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Convex Dominating-Geodetic Partitions in Graphs

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Abstract. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called $u - v$ geodesic. A set X is convex in G if vertices from all $a - b$ geodesics belong to X for every two vertices $a, b \in X$. A set of vertices D is dominating in G if every vertex of $V - D$ has at least one neighbor in D. The convex domination number $\gamma_{con}(G)$ of a graph G equals the minimum cardinality of a convex dominating set in G. A set of vertices S of a graph G is a geodetic set of G if every vertex $v \notin S$ lies on a $x - y$ geodesic between two vertices x, y of S. The minimum cardinality of a geodetic set of G is the geodetic number of G and it is denoted by $g(G)$. Let D, S be a convex dominating set and a geodetic set in G, respectively. The two sets D and S form a convex dominating-geodetic partition of G if $|D| + |S| = |V(G)|$. Moreover, a convex dominating-geodetic partition of G is called optimal if D is a $\gamma_{con}(G)$-set and S is a $g(G)$-set. In the present article we study the (optimal) convex dominating-geodetic partitions of graphs.

1. Introduction

Let $G = (V(G), E(G))$ be a connected simple graph. Given two vertices $u, v \in V(G)$, $u \sim v$ denotes two adjacent vertices in G, or equivalently, that $uv \in E(G)$. The *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v in G, *i.e.*, $N_G(v) = \{u \in V : u \sim v\}$. For a set $X \subseteq V(G)$, the *open neighborhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and the *closed neighborhood* is $N_G[X] = N_G(X) \cup X$. The degree of a vertex v is $deg_G(v) = |N_G(v)|$. A vertex of degree one in G is called an *end-vertex* in G. The set of end-vertices of G is denoted by $\Omega(G)$.

The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d_G(u, v)$ is called a $u - v$ *geodesic*. A set X is *convex* in G if all the vertices belonging to every $a - b$ geodesic are contained in X, for any two vertices $a, b \in X$. A subset D of $V(G)$ is *dominating* in G, if every vertex of $V(G) - D$ has at least one neighbor in D. A set $X \subseteq V$ is a *convex dominating set* if X is convex and dominating. The *convex domination number* $\gamma_{con}(G)$ of a graph G equals the minimum cardinality of a convex dominating set. By a $\gamma_{con}(G)$-set we mean a convex dominating set of cardinality $\gamma_{con}(G)$ and is called a *minimum convex dominating set*. The convex domination number was first introduced in [17].

We define $I[u, v]$ to be the set of all vertices lying on some $u - v$ geodesic of G and for a nonempty set of vertices $S \subseteq V(G)$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. A set $S \subseteq V(G)$ is a *geodetic set* if $I[S] = V(G)$. The cardinality of

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any geodetic set in G is called the *geodetic number* of G and it is denoted by $g(G)$. By a $s(G)$-set we mean a geodetic set of cardinality $s(G)$, which is called a *minimum geodetic set*. Geodetic sets were introduced first in [6].

The graph partition problem has been extensively studied in graph theory. Roughly speaking, such problem deals with with partitioning the vertex set $V(G)$ of a graph G in such a way that every set of the partition satisfies a specific property. Sometimes the partition problem is related with finding the minimum number of sets in the partition, sometimes with finding the maximum number of sets in the partition, sometimes with computing how many different partitions can be constructed having the same quantity of "similar" elements in each set of the partition, etc. One particular and interesting case of such problems regards with finding a vertex partition of a graph G into two sets such that both sets satisfy some properties. A specific example is, for instance, the balanced graph partitioning problem [1]. Moreover, partitions into two independent sets, two dominating sets, two alliances, among other ones have been widely studied (see [2, 14]). On the other hand, the partition problem has an enormous range of applications in several areas like computer engineering or social networks.

A simple yet fundamental observation in domination theory made by Ore [15] is that every graph of minimum degree at least one contains two disjoint dominating sets. Thus, the vertex set of every graph without isolated vertices can be partitioned into two dominating sets. Several equivalent results for other graphs parameters can be found in the literature. For example, a characterization of graphs with disjoint dominating and total dominating sets is given in [8], [9] and [13].

