



Sobolev Type Spaces Based on Lorentz-Karamata Spaces

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Abstract. In this paper, firstly Lorentz-Karamata-Sobolev spaces $W_{L(p,q;b)}^k(\mathbb{R}^n)$ of integer order are introduced and some of their important properties are emphasized. Also, Banach spaces $A_{L(p,q;b)}^k(\mathbb{R}^n) = L^1(\mathbb{R}^n) \cap W_{L(p,q;b)}^k(\mathbb{R}^n)$ (Lorentz-Karamata-Sobolev algebras) are studied. Using a result of H.C.Wang, it is showed that Banach convolution algebras $A_{L(p,q;b)}^k(\mathbb{R}^n)$ don't have weak factorization and the multiplier algebra of $A_{L(p,q;b)}^k(\mathbb{R}^n)$ coincides with the measure algebra $M(\mathbb{R}^n)$ for $1 < p < \infty$ and $1 \leq q < \infty$.

1. Introduction and Preliminaries

A new generalization of Lebesgue, Lorentz, Zygmund, Lorentz-Zygmund and generalized Lorentz-Zygmund spaces was studied by D.E.Edmunds, R.Kerman and L.Pick in [12]. By using Karamata theory, they introduced Lorentz-Karamata (or briefly *LK*) spaces and compared corresponding quasinorms on these spaces. Also J.S.Neves studied *LK* spaces $L(p, q; b)(R, \mu)$ in [22] where $p, q \in (0, \infty]$, b is a slowly varying function on $[1, \infty)$ and (R, μ) is a measure space. These spaces include the spaces generalized Lorentz-Zygmund spaces $L_{p,q;\alpha_1, \dots, \alpha_m}(R)$ (introduced in [11]), Lorentz-Zygmund spaces $L^{p,q}(\log L)^\alpha(R)$ (introduced in [2]), Zygmund spaces $L^p(\log L)^\alpha(R)$ (introduced in [3, 25]), Lorentz spaces $L(p, q)(R)$ and Lebesgue spaces $L^p(R)$ under convenient choices of slowly varying functions and parameters p, q . In [10] and [22], it is proved that $L(p, q; b)(R, \mu)$ space endowed with a convenient norm, is a rearrangement-invariant Banach function space and has an associate space $L(p', q'; b^{-1})(R, \mu)$ if (R, μ) is a resonant measure space, $p \in (1, \infty)$ and $q \in [1, \infty]$. Also it is showed that when $p \in (1, \infty)$ and $q \in [1, \infty)$, *LK* spaces have absolutely continuous norm.

If one looks for "Sobolev algebras" in literature, he sees that there are a lot of published papers about Sobolev algebras obtained by using different function spaces that are defined over different groups or sets. These spaces have been investigated under several respects, and mostly applied to the study of strongly nonlinear variational problems and partial differential equations.

In the sense of our study, we attach importance to [7–9, 15]. In [8], Orlicz-Sobolev spaces that are multiplicative Banach algebras are characterized. In [9], it is showed that the space $L_\alpha^p(G) \cap L^\infty(G)$ is an algebra with respect to pointwise multiplication where G is a connected unimodular Lie group. Also,

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sufficient conditions for the Sobolev spaces to form an algebra under pointwise multiplication have been given.

In [7], Chu defined $A_k^p(\mathbb{R}^n) = L^1(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ spaces and showed some algebraic properties of these spaces (Segal algebras). Moving from this paper, the results are carried to Sobolev-Lorentz spaces in [15]. In this paper, we will generalize his results to Lorentz-Karamata-Sobolev spaces and Lorentz-Karamata-Sobolev algebras.

Throughout this paper, G, \mathbb{R}^n and dx will stand for a non-compact locally compact abelian group, Euclidean n -dimensional space and Lebesgue measure, respectively. Besides these, $C_0(\mathbb{R}^n)$ will denote the space of all continuous functions that vanish at infinity. For any two non-negative expressions (i.e. functions or functionals), A and B , the symbol $A \lesssim B$ means that $A \leq cB$, for some positive constant c independent of the variables in the expressions A and B . If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. Certain well-known terms such as Banach function space, rearrangement invariant Banach function space, associate space, absolutely continuous norm, etc. will be used frequently in the sequel without their definitions. However, the reader may be found their definitions e.g., in [1, 3, 10, 23, 24] and [25]. For the convenience of the reader, we now review briefly what we need from the theory of Lorentz-Karamata spaces.

Definition 1.1. Let f be a measurable function defined on a measure space (X, μ) and finite valued almost everywhere. The distribution function λ_f of f is defined by

$$\lambda_f(y) = \mu \{x \in X : |f(x)| > y\}.$$

The nonnegative rearrangement of f is given by

$$f^*(t) = \inf \{y > 0 : \lambda_f(y) \leq t\} = \sup \{y > 0 : \lambda_f(y) > t\}, \quad t \geq 0$$

where we assume that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. Also the average(maximal) function of f on $(0, \infty)$ is given by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds.$$

Note that $\lambda_f(\cdot)$, $f^*(\cdot)$ and $f^{**}(\cdot)$ are nonincreasing and right continuous functions [6, 20].

