x) : x \in \aleph_p\} < \infty.$$

Then there exists a positive constant C_1, with probability 1

$$\sup_{x\in\mathfrak{D}_n}\Big|(F_n(x) - F(x)) - (F_n(\xi_p) - p)\Big| \le (1+d)\left(\frac{C_1[\log(g(n)+1) + \log n]}{n}\right)^{1/2}, \tag{7}$$

for all n sufficiently large, where

$$\mathfrak{D}_n = [\xi_p - \tau_n, \xi_p + \tau_n], \quad \tau_n = \sqrt{C_1}\frac{[\log(g(n)+1) + \log n]^{1/2}\log n}{n^{1/2}(\log\log n)^{1/2}}.$$

Theorem 3.2. *Assume that conditions of Theorem* 3.1 *hold true. Then there exists a positive constant* C_2*, with probability* 1*,*

$$\sup_{x\in\mathfrak{D}_n}|F_n(x)-F(x)|\le(1+d)\left(\frac{C_2[\log(g(n)+1)+\log n]}{n}\right)^{1/2},\tag{8}$$

for all n sufficiently large.

Theorem 3.3. *Suppose that conditions of Theorem* 3.1 *are satisfied,* $F'(\xi_p)=f(\xi_p)>0$ *and* $f'(x)$ *is bounded in some neighborhood of* ξ_p*. Then with probability* 1*,*

$$\xi_{p,n}=\xi_p-\frac{F_n(\xi_p)-p}{f(\xi_p)}+O\left(\frac{[\log(g(n)+1)+\log n]^{1/2}}{n^{1/2}}\right),\ \ n\to\infty.\tag{9}$$

Remark 3.4. Ling [10] investigated the Bahadur representation for sample quantiles under NA sequences, Li et al. [9] generalized and improved the results of Ling [10] to the case of NOD sequences, and obtained the bound as $O(\frac{(\log n)^{1/2}}{n^{1/2}})$, *a.s.* (see Theorems 2.1-2.3 of Li et al. [9]). Meanwhile, Xu et al. [31] obtained the bound $o(\frac{(\log n)^{1/2}}{n^{1/2}})$, *a.s.*, for the Bahadur representation of sample quantiles under NA sequences (see Theorem 2.1 of Xu et al. [31]). On the other hand, if $g_u(n)=g_l(n)=M>0$, then WOD random variables are END random variables. In particulary, if $M=1$, then END random variables form NOD random variables. So by taking $g(n)=g_u(n)=g_l(n)=M=1$ in our results of Theorems 3.1-3.3, one can get the bound as $O(\frac{(\log n)^{1/2}}{n^{1/2}})$, *a.s.*, which coincides with the corresponding one of Li et al. [9]. Therefore, our results generalize the corresponding ones of Ling [10], Xu et al. [31] and Li et al. [9] to the case of WOD random variables.

Proof of Theorem 3.1. For some $C_1>0$, whose value will be given later, let

$$t_n=\sqrt{C_1}[\log(g(n)+1)+\log n]^{1/2}/n^{1/2},\ \ \eta_{r,n}=\xi_p+rt_n,\ \ \text{for } n>2,$$

$$\Delta_{r,n}=F_n(\eta_{r,n})-F(\eta_{r,n})-F_n(\xi_p)+p\ \ \text{for } r=0,\pm1,\pm2,\cdots\pm\lceil b_n\rceil,$$

where $b_n=\log n/(\log\log n)^{1/2}$. For $x\in\mathfrak{D}_n$, denote

$$g(x)=F_n(x)-F(x)-F_n(\xi_p)+p.$$

Then, for all $x\in[\eta_{r,n},\eta_{r+1,n}]$, $r=0,\pm1,\pm2,\cdots\pm\lceil b_n\rceil$,

$$g(x)\le F_n(\eta_{r+1,n})-F(\eta_{r,n})-F_n(\xi_p)+p\le\Delta_{r+1,n}+dt_n.\tag{10}$$