In this work we consider similar problems for convex dominating sets and geodetic sets. Let D, S be a convex dominating set and a geodetic set in G, respectively. The sets D and S form a *convex dominating-geodetic partition* of G if $|D| + |S| = |V(G)|$. Moreover, a convex dominating-geodetic partition of G is called *optimal* if D is a $\gamma_{con}(G)$-set and S is a $g(G)$-set. Thus, if G has an optimal convex dominating-geodetic partition, then $\gamma_{con}(G) + g(G) = n(G)$. Notice that there exist graphs G such that $\gamma_{con}(G) + g(G) > n$. For instance, if G is a cycle graph of even order, then $\gamma_{con}(G) + g(G) = n + 2$. Moreover, there are graphs in which the convex domination number equals its order, for instance, in [12] was presented an infinite family $\mathcal{F}$ of graphs with diameter two and convex domination number equal to its order. Since the geodetic number is greater than two for every graph G, we have that any graph $G \in \mathcal{F}$ satisfies $\gamma_{con}(G) + g(G) \geq n + 2$.

If a graph G has an a partition into a convex dominating set D and a geodetic set S, then we say that G is a *convex dominating-geodetic graph* (CD-GDT graph for short). In addition, if $|D| = \gamma_{con}(G)$ and $|S| = g(G)$, then we say that G is an *optimal convex dominating-geodetic graph* (OCD-GDT graph for short). An example of an OCD-GDT graph can be a path P_n with at least three vertices, where $S = \Omega(P_n)$ is a $g(P_n)$-set and $V(P_n) - \Omega(P_n)$ is a $\gamma_{con}(P_n)$ set. The next remarks are straightforward.

Remark 1.1. *If $\gamma_{con}(G) + g(G) > n$, then G is not a CD-GDT graph.*

Remark 1.2. *If G is a CD-GDT graph, then $\gamma_{con}(G) + g(G) \leq n$.*

We must remark that there are graphs G such that $\gamma_{con}(G) + g(G) \leq n$ and that are not CD-GDT graphs. For instance, the hypercube graph Q_3, which has order 8, satisfies that $g(Q_3) = 2$ and $\gamma_{con}(Q_3) = 4$. So $\gamma_{con}(Q_3) + g(Q_3) = 6 < 8$. However Q_3 is not partitionable into a convex dominating set and a geodetic set.

Clearly, every OCD-GDT graph is a CD-GDT graph and the converse is not necessarily true. As an example of a graph which is a CD-GDT graph and not an OCD-GDT graph, we can consider the graph H from Figure 1. The set $S = \{a, b, c, d, e, f\}$ is a (not minimum) geodetic set in H and $V - S$ is a minimum convex dominating set of H.

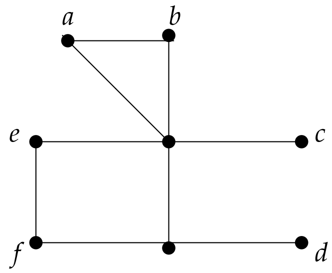

Figure 1: The graph H is a CD-GDT graph and is not an OCD-GDT graph.

2. Results

The vertex $y \in V(G)$ is an *extreme vertex* in G if the subgraph induced by $N_G[y]$ is isomorphic to a complete graph. Let $\mathcal{T}(G)$ be the set of extreme vertices of G. We begin with some almost straightforward remarks. Let $\mathcal{P}$ denotes the property of a graph G such that every non extreme vertex of G is on some $x - y$ geodesic, where $x, y \in \mathcal{T}(G)$.

Remark 2.1. *If S is a geodetic set, then every extreme vertex belongs to S.*

Remark 2.2. *If $G \neq K_n$, then no extreme vertex belongs to a $\gamma_{con}(G)$-set.*

Lemma 2.3. *If a connected graph G has the property $\mathcal{P}$, then G is CD-GDT graph.*

Proof. Let S be a $g(G)$-set. From Remark 2.1, every extreme vertex belongs to S. Since G has the property $\mathcal{P}$, it follows $S = \mathcal{T}(G)$. Notice that $V - S$ is a dominating set of G. Now, suppose $V - S$ is not convex. Thus, for some vertices $x, y \in V - S$, there exists $z \in S$ such that z belongs to some $x - y$ geodesic. Since $z \in S$, we have that z is an extreme vertex, which means $xy \in E(G)$. So, z does not belong to a $x - y$ geodesic, a contradiction. □

The example of a graph H from Figure 1 shows that the converse is not true. Graph H is a CD-GDT graph, the set $S = \{a, b, c, d, e, f\}$ is a geodetic set and $V(G) - S$ is a convex dominating set, but vertices e, f are non-extreme vertices and none of them is on any $u - v$ geodesic, where $u, v \in \mathcal{T}(G)$.

