



## Combinatorial Identities Associated with Bernstein Type Basis Functions

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**Abstract.** In this paper, we give some identities and relations for the Bernstein basis functions and the beta type polynomials. Integrating these identities, we derive many identities and formulas, some old and some new, for combinatorial sums involving binomial coefficients and the Catalan numbers. We also give remarks and comments on these identities.

### 1. Introduction and Main Definition

Combinatorial sums and combinatorial problems appear in many areas of mathematics, notably in algebra, in probability theory, in topology, in geometry, in mathematical optimization, in computer science, in ergodic theory and also in statistical physics. Combinatorics is also used in computer science to derive formulas and estimates in the analysis of algorithms. Recently, In order to study combinatorial sums and combinatorial problems in all branch of mathematics, many different methods have been developed (cf. [25]).

In [19] Gould and Srivastava, studied some combinatorial identities associated with the Vandermonde convolution. They presented a unification and generalization of some combinatorial identities associated with the familiar Vandermonde convolution. They also gave  $q$ -extension of the combinatorial identity. In [20], by using double-series identities and hypergeometric series, Srivastava and Raina derived three new classes of combinatorial series identities.

Recently, by using generating functions, special functions, combinatorial sums involving binomial coefficients and inverses of binomial coefficients have studied by many Mathematicians (cf. [1]-[24] and see also the references cited in each of these earlier works ). In this paper, by using an approach similar to that of the author as well as some known properties of generating functions for the Bernstein basis functions and the beta type polynomials and their identities [11]-[17], we shall derive some known and some new identities and formulas for the combinatorial sums involving binomial coefficients and inverses of binomial coefficients and the Catalan numbers.

Bernstein [1] first introduced and investigated the extended form of the polynomials, which are known as the Bernstein polynomials. By using the Bernstein polynomials, *Bezier Curves and Surfaces* are constructed.

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These polynomials are used to approximate a curve and also derive many combinatorial identities (cf. [17], [15], [14]). Recently, the Bernstein polynomials have many applications: in approximations of functions, in statistics, in numerical analysis, in  $p$ -adic analysis, in the solution of differential equations and in Computer Aided Geometric Design (CAGD). In CAGD, polynomials are often expressed in terms of the Bernstein basis functions. These polynomials are called Bezier curves (cf. [1], [3], [4], [5], [6], [11], [12], [13], [14], [15], [16], [17] and see also the references cited in each of these earlier works).

The Bernstein basis functions  $B_k^n(x)$  are defined as follows (cf. [1], [3], [4]):

**Definition 1.1.** Let  $x \in [0, 1]$ . Let  $n$  and  $n$  be non-negative integers. The Bernstein basis functions  $B_k^n(x)$  can be defined by

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{1}$$

where

$$k = 0, 1, \dots, n,$$

and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

**Remark 1.2.** Definition 1.1 is modified by Goldman [4, p. 384, Eq.(24.6)]. If the interval  $[0, 1]$  is extended to arbitrary intervals  $[a, b]$  and  $x$  is replaced by  $\frac{x-a}{b-a}$ , then Definition 1.1 yields the corresponding well known results concerning the Bernstein basis functions  $B_k^n(x, a, b)$ :

$$B_k^n(x; a, b) = \binom{n}{k} \left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k}, \tag{2}$$

where  $k = 0, 1, \dots, n$  and  $x \in [a, b]$  (cf. [1], [4, p. 384, Eq.(24.6)], [14]).

In [15], we defined a new class polynomials which are related to the family of the Bernstein polynomials, the unification of the Bernstein type polynomials and the beta polynomials:

**Definition 1.3.** Let  $b, n$  and  $k$  be nonnegative integers. Then we define

$$Y_k^n(x; b) = \binom{n}{k} \frac{x^k (1+x)^{n-k-b}}{2^{b-1}}. \tag{3}$$

where  $n \geq k + b$ ,

$$\binom{n}{b} = \frac{n!}{b!(n-b)!}, n \geq b,$$

$k = 0, 1, 2, \dots, n$  and  $b = 0, 1, 2, \dots, n$ .