Similarly,

$$g(x)\ge F_n(\eta_{r,n})-F(\eta_{r+1,n})-F_n(\xi_p)+p\ge\Delta_{r+1,n}-dt_n.\tag{11}$$

Therefore, by (10) and (11), we have

$$\sup_{x\in\mathfrak{D}_n}|F_n(x)-F(x)-F_n(\xi_p)+p|\le\max_{0\le|r|\le\lceil b_n\rceil}|\Delta_{r,n}|+dt_n.\tag{12}$$

By the notations above, it follows

$$\begin{aligned}P(|\Delta_{r,n}|>t_n)&\le P\left(|F_n(\eta_{r,n})-F(\eta_{r,n})|>\frac{t_n}{2}\right)+P\left(|F_n(\xi_p)-p|>\frac{t_n}{2}\right)\\&\doteq I_{n1}+I_{n2}.\end{aligned}\tag{13}$$

For $i=1,2,\cdots,n$, $r=0,\pm1,\pm2,\cdots,\pm\lceil b_n\rceil$, denote

$$\xi_i=I(X_i\le\eta_{r,n})-E(I(X_i\le\eta_{r,n})),\ 1\le i\le n.$$

From Lemma 2.1(1), it can be seen that $\{\xi_i\}_{1\le i\le n}$ are also WOD random variables with the dominating coefficients $g(n) = \max\{g_u(n), g_l(n)\}$. By the fact $|\xi_i| \le 1$ and $t_n/2 \to 0$ as $n \to \infty$, there exists a positive constant $0 < C_2 < 1$ such that $0 < t_n/2 < \frac{C_2}{1-C_2}$ for all n sufficiently large. Hence, by Lemma 2.3, we have for all n sufficiently large that,

$$\begin{aligned} I_{n1} &= P\left(|F_n(\eta_{r,n}) - F(\eta_{r,n})| > \frac{t_n}{2}\right) = P\left(\left|\sum_{i=1}^{n} \xi_i\right| > nt_n/2\right) \\ &\le 2g(n)\exp\left\{-\frac{nt_n^2}{16K_1}\right\} \le \frac{2g(n)}{[(g(n)+1)n]^{\frac{C_1}{16K_1}}}, \end{aligned} \tag{14}$$

where $K_1 = \frac{1}{2(1-C_2)}$. Likewise, we have for all n sufficiently large that,

$$\begin{aligned} I_{n2} &= P\left(|F_n(\xi_p) - p| > \frac{t_n}{2}\right) \le 2g(n)\exp\left\{-\frac{nt_n^2}{16K_1}\right\} \\ &\le \frac{2g(n)}{[(g(n)+1)n]^{\frac{C_1}{16K_1}}}. \end{aligned} \tag{15}$$

Taking $C_1 = 17K_1$, we have by (13), (14) and (15) that for all n sufficiently large,

$$\begin{aligned} \sum_{n=1}^{\infty} P\left(\max_{0\le r\le \lceil b_n \rceil} |\Delta_{r,n}| > t_n\right) &\le \sum_{n=1}^{\infty} \sum_{r=-\lceil b_n\rceil}^{\lceil b_n\rceil} P(|\Delta_{r,n}| > t_n) \\ &\le C_3 \sum_{n=1}^{\infty} \frac{g(n)\log n}{(\log\log n)^{1/2} n^{17/16} (g(n)+1)^{17/16}} \\ &\le C_3 \sum_{n=1}^{\infty} \frac{\log n}{(\log\log n)^{1/2} n^{17/16} (g(n)+1)^{1/16}} \\ &\le C_3 \sum_{n=1}^{\infty} \frac{\log n}{(\log\log n)^{1/2} n^{17/16}} < \infty. \end{aligned}$$

By Borel-Cantelli lemma, it follows that with probability 1, the relations $\max_{0\le |r|\le\lceil b_n\rceil} |\Delta_{r,n}| > t_n$ hold true for only finitely many n. Together with Equations (12) and (13), with probability 1, (7) holds true. □