A *clique* in a graph G is a set of vertices X such that the subgraph $G[X]$ induced by X is isomorphic to a complete graph. The maximum order of a clique in a graph is the *clique number* and is denoted by $\omega(G)$.

Proposition 2.4. *Let G be a connected graph.*

(i) *If there exists a clique in G which is a dominating set, then G is a CD-GDT graph.*
(ii) *If there exists an $\omega(G)$-set which is a $\gamma(G)$-set, then G is an OCD-GDT graph.*

Proof. Let S be a clique which is a dominating set in G. If $S \cap \mathcal{T}(G) \neq \emptyset$, then let $A \subset S$ such that $A \subset \mathcal{T}(G)$ and let $B = S - A$. Notice that every vertex of A belongs to every $g(G)$-set and also that $S - A$ is still a dominating set in G. Thus, B is a convex dominating set in G. Since every vertex in B belongs to a $u - v$ geodesic such that $u, v \in V(G) - B$ we have that $V(G) - B$ is a geodetic set. On the contrary, if $S \cap \mathcal{T}(G) = \emptyset$, then S is a convex dominating set and $V(G) - S$ is a geodetic set of G. Thus (i) is proved. Moreover, if S is an $\omega(G)$-set and a $\gamma(G)$-set, then $A = \emptyset$, $B = S$ and $V(G) - B$ is a $g(G)$-set. Therefore (ii) follows. □

2.1. Generalized trees

A *cut vertex* in G is a vertex $x \in V(G)$ such that the number of components of $G - \{x\}$ is bigger than the number of components of G. We consider now the family of graphs G_i obtained in the following way. We begin with a complete graph K_{n_1}, $n_1 \geq 2$, and G_i, $i \geq 2$, is obtained recursively from G_{i-1} by adding a complete graph K_{n_i}, $n_i \geq 2$, and identifying a vertex of G_{i-1} with a vertex in K_{n_i}. From now on we say that a connected graph G is a *generalized tree* if and only if there is a sequence of complete graphs $K_{n_1}, K_{n_2}, ..., K_{n_r}$ such that $G_r \cong G$ for some $r \geq 1$. Note that if every K_{n_i} is isomorphic to K_2, then G_r is a tree, which justifies the terminology used. The next remarks follow immediately from the definition of a generalized tree.

Remark 2.5. *In a generalized tree G every vertex is a cut vertex or an extreme vertex.*

Remark 2.6. *If D is a convex dominating set of a graph G, then every cut-vertex of G belongs to D.*

Proposition 2.7. *Every generalized tree different from the complete graph is an OCD-GDT graph.*

Proof. Let G be a generalized tree different from the complete graph. Let S be a $g(G)$-set and let D be a $\gamma_{con}(G)$-set. From Remark 2.1, every extreme vertex belongs to S and from Remark 2.6, every cut-vertex belongs to D. From Remark 2.5, $S \cup D = V(G)$. So G is a OCD-GDT graph. □

Corollary 2.8. *Every tree of order at least three is an OCD-GDT graph.*

2.2. Unicyclic graphs

We begin this section with the case of cycle graphs, which are the most simple examples of unicyclic graphs.

Lemma 2.9. *Cycle graphs are not CD-GDT graphs.*

Proof. If $n = 3$, then $\gamma_{con}(C_n) = 1$ and $g(C_n) = 3$. So, $\gamma_{con}(C_n) + g(C_n) = 4 > 3 = n$ and the result follows from Remarkss 1.1 and 1.2. If $n = 4$, then $\gamma_{con}(C_n) + g(C_n) = 4 = n$. Nevertheless, there is no convex dominating set D of C_n such that $V - D$ is a geodetic set. If $n = 5$, then $\gamma_{con}(C_n) = 3$ and $g(C_n) = 3$. So, $\gamma_{con}(C_n) + g(C_n) = 6 > 5 = n$. Finally, if $n \geq 6$, then $\gamma_{con}(C_n) = n$ and $g(C_n) \geq 2$. So, $\gamma_{con}(C_n) + g(C_n) > n$ and again, from Remarks 1.1 and 1.2, we are done. □

From now on we shall denote by G_t, a unicyclic graph different from a cycle whose unique cycle C_t has vertex set $\{u_0, u_1, ..., u_{t-1}\}$ with $t \geq 3$ an $u_i \sim u_{i+1}$, for every $i \in \{0, ..., t-1\}$. From now on, operations with the subindex i are done modulo t. If there exists $j \in \{0, ..., t-1\}$, such that for every $u_i \in \{u_j, u_{j+1}, ..., u_{j+\lfloor \frac{t}{2} \rfloor - 1}\}$, the vertex u_i has degree two in G_t, then we call G_t as a *not balanced* unicyclic graph. Otherwise, we say that G_t is a *balanced* unicyclic graph.

