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A Fixed Point Technique for Approximate a Functional Inequality in Normed Modules over C^*-algebras

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Abstract. In this paper, we apply fixed point technique to investigate the following additive functional inequality:

$$\|f(x) + f(y) + f(z) + f(w)\| \leq \|f(x + y) + f(z + w)\|$$

in normed modules over a C^*-algebra, which is also applied to understand homomorphisms in C^*-algebras. Our results improve and generalize some results given by some authors. Especially, we get a better error estimation of An's main result.

1. Introduction

Ulam [20] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms:

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

Recently, an affirmative answer has been given in several cases and some interesting variations of the problem have also been investigated.

Hyers [10] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies the *Hyers inequality*:

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

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for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping satisfying the following:

$$\|f(x) - L(x)\| \leq \epsilon.$$

No the continuity conditions are required for this result, but, if $f(tx)$ is continuous in the real variable t for each fixed x, then L is linear and, if f is continuous at a single point of E, then $L : E \to E'$ is also continuous.

In 1982 and 1994, a generalization of this result was proved by Rassias [17]. Also, he introduced the following weaker condition (or weaker inequality or Cauchy inequality):

$$\|f(x + y) - f(x) - f(y)\| \leq \theta\|x\|^p\|y\|^q$$

for all x, y in E, which was controlled by (or involving) a product of different powers of norms, where $\theta \geq 0$ and real $p,\ q : r = p + q \neq 1$, and retained the condition of the continuity of $f(tx)$ in t for fixed x. Besides, he investigated that it is possible to replace ϵ in the above Hyers inequality by a non-negative real-valued function such that the pertinent series converges and other conditions hold and still obtain the stability results. In all the cases investigated in these results, the approach to the existence question was to prove asymptotic type formulas of the form:

$$L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

or

$$L(x) = \lim_{n\to\infty} 2^{n} f(2^{-n} x).$$

Theorem 1.1. ([17]) *Let X be a real normed linear space and Y be a real Banach space. Assume, in addition, that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the Cauchy-Rassias inequality:*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta\|x\|^p\|y\|^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying the following:

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|}\|x\|^r$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Gilányi [8] showed that, if f satisfies the functional inequality:

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\|, \tag{1}$$

then f satisfies the Jordan-von Neumann functional equation:

$$2f(x) + 2f(y) = f(x + y) + f(x - y)$$

(see also [18]). Fechner [7] and Gilányi [9] proved the generalized Hyers-Ulam stability of the functional inequality (1). Park et al. [16] investigated the functional inequality:

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| \tag{2}$$

in Banach spaces and proved the generalized Hyers-Ulam stability of the functional inequality (2) in Banach spaces.

Throughout this paper, let A be a unital C^*-algebra with the unitary group $U(A)$ and the unit e and B be a C^*-algebra. Assume that X is a normed A-module with the norm $\|\cdot\|_X$ and Y is a normed A-module with the norm $\|\cdot\|_Y$.

In this paper, we investigate an A-linear mapping associated with the functional inequality:

$$\|f(x) + f(y) + f(z) + f(w)\| \le \|f(x+y) + f(z+w)\|. \tag{3}$$

This is also applied to understand homomorphisms in C^*-algebras.

Following Luxemburg [12], the concept of a *generalized complete metric space* may be introduced as in this quotation:

Let X be a nonempty set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if, for any $x, y, z \in X$,

(1) $d(x,y) = 0$ if and only if $x = y$,
(2) $d(x,y) = d(y,x)$,
(3) $d(x,z) \le d(x,y) + d(y,z)$.

This concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such a space a *generalized complete metric space*.

Next, Diaz and Margolis [3] proved a *theorem of the alternative* for any contraction mapping on a generalized complete metric space X. For similar and new results, see [4, 5, 13, 15, 19, 21]. The conclusion of the theorem, speaking in general terms, asserts that either all consecutive pairs of the sequence of successive approximations (starting from an element $x_0 \in X$) are infinitely far apart or the sequence of successive approximations with initial element x_0 converges to a fixed point of T (what particular fixed point depends, in general, on the initial element x_0).