Proof of Theorem 3.2. Let $C_2 > 0$, whose value will be given later. For $n > 2$, let $t_n = (\frac{C_2[\log(g(n)+1)+\log n]}{n})^{1/2}$ and

$$s_{r,n} = \xi_p + rt_n, d_{r,n} = F_n(s_{r,n}) - F(s_{r,n}),$$

for $r = 0, \pm1, \pm2, \cdots \pm \lceil b_n\rceil$ and $b_n = \sqrt{C_1/C_2}\log n/(\log\log n)^{1/2}$, where C_1 is defined in Theorem 3.1. Then for any $x \in [\xi_p + rt_n, \xi_p + (r+1)t_n], r = 0, \pm0, \pm1, \pm2, \cdots \pm \lceil b_n\rceil$, it has that

$$d_{r,n} - dt_n \le F_n(x) - F(x) \le d_{r+1,n} + dt_n.$$

Hence

$$\sup_{|x-\xi_p|\le\tau_n} |F_n(x) - F(x)| \le \max_{0\le|r|\le\lceil b_n\rceil} |d_{r,n}| + dt_n,$$

where τ_n is defined in Theorem 3.1. Let

$$\eta_i \doteq I(X_i \le \xi_p + rt_n) - EI(X_i \le \xi_p + rt_n),\ i = 1, 2, \cdots, n.$$

Obviously, by Lemma 2.1(1), $\{\eta_i\}_{1\le i\le n}$ are also WOD random variables with the dominating coefficients $g(n) = \max\{g_u(n), g_l(n)\}$. On the other hand, by the fact $|\eta_i| \le 1$ and $t_n \to 0$ as $n \to \infty$, there exists a positive constant $0 < C_3 < 1$ such that $0 < t_n < \frac{C_3}{1-C_3}$ for all n sufficiently large. We have by Lemma 2.3 that

$$\begin{aligned} P(|d_{r,n}| > t_n) &= P(|F_n(\xi_p + rt_n) - F(\xi_p + rt_n)| > t_n) \\ &= P\left(\left|\sum_{i=1}^{n} \eta_i\right| > nt_n\right) \le 2g(n)\exp\left\{-\frac{nt_n^2}{4K_2}\right\} \\ &\le \frac{2g(n)}{[(g(n)+1)n]^{C_2/(4K_2)}}, \end{aligned}$$

where $K_2 = \frac{1}{2(1-C_3)}$. By taking $C_2 = 5K_2$, it has

$$P(|d_{r,n}| > t_n) \le \frac{2g(n)}{[(g(n)+1)n]^{5/4}}.$$

Consequently,

$$\sum_{n=1}^{\infty} P(\max_{0\le r\le \lceil b_n \rceil} |d_{r,n}| > t_n) \le C_4 \sum_{n=1}^{\infty} \frac{g(n)\log n}{[(g(n)+1)n]^{5/4}(\log\log n)^{1/2}} < \infty.$$

By Borel-Cantelli lemma, with probability 1, the relations $\max_{0\le |r|\le \lceil b_n \rceil} |d_{r,n}| > t_n$ hold true for only finitely many n. Therefore, with probability 1, (8) holds true. □

Proof of Theorem 3.3. Applying Lemma 2.5, we obtain that with probability 1, for all n sufficiently large,

$$|\xi_{p,n} - \xi_p| \le \frac{\sqrt{\lambda+1}[\log(g(n)+1) + \log n]^{1/2}}{f(\xi_p)n^{1/2}}, \tag{16}$$

where λ is a positive constant. So, with probability 1, $\xi_{p,n} \in \mathfrak{D}_n$ for all n sufficiently large. By Theorem 3.1, with probability 1, for all n sufficiently large, we have that

$$F_n(\xi_p) - F(\xi_p) = F_n(\xi_{p,n}) - F(\xi_{p,n}) + O\left(\frac{[\log(g(n)+1) + \log n]^{1/2}}{n^{1/2}}\right),\ n \to \infty. \tag{17}$$