Lemma 2.10. *Let G_t be a unicyclic graph different from a cycle and $t \geq 4$. Then,*

(i) *$\Omega(G_t)$ is a $g(G_t)$-set if and only if G_t is a balanced unicyclic graph.*
(ii) *If G_t is not a balanced unicyclic graph, then every $g(G_t)$-set D has the form $D = S \cup \Omega(G_t)$, where $S \subset V(C_t)$ and $|S| \leq 2$.*

Proof. Suppose G_t is a balanced unicyclic graph. Let X be the set of vertices in $V(C_t)$ having degree greater than two. It is clear that $|X| \geq 2$. Now, for every vertex $x \in X$, let $U_x \subset \Omega(G_t)$ such that for every $u_x \in U_x$, $d(u_x, x) = \min_{x' \in V(C_t)}\{d(u_x, x')\}$. Let v be a vertex of G_t. If $v \in V(C_t)$, then since G_t is a balanced unicyclic graph, there exist two distinct vertices $a, b \in X$ such that $d(a, b) \leq \left\lfloor \frac{t}{2} \right\rfloor$ and $d(a, b) = d(a, v) + d(v, b)$. Thus, there exist two vertices $u_a \in U_a$ and $u_b \in U_b$ such that $d(u_a, u_b) = d(u_a, a) + d(a, b) + d(b, u_b) = d(u_a, a) + d(a, v) + d(v, b) + d(b, u_b) = d(u_a, v) + d(v, u_b)$ and, as a consequence, $v \in I[u_a, u_b]$. On the other hand, if $v \notin V(C_t)$, then let $y \in V(C_t)$ and $u_y \in U_y$ such that $v \in I[y, u_y]$. Then for any vertex $w \in X$ such that $w \neq y$, there exists a vertex $u_w \in U_w$ such that $v \in I[u_y, u_w]$. Therefore, $\Omega(G_t)$ is a geodetic set in G_t. Finally, since every geodetic set of a graph contains all their end-vertices we obtain that $\Omega(G_t)$ is a $g(G_t)$-set.

Now, let G_t be a unicyclic graph different from a cycle such that $g(G_t) = |\Omega(G_t)|$. Notice that the only $g(G_t)$-set S of G_t is $\Omega(G_t)$. So, $S \cap V(C_t) = \emptyset$. Let $V(C_t) = \{u_0, u_1, ..., u_{t-1}\}$ where $u_i \sim u_{i+1}$ (operations with the subindex i is done modulo t), for every $i \in \{0, ..., t-1\}$. Suppose G_t is not a balanced unicyclic graph. Thus, there exists $j \in \{0, ..., t-1\}$ such that for every $u_i \in \{u_j, u_{j+1}, ..., u_{j+\lfloor \frac{t}{2} \rfloor - 1}\}$, u_i has degree two in G_t. Moreover, every $u_i \in \{u_j, u_{j+1}, ..., u_{j+\lfloor \frac{t}{2} \rfloor - 1}\}$ satisfies $u_i \notin I[S]$, which is a contradiction. Therefore, G_t is a balanced unicyclic graph and the proof of (i) is completed.