Theorem 1.2. ([2, 3]) *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with a Lipschitz constant $L < 1$. Then, for each $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for each $n \ge 0$ or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$ *for each $n \ge n_0$,*
(2) *the sequence $(J^n x)$ converges to a fixed point y^* of J,*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$,*
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)b$ *for all $y \in Y$.*

2. Functional Inequalities in Normed Modules over C^*-Algebras

Theorem 2.1. ([1]) *Let $f : X \to Y$ be a mapping such that*

$$\|uf(x) + f(y) + f(z) + f(w)\|_Y \le \|f(ux+y) + f(z+w)\|_Y \tag{4}$$

for all $x, y, z, w \in X$ and all $u \in U(A)$. Then the mapping $f : X \to Y$ is A-linear.

Corollary 2.2. *Let $f : A \to B$ be a multiplicative mapping such that*

$$\|\mu f(x) + f(y) + f(z) + f(w)\| \le \|f(\mu x + y) + f(z+w)\| \tag{5}$$

for all $x, y, z, w \in A$ and $\mu \in \mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Proof. By Theorem 2.1, the multiplicative mapping $f : A \to B$ is $\mathbb{C}$-linear since C^*-algebras are normed modules over $\mathbb{C}$. Therefore, the multiplicative mapping $f : A \to B$ is a C^*-algebra homomorphism. $\square$

3. Generalizations of Cauchy-Rassias Inequalities

Theorem 3.1. *Let X be a real normed linear space and Y be a real Banach space. Assume, in addition, that $f : X \to Y$ is an approximately additive odd mapping satisfies the general Cauchy-Rassias inequality:*

$$\|f(x) + f(y) + f(z) + f(w)\| \leq \|f(x + y + z + w)\| + \varphi(x, y, z, w) \tag{6}$$

for all $x, y, z, w \in X$, where $\varphi : X^4 \to [0, \infty)$ is a given function. If there exists $0 < L < 1$ such that

$$\varphi(x, y, z, w) \leq \frac{1}{3} L\varphi(3x, 3y, 3z, 3w) \tag{7}$$

for all $x, y, z, w \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying the following:

$$\|f(x) - A(x)\| \leq \frac{L}{3 - 3L} \varphi(x, x, x, -3x) \tag{8}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then A is an $\mathbb{R}$-linear mapping.

Proof. Since f is odd, $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$. Consider the set $S := \{g : X \to Y\}$ and introduce the generalized metric on S:

$$d(g, h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq C\varphi(x, x, x, -3x), \ \forall x \in X\}.$$

Now, we show that (S, d) is complete. Let $\{h_n\}$ be a Cauchy sequence in (S, d). Then, for any $\varepsilon > 0$, there exists an integer $N_\varepsilon > 0$ such that $d(h_m, h_n) < \varepsilon$ for all $m, n \geq N_\varepsilon$. Then there exists $C \in (0, \varepsilon)$ such that

$$\|h_m(x) - h_n(x)\| \leq C\varphi(x, x, x, -3x) \leq \varepsilon\varphi(x, x, x, -3x) \tag{9}$$

for all $m, n \geq N_\varepsilon$ and $x \in X$. Since Y is complete, $\{h_n(x)\}$ converges for all $x \in X$. Thus a mapping $h : X \to Y$ can be defined by

$$h(x) := \lim_{n\to\infty} h_n(x) \tag{10}$$

for all $x \in X$. Letting $n \to \infty$ in (9), we have

$$\begin{aligned} m \geq N_\varepsilon \quad &\Longrightarrow \quad \|h_m(x) - h_n(x)\| \leq \varepsilon\varphi(x, x, x, -3x) \\ &\Longrightarrow \quad \varepsilon \in \{C \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq C\varphi(x, x, x, -3x), \ \forall x \in X\} \\ &\Longrightarrow \quad d(h_m, h) \leq \varepsilon \end{aligned}$$

for all $x \in X$. This means that the Cauchy sequence $\{h_n\}$ converges to h in (S, d). Hence (S, d) is complete.