Definition 1.2. A positive and Lebesgue measurable function b is said to be slowly varying (s.v.) on $(0, \infty)$ in the sense of Karamata if, for each $\varepsilon > 0$, $t^\varepsilon b(t)$ is equivalent to a non-decreasing function and $t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function on $(0, \infty)$ [12].

Given a s.v. function b on $(0, \infty)$, we denote by γ_b the positive function defined by

$$\gamma_b(t) = \begin{cases} b\left(\frac{1}{t}\right), & 0 < t < 1 \\ b(t), & t \geq 1 \end{cases}.$$

It is known that any s.v. function b on $(0, \infty)$ is equivalent to a s.v. continuous function \tilde{b} on $(0, \infty)$. Consequently, without loss of generality, we assume that all s.v. functions in question are continuous functions on $(0, \infty)$ [17]. The detailed study of Karamata Theory, properties and examples of s.v. functions can be found in [4, 10–12, 21, 22] and chapter 5 of [25].

Definition 1.3. Let $p, q \in (0, \infty]$ and b be a s.v. function on $(0, \infty)$. The Lorentz-Karamata (LK) space $L(p, q; b)(G)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{(p,q;b)} := \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^*(t) \right\|_{q;(0,\infty)} \tag{1}$$

is finite. Here $\|\cdot\|_{q;(0,\infty)}$ stands for the usual L_q (quasi-) norm over the interval $(0, \infty)$.

Let us introduce the functional $\|f\|_{p,q;b}$ defined by

$$\|f\|_{p,q;b} := \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^{**}(t) \right\|_{q;(0,\infty)}; \tag{2}$$

this is identical with that defined in (1) except that f^* is replaced by f^{**} . It is easy to see that LK spaces $L(p, q; b)(G)$ endowed with a convenient norm (2), are rearrangement-invariant Banach function spaces and have absolutely continuous norm when $p \in (1, \infty)$ and $q \in [1, \infty)$. It is clear that, for $0 < p < \infty$, LK spaces contain the characteristic function of every measurable subset of G with finite measure and hence, by linearity, every μ -simple function. From the definition of $\|\cdot\|_{(p,q;b)}$, it follows that if $f \in L(p, q; b)(G)$ and $p, q \in (0, \infty)$, then the function $\lambda_f(y)$ is finite valued. In this case, with a little thought, it is easy to see that it is possible to construct a sequence of (simple) functions which satisfies Lemma 1.1 in [4]. Therefore, if we use the same method as employed in the proof of Proposition 2.4 in [20], we can show that Lebesgue dominated convergence theorem holds and so the set of simple functions is dense in LK spaces. Also, we can see the density of $C_c(G)$, the set of all continuous and complex-valued functions with compact support.

Lemma 1.4. [[22], Lemma 3.1] *Let b be a s.v. function on $(0, \infty)$. Then*

- (i) b^r is also a s.v. function on $(0, \infty)$ for any $r \in \mathbb{R}$ and $\gamma_{b^r}(t) = \gamma_b^r(t)$ for all $t > 0$.
- (ii) Given positive numbers ε and κ , $\gamma_b(\kappa t) \approx \gamma_b(t)$, i.e., there are positive constants c_ε and C_ε such that

$$c_\varepsilon \min\{\kappa^{-\varepsilon}, \kappa^\varepsilon\} \gamma_b(t) \leq \gamma_b(\kappa t) \leq C_\varepsilon \max\{\kappa^{-\varepsilon}, \kappa^\varepsilon\} \gamma_b(t)$$

for all $t > 0$.

- (iii) Let $\alpha > 0$. Then

$$\int_0^\tau t^{\alpha-1} \gamma_b(t) dt \approx \sup_{0 < t < \tau} t^\alpha \gamma_b(t) \approx \tau^\alpha \gamma_b(\tau) \text{ for all } \tau > 0;$$

$$\int_\tau^\infty t^{-\alpha-1} \gamma_b(t) dt \approx \sup_{\tau < t < \infty} \tau^{-\alpha} \gamma_b(t) \approx \tau^{-\alpha} \gamma_b(\tau) \text{ for all } \tau > 0.$$

Lemma 1.5. [[10], Lemma 3.4.49] *Let $1 < p < \infty$, $1 \leq q < \infty$ and b be a s.v. function. Then $C_0^\infty(\mathbb{R}^n)$, the space of all smooth functions with compact support, is dense in $L(p, q; b)(\mathbb{R}^n)$.*

The following lemma is a generalization of Lemma 4.1 of [19].

Lemma 1.6. *Let G be a non-compact locally compact abelian group and $1 < p < \infty$, $1 \leq q < \infty$. If $f \in L(p, q; b)(G)$, then*

$$\lim_{s \rightarrow \infty} \|f + L_s f\|_{p,q;b} = 2^{\frac{1}{p}} \|f\|_{p,q;b}$$

where $L_s f(x) = f(x - s)$.