To prove (ii), let $j \in \{0, ..., t-1\}$ such that for every $u_i \in \{u_j, u_{j+1}, ..., u_{j+\lfloor \frac{t}{2} \rfloor - 1}\}$, u_i has degree two in G_t. We can suppose without loss of generality that $j = 0$ and that $|d(u_0, w_1) - d(u_{\lfloor \frac{t}{2} \rfloor - 1}, w_2)| \leq 1$ where $w_1, w_2 \in V(C_t)$

are the nearest vertices to $u_0, u_{\lfloor \frac{t}{2} \rfloor -1}$ in C_t, respectively, having degree greater than two in G_t. Since every $u_l - u_k$ geodesic, $l, k \notin \{0, ..., \lfloor \frac{t}{2} \rfloor - 1\}$, is formed by vertices from $V(C_t) - \{u_0, u_1, ..., u_{\lfloor \frac{t}{2} \rfloor -1}\}$ we have that at least one vertex of the set $\{u_0, u_1, ..., u_{\lfloor \frac{t}{2} \rfloor -1}\}$ must belong to a $g(G_t)$-set. Now, if $\lfloor \frac{t}{2} \rfloor$ is even, then we take $S = \{u_{\lfloor \frac{t}{2} \rfloor -1}, u_{\lfloor \frac{t}{2} \rfloor}\}$. On the contrary, if $\lfloor \frac{t}{2} \rfloor$ is odd, then we take $S = \{u_{\lfloor \frac{t}{2} \rfloor}\}$. Since $d(u_j, w_1) \le \lfloor \frac{t}{2} \rfloor - 1$ and $d(u_{j+\lfloor \frac{t}{2} \rfloor -1}, w_2) \le \lfloor \frac{t}{2} \rfloor - 1$, it is straightforward to observe that $\Omega(G_t) \cup S$ is a geodetic set in G_t. If $\Omega(G_t) \cup S$ is not a $g(G_t)$-set, then $\Omega(G_t) \cup S$ contains a $g(G_t)$-set X such that $\Omega(G_t) \subset X$ and $X \cap V(C_t) \neq \emptyset$. Therefore, the proof of (ii) is completed. □

Note that the lemma above leads to the following result.

Corollary 2.11.

(i) *In every balanced unicyclic graph G_t, $\Omega(G_t)$ is the unique $g(G_t)$-set.*

(ii) *If G_t is not a balanced unicyclic graph, then every $g(G_t)$-set S satisfies $1 \le |S \cap V(C_t)| \le 2$ and $S - \Omega(G_t) = S \cap V(C_t)$.*

Notice that if $t = 3$ in a unicyclic graph G_t, then G_3 is a generalized tree. Thus, from Proposition 2.7, we obtain that G_3 is an OCD-GDT graph. From now on we will study unicyclic graphs G_t according to the length t of the unique cycle C_t of G_t.

Theorem 2.12. *Let $t = 4$.*

(i) *If G_t is not a balanced unicyclic graph, then G_t is a CD-GDT graph.*

(ii) *If G_t is a balanced unicyclic graph, then G_t is an OCD-GDT graph.*

Proof. If G_4 is not a balanced unicyclic graph, then there exist two consecutive vertices v, w of C_4 such that they have degree two. Then $S = \Omega \cup \{v, w\}$ is a geodetic set of G_4 but S is not a $g(G_4)$-set. Since $V - S$ is a convex dominating set of G_4, we obtain that G_4 is a CD-GDT graph and (i) is proved.

Now, if G_4 is a balanced unicyclic graph, then it is clear that $\Omega(G_4)$ is a $g(G_4)$-set and $V(G_4) - \Omega(G_4)$ is a $\gamma_{con}(G_4)$-set. Thus (ii) follows. □

Theorem 2.13. *Let $t = 5$.*

(i) *If G_t is a balanced unicyclic graph, then G_t is an OCD-GDT graph.*

(ii) *If G_t is not a balanced unicyclic graph and there is only one vertex in C_t with degree greater than two, then G_t is an OCD-GDT graph.*

(iii) *If G_t is not a balanced unicyclic graph and there is more than one vertex in C_t with degree greater than two, then G_t is a CD-GDT graph.*

Proof. If G_5 is a balanced unicyclic graph, then it is clear that $\Omega(G_5)$ is a $g(G_5)$-set and $V(G_5) - \Omega(G_5)$ is a $\gamma_{con}(G_5)$-set. Thus (i) follows. Now, if G_5 is not a balanced unicyclic graph, then there exist at least two consecutive vertices v, w of C_t such that they have degree two. Moreover, if there is only one vertex in C_5 with degree greater than two, then $S = \{v, w\} \cup \Omega(G_5)$ and $D = V(G_5) - S$ form an optimal convex dominating-geodetic partition of G_5 and (ii) follows. On the contrary, if there is more than one vertex in C_5 with degree greater than two, then $S = \{v, w\} \cup \Omega(G_5)$ and $D = V(G_5) - S$ form a convex dominating-geodetic partition of G_5 which is not optimal and (iii) follows. □