Meanwhile, by (16), assumption on $f'(x)$, Taylor's expansion and Theorem 3.2, we can get that with probability 1, for all n sufficiently large,

$$\begin{aligned} |F_n(\xi_{p,n}) - p| &\le |F_n(\xi_{p,n}) - F(\xi_{p,n})| + |F(\xi_{p,n}) - F(\xi_p)| \\ &\le \sup_{x\in\mathfrak{D}_n} |F_n(x) - F(x)| + f(\xi_p)|\xi_{p,n} - \xi_p| + o(|\xi_{p,n} - \xi_p|) \\ &= O\left(\frac{[\log(g(n)+1) + \log n]^{1/2}}{n^{1/2}}\right). \end{aligned} \tag{18}$$

On the other hand, from the assumption on $f'(x)$, by Taylor's expansion again and Equations (17) and (18), we obtain that with probability 1, for all n sufficiently large,

$$\begin{aligned} F_n(\xi_p) - F(\xi_p) &= F(\xi_p) - F(\xi_{p,n}) + O\left(\frac{[\log(g(n)+1) + \log n]^{1/2}}{n^{1/2}}\right) \\ &= -f(\xi_p)(\xi_{p,n} - \xi_p) - \frac{1}{2}f'(\omega_n)(\xi_{p,n} - \xi_p)^2 + O\left(\frac{[\log(g(n)+1) + \log n]^{1/2}}{n^{1/2}}\right) \\ &= -f(\xi_p)(\xi_{p,n} - \xi_p) + O\left(\frac{[\log(g(n)+1) + \log n]^{1/2}}{n^{1/2}}\right), \end{aligned}$$

where ω_n is a random variable between $\xi_{p,n}$ and ξ_p. Reorganizing the terms in the above equality, we can get that with probability 1,

$$\xi_{p,n} - \xi_p = -\frac{F_n(\xi_p) - p}{f(\xi_p)} + O\left(\frac{[\log(g(n)+1) + \log n]^{1/2}}{n^{1/2}}\right), \quad n \to \infty.$$

So (9) holds true. □

Acknowledgements

The authors are deeply grateful to Section Editor and anonymous referee for their careful reading and insightful comments.