In this paper, we also use the following notations:

$$\mathbb{N} := \{1, 2, 3, \dots\}$$

and

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Combinatorial sums involving binomial coefficients are very important applications in all branches of Mathematics. In order to find identities and formulas, some old and some new, for the Catalan numbers and combinatorial sums involving binomial coefficients and inverse binomial coefficients, we give some identities and relations for the Bernstein basis functions and the beta type polynomials. In order to give identities related to the Bernstein basis functions and combinatorial sums, we need the the following lemma.

**Lemma 1.4.** (Generalized multinomial identity [2, p. 41, Equation (12m)]) If  $x_1, x_2, \dots, x_m$  are commuting elements of a ring ( $\Leftrightarrow x_i x_j = x_j x_i, 1 \leq i < j \leq m$ ), then we have for all real or complex variable  $\alpha$ :

$$(x_1 + x_2 + \dots + x_m)^\alpha = \sum_{v_1, v_2, \dots, v_m \geq 0} C_{v_1, \dots, v_m}^\alpha x_1^{v_1} x_2^{v_2} \dots x_m^{v_m}, \tag{4}$$

the last summation takes places over all positive or zero integers  $v_i \geq 0$ , where

$$C_{v_1, \dots, v_m}^\alpha = \binom{\alpha}{v_1, v_2, \dots, v_m} := \frac{(\alpha)_{v_1+v_2+\dots+v_m}}{v_1! v_2! \dots v_m!}$$

are called generalized multinomial coefficients, where  $(x)_0 = 1$ .

**Remark 1.5.** The following multinomial identity is equivalent to (4):

$$\begin{aligned} (x_1 + x_2 + \dots + x_v)^k &= \sum_{m_1+m_2+\dots+m_{v-1}=k} \frac{x_1^{m_1} x_2^{m_2} \dots x_v^{k-(m_1+m_2+\dots+m_{v-1})}}{m_1! m_2! \dots (k-(m_1+m_2+\dots+m_{v-1}))!} \\ &= \sum_{m_1+m_2+\dots+m_{v-1}=k} C_{m_1, \dots, m_{v-1}}^k x_1^{m_1} x_2^{m_2} \dots x_v^{k-(m_1+m_2+\dots+m_{v-1})}, \end{aligned}$$

where

$$\sum_{m_1+m_2+\dots+m_{v-1}=k} = \sum_{m_1=0}^k \sum_{m_2=0}^{k-m_1} \dots \sum_{m_{v-1}=0}^{k-m_1-m_2-\dots-m_{v-2}}.$$

In order to find combinatorial sums in Section 2, we also need the following identities and relations for the Bernstein basis functions, the gamma function and the beta function.

In [14], we proved the following theorems:

**Theorem 1.6.** ([14, p. 475, Theorem 2.1.]) Let  $n \geq k_1 + \dots + k_v$ . Then we have

$$\begin{aligned} B_{k_1+\dots+k_v}^n(x) &= \frac{v^{k_1+\dots+k_v-n} k_1! \dots k_v!}{(k_1 + \dots + k_v)!} \\ &\times \sum_{m_1+\dots+m_{v-1}=n} C_{m_1, \dots, m_{v-1}}^n B_{k_v}^{m_{v-1}}(x) B_{k_{v-1}}^{m_{v-2}}(x) \dots B_{k_1}^{m_1}(x) B_{k_2}^{n-m_1-\dots-m_{v-1}}(x). \end{aligned} \tag{5}$$

By substituting  $v = 2$  into (5), we have

$$B_{k_1+k_2}^n(x) = \frac{2^{k_1+k_2-n} k_1! k_2!}{(k_1 + k_2)!} \sum_{m_1=0}^n \binom{n}{m_1} B_{k_1}^{m_1}(x) B_{k_2}^{n-m_1}(x)$$

(cf. [13, Theorem 4.1], [14]).

The gamma function  $\Gamma(s)$  is defined for all complex numbers except the negative integers and zero. Let  $s$  be a complex numbers with a positive real part. The gamma function is defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

(cf. [21]).

The Beta function  $B(\alpha, \beta)$  is a function of two complex variables  $\alpha$  and  $\beta$ , defined by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = B(\beta, \alpha), \tag{6}$$

where  $\alpha, \beta \in \mathbb{C}$  with positive real parts (cf. [21, p. 9, Eq-(60)], [22]).