Theorem 2.14. *Let $t \ge 6$. Then a unicyclic graph G_t is an OCD-GDT graph if and only if G_t is a balanced unicyclic graph.*

Proof. If G_t is a balanced unicyclic graph, then by Corollary 2.11 (i) we have that the only $g(G_t)$-set is $\Omega(G_t)$. Also, notice that $V(G_t) - \Omega(G_t)$ is a $\gamma_{con}(G_t)$-set. Thus, G_t is an OCD-GDT graph. On the other hand, suppose G_t is an OCD-GDT graph. So, G_t contains an optimal convex dominating-geodetic partition given by the convex dominating set D and the geodetic set S. If G_t is not a balanced unicyclic graph, then there exists $j \in \{0, ..., t-1\}$ such that for every $u_i \in \{u_j, u_{j+1}, ..., u_{j+\lfloor \frac{t}{2} \rfloor -1}\}$, u_i has degree two in G_t. Also, by Corollary 2.11 (ii), every $g(G_t)$-set S contains a vertex $u_l \in \{u_j, u_{j+1}, ..., u_{j+\lfloor \frac{t}{2} \rfloor -1}\} \subset V(C_t)$ such that at most one of its two neighbors, say u_{l-1}, is in S. Since $t \geq 6$, we have that the $u_{l-2} - u_{l+1}$ geodesic contains the vertices $u_{l-1}, u_l \in S$. Thus, $V(G_t) - S$ is not convex. Similar arguments can be described for each $g(G_t)$-set, which leads to a contradiction since G_t is an OCD-GDT graph. Therefore, G_t is a balanced unicyclic graph. □

Theorem 2.15. *If $t \geq 6$ and G_t is not a balanced unicyclic graph, then G_t is not a CD-GDT graph.*

Proof. By using similar arguments to the proof of Theorem 2.14, we obtain that for every $g(G_t)$-set S, $V(G_t) - S$ is not convex. Moreover, if we add more vertices of the cycle to the set S, in order to achieve the convexity for $V(G_t) - S$, then $V(G_t) - S$ is not dominating. So, G_t does not contain any convex dominating-geodetic partition and we are done. □

2.3. Product graphs

Next we study some different kind of product graphs. Given two graphs G and H with set of vertices $V_1 = \{v_1, v_2, ..., v_{n_1}\}$ and $V_2 = \{u_1, u_2, ..., u_{n_2}\}$, respectively, the Cartesian product of G and H is the graph $G \Box H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (v_i, u_j) and (v_k, u_l) are adjacent in $G \Box H$ if and only if ($v_i = v_k$ and $u_j \sim u_l$), or ($v_i \sim v_k$ and $u_j = u_l$). Given a set S of vertices of $G \Box H$, the *projection* of S over G is the set $P_G(S) = \{u \in V(G) : (u, x) \in S \text{ for some } x \in V(H)\}$. The following results will be useful to prove our results.

Lemma 2.16. [11] *Let G and H be connected graphs. A subset C of $V(G \Box H)$ is a convex dominating set in $G \Box H$ if and only if $C = C_1 \times C_2$ and*

(i) *C_1 is a convex dominating set in G and $C_2 = V(H)$, or*
(ii) *C_2 is a convex dominating set in H and $C_1 = V(G)$.*

Lemma 2.17. [3, 10] *Let G, H be a connected graphs of order greater than two. If S_1 and S_2 are geodetic sets in G and H, respectively, then $S_1 \times S_2$ is a geodetic set in $G \Box H$ and $S_1 \times S_2$ is not a $g(G \Box H)$-set.*

From the result above, it is deduced that if S is a geodetic set of a graph G, then for any graph H, the set $S \times V(H)$ is a geodetic set in $G \Box H$ and it is not a $g(G \Box H)$-set. This observation together with Lemmas 2.16 and 2.17 lead to the following results.

Proposition 2.18. *If the graphs G and H have order at least two and three, respectively, then the Cartesian product graph $G \Box H$ is not an OCD-GDT graph.*

Proposition 2.19. *If G is an OCD-GDT graph or a CD-GDT graph, then for every graph H, $G \Box H$ is a CD-GDT graph.*

The strong product $G \boxtimes H$ of the graphs G and H has vertex set equal to the Cartesian product of the vertex sets of the factors. Two distinct vertices (v_i, u_j) and (v_k, u_l) of $G \boxtimes H$ are adjacent if and only if ($v_i = v_k$ and $u_j \sim u_l$), or ($v_i \sim v_k$ and $u_j = u_l$), or ($v_i \sim v_k$ and $u_j \sim u_l$). The following results on geodetic sets in strong product graphs were obtained in [4].