Now, we consider the linear mapping $\Lambda : S \to S$ such that

$$\Lambda g(x) := 3g\left(\frac{x}{3}\right) \tag{11}$$

for all $x \in X$. We show that Λ is a strictly contractive mapping on S. For any $g, h \in S$, let $C_{g,h} \geq 0$ be an arbitrary constant with $d(g, h) \leq C_{g,h}$. Then we have

$$\begin{aligned} d(g, h) \leq C_{g,h} \quad &\Longrightarrow \quad \|g(x) - h(x)\| \leq C_{g,h}\varphi(x, x, x, -3x) \\ &\Longrightarrow \quad \left\|3g\left(\frac{x}{3}\right) - 3h\left(\frac{x}{3}\right)\right\| \leq 3C_{g,h}\varphi\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}, -x\right) \\ &\Longrightarrow \quad \left\|3g\left(\frac{x}{3}\right) - 3h\left(\frac{x}{3}\right)\right\| \leq LC_{g,h}\varphi(x, x, x, -3x) \end{aligned}$$

for all $x \in X$. This means $d(\Lambda g, \Lambda h) \leq LC_{g,h}$. Hence we see that $d(\Lambda g, \Lambda h) \leq Ld(g,h)$ for any $g, h \in S$. Therefore, Λ is a strictly contractive mapping on S with the Lipschitz constant $0 < L < 1$. Letting $y = z = x$ and $w = -3x$ in (6), we get

$$\|3f(x) - f(3x)\| \leq \varphi(x, x, x, -3x) \tag{12}$$

for all $x \in X$ and so

$$\left\| f(x) - 3f\left(\frac{x}{3}\right) \right\| \leq \varphi\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}, -x\right) \leq \frac{1}{3} L\varphi(x, x, x, -3x)$$

for all $x \in X$. Thus we have

$$d(f, \Lambda f) \leq \frac{L}{3}.$$

Therefore, it follows of Theorem 1.2 that the sequence $\{\Lambda^n f\}$ converges to an unique fixed point A of Λ, i.e.,

$$A(x) = (\Lambda f)(x) =:= \lim_{k\to\infty} 3^k f\left(\frac{x}{3^k}\right)$$

and $A(3x) = 3A(x)$ for all $x \in X$. Also, we have

$$d(A, f) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{L}{3-3L},$$

this means that (8) holds.

Next, assume that $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. By the same reasoning as in the proof of Theorem 2.1, one can prove that A is an $\mathbb{R}$-linear mapping. This completes the proof. □

In the next corollary, we get a better error estimation of main result of [1].

Corollary 3.2. *Let X be a real normed linear space and Y be a real Banach space. Assume, in addition, that $f : X \to Y$ is an approximately additive odd mapping for which there exist constants $\theta \geq 0$ and $p \in \mathbb{R}$ such that $4p \neq 1$ and f satisfies the general Cauchy-Rassias inequality:*

$$\|f(x) + f(y) + f(z) + f(w)\| \leq \|f(x + y + z + w)\| + \theta\|x\|^p\|y\|^p\|z\|^p\|w\|^p \tag{13}$$

for all $x, y, z, w \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying the following:

$$\|f(x) - L(x)\| \leq \frac{3^{p-1}\theta}{|81^p - 3|}\|x\|^{4p} \tag{14}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then A is an $\mathbb{R}$-linear mapping.

Proof. In Theorem 3.1, take

$$\varphi(x, y, z, w) := \theta\|x\|^p\|y\|^p\|z\|^p\|w\|^p$$

for all $x, y, z, w \in X$. Then we can choose $L = 81^{1-p}$ and so we have desired result. □

Corollary 3.3. *Let X be a real normed linear space and Y be a real Banach space. Assume, in addition, that $f : X \to Y$ is an approximately additive odd mapping for which there exist constants $\theta \geq 0$ and $p \in \mathbb{R}$ such that $4p \neq 1$ and f satisfies the general Cauchy-Rassias inequality:*

$$\begin{aligned} &\|f(x) + f(y) + f(z) + f(w)\| \\ \leq\ &\|f(x + y + z + w)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) \end{aligned} \tag{15}$$

for all $x, y, z, w \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{(3^p + 3)\theta}{|3^p - 3|}\|x\|^p \tag{16}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then A is an $\mathbb{R}$-linear mapping.

Proof. In Theorem 3.1, take

$$\varphi(x, y, z, w) := \theta\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p$$

for all $x, y, z, w \in X$. Then we can choose $L = 3^{1-p}$ and so we have desired result. □

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