A relation between the gamma function and the beta function is given by

$$B(n, m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} = \frac{(n-1)!(m-1)!}{(n+m-1)!}, \tag{7}$$

where  $m$  and  $n$  are positive integers (cf. [21, p. 9, Eq-(62)], [22]). Integral representation of the Bernstein basis functions is given by the following theorem:

**Theorem 1.7.** ([14])

$$\int_a^b B_k^n(x; a, b) dx = \binom{n}{k} (b-a) B(k+1, n-k+1), \tag{8}$$

where  $B_k^n(x; a, b)$  is defined in (2).

**Remark 1.8.** In [21, p. 10, Eq-(69)], Srivastava and Choi defined the following integral related to the Beta function:

$$\int_a^b (x-a)^{\alpha-1} (b-x)^{\beta-1} dx = (b-a)^{\alpha+\beta-1} B(\alpha, \beta).$$

The organization of this paper is given as follows:

In Section 2, by using beta function, the Bernstein basis functions, beta type polynomials and also the Marsden identity, we derive many combinatorial sums and identities involving the binomial coefficients, inverse binomial coefficients and the Catalan numbers.

## 2. Combinatorial Sums Involving Binomial Coefficients and the Catalan Numbers

Combinatorial sums and binomial coefficients have been used all branches of Mathematics, particularly including Statistics, Probability, Combinatorics, Analytic Number Theory and CAGD. In this section, by applying the Riemann integral, the beta function and the gamma function to the identities for the Bernstein basis functions and the beta type polynomials, we derive many combinatorial sums involving binomial coefficients and the Catalan numbers.

Integrating both sides of Equation (5) from 0 to 1 and using Equation (8), we arrive at the following identity:

**Theorem 2.1.** Let  $n \in \mathbb{N}$ . Then we have

$$\begin{aligned} & \sum_{m_1+\dots+m_{v-1}=n} C_{m_1, \dots, m_{v-1}}^n \binom{m_{v-1}}{k_v} \dots \binom{m_2}{k_1} \binom{n-m_1-\dots-m_{v-1}}{k_2} \\ &= \frac{v^{n-(k_1+\dots+k_v)} (k_1+\dots+k_v)!}{k_1! \dots k_v!} \binom{n}{k_1+\dots+k_v}. \end{aligned} \tag{9}$$

Substituting  $v = 2$  into (9), we obtain the following known result due to the author [16, Theorem 5.2]:

**Corollary 2.2.** Let  $n \in \mathbb{N}$ . Then we have

$$\sum_{m_1=0}^n \binom{n}{m_1} \binom{m_1}{k_1} \binom{n-m_1}{k_2} = \frac{2^{n-k_1-k_2} (k_1+k_2)!}{k_1! k_2!} \binom{n}{k_1+k_2}. \tag{10}$$

If we replace  $k_1$  by 1 and  $k_2$  by 1 in (10), we obtain the following result:

**Corollary 2.3.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ . Then we have*

$$\sum_{m_1=1}^n m_1 (n - m_1) \binom{n}{m_1} = 2^{n-1} \binom{n}{2}.$$

If we replace  $k_1$  by  $n$ ,  $k_2$  by 1, and  $n$  by  $2n$  in (10), we obtain the following result:

**Corollary 2.4.** *Let  $n \in \mathbb{N}$ . Then we have*

$$\sum_{m_1=n}^{2n} \binom{2n}{m_1} \binom{m_1}{n} (2n - m_1) = \frac{2^{n-1}(n+1)!}{n!} \binom{2n}{n+1}.$$

If we replace  $k_1$  by  $n$ ,  $k_2$  by  $n$ , and  $n$  by  $4n$  in (10), we obtain the following result:

**Corollary 2.5.** *Let  $n \in \mathbb{N}$ . Then we have*

$$\sum_{m_1=n}^{4n} \binom{4n}{m_1} \binom{m_1}{n} \binom{4n - m_1}{n} = \frac{2^{2n}(2n)!}{(n!)^2} \binom{4n}{2n}. \tag{11}$$

By using (11), we obtain the following formula for the Catalan numbers  $C_n$ , which are defined by

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

(cf. [2], [16] and see also the references cited in each of these earlier works).