Lemma 2.20. [4] *If P is a shortest $(a, b) - (c, d)$ path in a strong product graph $G \boxtimes H$ such that $d_{G \boxtimes H}((a, b), (c, d)) = d_G(a, b) = p$, then the projection of P over the graph G is a shortest $a - c$ path of length p.*

Lemma 2.21. [4] *For any two graphs G and H, $\min\{g(G), g(H)\} \leq g(G \boxtimes H) \leq g(G)g(H)$.*

Lemma 2.22. [4] *For any trees T_1 and T_2 with l_1 and l_2 leaves respectively, $g(T_1 \boxtimes T_2) = l_1 \cdot l_2$. Moreover, the only geodetic set of $T_1 \boxtimes T_2$ is formed by vertices (x, y) such that x and y are leaves in T_1 and T_2, respectively.*

Theorem 2.23. *The strong product graph $T_1 \boxtimes T_2$ of two trees T_1 and T_2 is a CD-GDT graph.*

Proof. From Lemma 2.22, the only geodetic set of $T_1 \boxtimes T_2$ is formed by the set of vertices $L = L_1 \times L_2$ where L_1 and L_2 are the set of leaves of T_1 and T_2, respectively. Since every vertex in L is an extreme vertex we have that $T_1 \boxtimes T_2$ satisfies property $\mathcal{P}$. Thus, by Lemma 2.3, we obtain that $T_1 \boxtimes T_2$ is a CD-GDT graph. □

Theorem 2.24. *If G and H are connected graphs satisfying one of the premises of Proposition 2.4 (i) or (ii), then $G \boxtimes H$ is a CD-GDT graph.*

Proof. The result is straightforward, since the product $A \times B$ of two cliques A and B being dominating sets in G and H, respectively, is also a clique which is a dominating set in $G \boxtimes H$. Thus the Proposition 2.4 (i) itself leads to the result. □

A *rooted graph* is a graph in which one vertex is labeled in a special way so as to distinguish it from other vertices. The special vertex is called the *root* of the graph. Let G be a labeled graph on n vertices. Let $\mathcal{H}$ be a sequence of n rooted graphs H_1, H_2,...,H_n. The *rooted product graph $G(\mathcal{H})$* is the graph obtained by identifying the root of H_i with the i^{th} vertex of G [5]. In this paper we consider a particular case of rooted product graphs, where $\mathcal{H}$ consists of n isomorphic rooted graphs [16]. More formally, assuming that $V(G) = \{u_1, ..., u_n\}$ and that the root vertex of H is v, we define the rooted product graph $G \circ_v H = (V, E)$, where $V = V(G) \times V(H)$ and

$$E = \bigcup_{i=1}^{n} \{(u_i, b)(u_i, y) : \; by \in E(H)\} \cup \{(u_i, v)(u_j, v) : \; u_i u_j \in E(G)\}.$$

Remark 2.25. *Let G be a connected graph of order $n \geq 2$ and let H be a non trivial graph. If v belongs to at least one $\gamma_{con}(H)$-set, then $\gamma_{con}(G \circ_v H) = n\gamma_{con}(H)$.*

Theorem 2.26. *Let G be a connected graph of order $n \geq 2$ and let H be a non trivial graph. Let v be a vertex of H.*

(i) *If v belongs to a $g(H)$-set, then $g(G \circ_v H) = n(g(H) - 1)$.*
(ii) *If v does not belong to any $g(H)$-set, then $g(G \circ_v H) = n \cdot g(H)$.*