References

[1] N. Asadian, V. Fakoor, A. Bozorgnia, Rosenthal's type inequalities for negatively orthant dependent random variables, Journal of the Iranian Statistical Society 5(1-2) (2006), 69–75.
[2] G.J. Babu, K. Singh, On deviations between empirical and quantile processes for mixing random variables, Journal of Multivariate Analysis 8(4) (1978), 532–549.
[3] R.R. Bahadur, A note on quantiles in large samples, The Annals of Mathematical Statistics 37(3) (1966), 577–580.
[4] Z.W. Cai, G.G. Roussas, Smooth estimate of quantiles under association, Statistics and Probability Letters 36(3) (1997), 275–287.
[5] Y. Chen, A. Chen, K.W. Ng, The strong law of large numbers for extend negatively dependent random variables, Journal of Applied Probability 47(4) (2010), 908–922.
[6] S.X. Chen, C.Y. Tang, Nonparametric inference of Value-at Risk for dependent financial returns, Journal of Financial Econometrics 3(2) (2005), 227–255.
[7] Y.B. Cheng, J.G. De Gooijer, Bahadur representation for the nonparametric M-estimator under α-mixing dependence, Statistics 43(5) (2009), 443–462.
[8] K. Joag-Dev, F. Proschan, Negative association of random variables with applications, The Annals of Statistics 11(1) (1983), 286–295.
[9] X.Q. Li, W.Z. Yang, S.H. Hu, X.J. Wang, The Bahadur representation for sample quantile under NOD sequence, Journal of Nonparametric Statistics 23(1) (2011), 59–65.
[10] N.X. Ling, The Bahadur representation for sample quantiles under negatively associated sequence, Statistics and Probability Letters 78(16) (2008), 2660–2663.
[11] L. Liu, Precise large deviations for dependent random variables with heavy tails, Statistics and Probability Letters 79(9) (2009), 1290–1298.
[12] X.J. Liu, Q.W. Gao, Y.B. Wang, A note on a dependent risk model with constant interest rate, Statistics and Probability Letters 82(4) (2012), 707–712.
[13] P. Matula, A note on the almost sure convergence of sums of negatively dependent random variables, Statistics and Probability Letters 15(3) (1992), 209–213.
[14] P.K. Sen, On Bahadur representation of sample quantile for sequences of ϕ-mixing random variables, Journal of Multivariate Analysis 2(1) (1972), 77–95.
[15] R.J. Serfling, Approximation Theorems of Mathematical Statistics, Wiley, New York, 1980.
[16] Q.M. Shao, A comparison theorem on moment inequalities between negatively associated and independent random variables, Journal of Theoretical Probability 13(2) (2000), 343–356.
[17] O.S. Sharipov, M. Wendler, Normal limits, nonnormal limits, and the bootstrap for quantiles of dependent data, Statistics and Probability Letters 83(4) (2013), 1028–1035.
[18] A.T. Shen, Probability inequalities for END sequence and their applications, Journal of Inequalities and Applications 2011, 2011:98.
[19] C. Su, L.C. Zhao, Y.B. Wang, Moment inequalities and weak convergence for negatively associated sequences, Science in China (A) 40(2) (1997), 172–182.
[20] S.X. Sun, The Bahadur representation for sample quantiles under weak dependence, Statistics and Probability Letters 76(12) (2006), 1238–1244.
[21] S.H. Sung, On the exponential inequalities for negatively dependent random variables, Journal of Mathematical Analysis and Applications 381(2) (2011), 538–545.
[22] Y.B. Wang, D.Y. Cheng, Basic renewal theorems for random walks with widely dependent increments, Journal of Mathematical Analysis and Applications 384(2) (2011), 597–606.
[23] Y.B. Wang, Z.L. Cui, K.Y. Wang, X.L. Ma, Uniform asymptotics of the fnite-time ruin probability for all times, Journal of Mathematical Analysis and Applications 390(1) (2012), 208–223.
[24] X.J. Wang, S.H. Hu, W.Z. Yang, N.X. Ling, Exponential inequalities and inverse moment for NOD sequence, Statistics and Probability Letters 80(5) (2010), 452–461.
[25] X.J. Wang, S.H. Hu, W.Z. Yang, The Bahadur representation for sample quantile under strongly mixing sequence, Journal of Statistical Planning and Inference 141(2) (2011), 655–662.

[26] X.J. Wang, T.-C. Hu, A. Volodin, S.H. Hu, Complete convergence for weighted sums and arrays of rowwise extended negatively dependent random variables, Communications in Statistics-Theory and Methods 42(13) (2013), 2391–2401.
[27] K.Y. Wang, Y.B. Wang, Q.W. Gao, Uniform asymptotics for the finite-time ruin probability of a dependent risk model with a constant interest rate, Methodology and Computing in Applied Probability 15(1) (2013), 109–124.
[28] M. Wendler, Bahadur representation for U-quantiles of dependent data, Journal of Multivariate Analysis, 102(6) (2011), 1064–1079.
[29] M. Wendler, U-processes, U-quantile processes and generalized linear statistics of dependent data, Stochastic Processes and their Applications 122(3) (2012), 787–807.
[30] W.B. Wu, On the Bahadur representation of sample quantiles for dependent sequences, The Annals of Statistics 33(4) (2005), 1934–1963.
[31] S.F. Xu, L. Ge, Y. Miao, On the Bahadur representation of sample quantiles and order statistics for NA sequences, Journal of the Korean Statistical Society 42(1) (2013), 1–7.
[32] K. Yoshihara, The Bahadur representation of sample quantile for sequences of strongly mixing random variables, Statistics and Probability Letters 24(2) (1995), 299–304.
[33] Q.C. Zhang, W.Z. Yang, S.H. Hu, On Bahadur representation for sample quantiles under α-mixing sequence, Statistical Papers 55(2) (2014), 285–299.