The Catalan numbers occur in the solutions of many counting problems. These numbers are related to the Riordan arrays, the Riordan group, and the ballot problem (cf. [16], [23] and see also the references cited in each of these earlier works).

**Corollary 2.6.** *Let  $n \in \mathbb{N}_0$ . Then the following identities holds true:*

$$C_{2n} = \frac{2^{-2n}(n!)^2}{(2n+1)!} \sum_{m_1=n}^{4n} \binom{4n}{m_1} \binom{m_1}{n} \binom{4n - m_1}{n}.$$

In [14], by using generating functions for the Bernstein basis functions, we proved the following Marsden identity:

**Theorem 2.7.**

$$(y - x)^n = \sum_{k=0}^n (-1)^k \binom{n}{k}^{-1} B_{n-k}^n(y) B_k^n(x). \tag{12}$$

By using the Marsden identity, we prove the well-known combinatorics identity involving inverse binomial coefficients.

Integrating both sides of Equation (12) with respect to  $x$  and  $y$  from 0 to 1, we get

$$\int_0^1 \int_0^1 (y - x)^n dx dy = \sum_{k=0}^n (-1)^k \binom{n}{k}^{-1} \int_0^1 \int_0^1 B_{n-k}^n(y) B_k^n(x) dx dy.$$

By substituting Equation (8) into the above equation, after some elementary calculation, we arrive at the following theorem:

**Theorem 2.8.** Let  $n \in \mathbb{N}$ . Then we have

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^{-1} = (1 + (-1)^n) \frac{n+1}{n+2}. \tag{13}$$

**Remark 2.9.** In [11], we defined the function  $\mathfrak{M}_{k,n}(x)$  as follows:

$$\mathfrak{M}_{k,n}(x) = x^k(1+x)^{n-k}.$$

If  $n \geq k$ , then we have the beta polynomials:

$$\mathfrak{M}_{k,n}(x) = \mathfrak{B}_{k,j}(x).$$

In [17], by using the following identity

$$\sum_{k=0}^n \mathfrak{M}_{k,n}(x) = (1+x)^{n+1} - x^{n+1},$$

we give another proof of (13). Proofs of (13) was given by Gould ([8, Vol. 3, Eq-(5.13)]). Gould [7, P. 18, Eq-(1)] gave proof of the following combinatorics identity

$$\sum_{k=0}^n (-1)^k \binom{x}{k}^{-1} = \left( 1 + (-1)^n \binom{x+1}{n+1}^{-1} \right) \frac{x+1}{x+2}.$$

Setting  $x = n$  in the above equation, one can also easily obtain (13).

In [15, Theorem 10], we approved the following identity:

$$\sum_{b=0}^n Y_k^n(x; b) = \sum_{j=0}^n \binom{n}{j} 2^{j-n+1} \mathfrak{B}_{k,j}(x),$$

where  $n \geq k + b$ .

Integrating both sides of the above equation with respect to  $x$  from  $-1$  to  $0$ , after some elementary calculations, we arrive at the following theorem:

**Theorem 2.10.** Let  $n \geq k + b$ . Then we have

$$\sum_{b=0}^n \frac{1}{2^{b-1}(n-b+1)} \frac{\binom{n}{b}}{\binom{n-b}{k}} = \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{j}{k}} \frac{2^{j-n+1}}{j+1}.$$

In [15, Theorem 11], we gave the following identity:

$$\sum_{b=0}^n (-1)^b Y_k^n(x; b) = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} 2^{j-n+1} \mathfrak{B}_{k,j}(x),$$

where  $n \geq k + b$ .

Integrating both sides of the above equation with respect to  $x$  from  $-1$  to  $0$ , we arrive at the following theorem:

**Theorem 2.11.** Let  $n \geq k + b$ . Then we have

$$\sum_{b=0}^n \frac{(-1)^b}{2^{b-1}(n-b+1)} \frac{\binom{n}{b}}{\binom{n-b}{k}} = \sum_{j=0}^n (-1)^{n-j} \frac{\binom{n}{j}}{\binom{j}{k}} \frac{2^{j-n+1}}{j+1}.$$

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