Proof. Let $V = \{u_1, u_2, ..., u_n\}$ be the vertex set of G. Suppose first v belongs to some $g(H)$-set A and let $B = \bigcup_{i=1}^{n}(\{u_i\} \times (A - \{v\}))$. We shall prove that B is a geodetic set of $G \circ_v H$. Let $(u_i, y) \notin B$. If $y \neq v$, then there exist vertices $a, b \in A$ such that $x \in I_H[a, b]$. So, if $a, b \neq v$, then $(u_i, y) \in I_{G\circ_v H}[(u_i, a), (u_i, b)]$. On the contrary, suppose $a = v$. In this case, for any vertex $u_j \in V$, $j \neq i$, and any vertex $c \in A - \{v\}$ we have that $(u_i, y) \in I_{G\circ_v H}[(u_j, c), (u_i, b)]$. On the other hand, if $y = v$, then for any vertex $c \in A - \{v\}$ and any vertex $u_j \in V$, $j \neq i$, we have that $(u_i, y) \in I_{G\circ_v H}[(u_j, c), (u_i, b)]$. Therefore, B is a geodetic set in $G \circ_v H$ and we obtain $g(G \circ_v H) \leq n(g(H) - 1)$.

Now, let S be a $g(G\circ_v H)$-set and let U be the vertex set of H. Now, for every $i \in \{1, ..., n\}$, let $S_i = S \cap (\{u_i\} \times U)$. Let the vertex $(u_i, x) \notin S_i$, $x \neq v$. So, $(u_i, x) \in I_{G\circ_v H}[(u_i, a), (u_j, b)]$ for some $u_i, u_j \in V$ and $a, b \in U$. If $u_i \neq u_j$, then $(u_i, x) \in I_{G\circ_v H}[(u_i, a), (u_i, v)]$. Thus we obtain that $S_i \cup \{(u_i, v)\}$ is a geodetic set of a graph induced by $\{u_i\} \times U$ and also, $g(H) \leq |S_i| + 1$. Therefore, $g(G \circ_v H) = |S| = \sum_{i=1}^{n} |S_i| \geq n(g(H) - 1)$ and the (i) is proved.

Now on suppose v does not belong to any $g(H)$-set. Let A be a $g(H)$-set. Hence, it is straightforward to prove that $B = \bigcup_{i=1}^{n}(\{u_i\} \times A)$ is a geodetic set in $G \circ_v H$ and we obtain $g(G \circ_v H) \leq n \cdot g(H)$. Suppose now that $g(G \circ_v H) < n \cdot g(H)$ and let S be a $g(G \circ_v H)$-set. So, there exists a set $S_i = S \cap (\{u_i\} \times U)$ such that $|S_i| < g(H)$. Nevertheless $S_i \cup \{(u_i, v)\}$ is a geodetic set in a graph induced by $\{u_i\} \times U \cong H$ with $|S_i \cup \{(u_i, v)\}| = g(H)$, a contradiction. Therefore $g(G \circ_v H) = n \cdot g(H)$ and (ii) is proved. □

The results above lead to the conclusion that no vertex of the graph G belongs to any $g(G \circ_v H)$-set. Therefore, we deduce the following result.

Theorem 2.27. *Let G be a connected graph of order greater than one and let H be a non trivial graph with vertex set U. Let $v \in U$ belonging to some $\gamma_{con}(H)$-set.*

(i) *If v belongs to a $g(H)$-set S and $U - S \cup \{v\}$ is a $\gamma_{con}(H)$-set, then $G \circ_v H$ is an OCD-GDT graph.*
(ii) *If v does not belong to any $g(H)$-set and H is an OCD-GDT graph, then $G \circ_v H$ is an OCD-GDT graph.*

Corollary 2.28. *Let G be a connected graph of order greater than one and let T be a non trivial tree with vertex set U. Then for every $v \in U$, $G \circ_v T$ is an OCD-GDT graph.*

The corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and $n = |V(G)|$ copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G.

Remark 2.29. *Let G be a connected graph of order $n \geq 2$ and let H be any graph. Then, $\gamma_{con}(G \odot H) = n$. Moreover, the only minimum convex dominating set of $G \odot H$ is formed by the vertex set of G.*

The following result was obtained in [18].

Lemma 2.30. [18] *Let G be a connected graph of order $n \geq 2$ and let H be any graph of order n_1. Then $g(G \odot H) = n \cdot n_1$ if and only if H is a graph in which every connected component is isomorphic to a complete graph.*

Remark 2.29 and Lemma 2.30 lead to the following result.

Theorem 2.31. *Let G be a connected graph of order $n \geq 2$ and let H be any graph. Then $G \odot H$ is always a CD-GDT graph. Moreover, $G \odot H$ is an OCD-GDT graph if and only if H is a graph in which every connected component is isomorphic to a complete graph.*

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