Proof. Suppose that $g = \sum_{j=1}^n c'_j \chi_{E_j}$ is a simple function where each E_j is measurable and compact with $\mu(E_j) > 0$ and $E_j \cap E_k = \emptyset$ for $j \neq k$. Let $d_0 = 0$ and $d_j = \mu(E_1) + \mu(E_2) + \dots + \mu(E_j)$ for $1 \leq j \leq n$. If we set $c_j = \lfloor c'_j \rfloor$, then $c_1 \geq c_2 \geq \dots \geq c_n \geq 0$ and

$$g^*(t) = \begin{cases} c_1, & 0 \leq t < d_1 \\ c_j, & d_{j-1} \leq t < d_j \\ 0, & d_n \leq t \end{cases}$$

for $1 \leq j \leq n$. Therefore, we get

$$\begin{aligned} \|g\|_{(p,q;b)}^q &= \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) g^*(t) \right\|_{q;(0,\infty)}^q = \int_0^\infty t^{\frac{q}{p}-1} \gamma_{b^q}(t) (g^*(t))^q dt \\ &= \int_0^{d_1} t^{\frac{q}{p}-1} \gamma_{b^q}(t) (g^*(t))^q dt + \int_{d_1}^{d_2} t^{\frac{q}{p}-1} \gamma_{b^q}(t) (g^*(t))^q dt + \dots + \int_{d_{n-1}}^{d_n} t^{\frac{q}{p}-1} \gamma_{b^q}(t) (g^*(t))^q dt \\ &\leq c_1^q d_1^{\frac{q}{p}} \gamma_{b^q}(d_1) \frac{2p}{2p+q} + \sum_{i=1}^{n-1} c_{i+1}^q \left(d_{i+1}^{\frac{q}{p}} \gamma_{b^q}(d_{i+1}) \frac{2p}{2p+q} - d_i^{\frac{q}{p}} \gamma_{b^q}(d_i) \frac{p}{p+q} \right) \end{aligned}$$

and similarly

$$\begin{aligned} \|g\|_{(p,q;b)}^q &= \int_0^{d_1} t^{\frac{q}{p}-1} \gamma_{b^q}(t) (g^*(t))^q dt + \int_{d_1}^{d_2} t^{\frac{q}{p}-1} \gamma_{b^q}(t) (g^*(t))^q dt + \dots + \int_{d_{n-1}}^{d_n} t^{\frac{q}{p}-1} \gamma_{b^q}(t) (g^*(t))^q dt \\ &\geq c_1^q d_1^{\frac{q}{p}} \gamma_{b^q}(d_1) \frac{p}{p+q} + \sum_{i=1}^{n-1} c_{i+1}^q \left(d_{i+1}^{\frac{q}{p}} \gamma_{b^q}(d_{i+1}) \frac{p}{p+q} - d_i^{\frac{q}{p}} \gamma_{b^q}(d_i) \frac{2p}{2p+q} \right) \end{aligned}$$

by using Lemma 1.4 and Proposition 2.2, Remark 2.3 of [18]. If $s \notin \cup_{j,k=1}^n E_j E_k^{-1}$, then the supports of g and $T_s g$ are disjoint and we get

$$g + L_s g = \sum_{j=1}^n c'_j \chi_{E_j \cup sE_j} \quad \text{and} \quad (E_j \cup sE_j) \cap (E_k \cup sE_k) = \emptyset \quad \text{for } j \neq k.$$

Also, we obtain

$$\begin{aligned} \tilde{d}_j &= \mu(E_1 \cup sE_j) + \mu(E_2 \cup sE_j) + \dots + \mu(E_j \cup sE_j) \\ &= 2(\mu(E_1) + \mu(E_2) + \dots + \mu(E_j)) = 2d_j \end{aligned}$$

and

$$(g + L_s g)^*(t) = \begin{cases} c_1, & 0 \leq t < \tilde{d}_1 \\ c_j, & \tilde{d}_{j-1} \leq t < \tilde{d}_j \\ 0, & \tilde{d}_n \leq t \end{cases}$$

where $c_j = |c'_j|$ and $c_1 \geq c_2 \geq \dots \geq c_n \geq 0$ for $1 \leq j \leq n$. Therefore, we have

$$\begin{aligned} \|g + L_s g\|_{(p,q;b)}^q &= \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) (g + L_s g)^*(t) \right\|_{q;(0,\infty)}^q = \int_0^\infty t^{\frac{q}{p}-1} \gamma_{b^q}(t) ((g + L_s g)^*(t))^q dt \\ &\leq c_1^q \tilde{d}_1^{\frac{q}{p}} \gamma_{b^q}(\tilde{d}_1) \frac{2p}{2p+q} + \sum_{i=1}^{n-1} c_{i+1}^q \left(\tilde{d}_{i+1}^{\frac{q}{p}} \gamma_{b^q}(\tilde{d}_{i+1}) \frac{2p}{2p+q} - \tilde{d}_i^{\frac{q}{p}} \gamma_{b^q}(\tilde{d}_i) \frac{p}{p+q} \right) \\ &\leq 2^{\frac{q}{p}+\varepsilon q} \|g\|_{(p,q;b)}^q \end{aligned}$$

and

$$\begin{aligned} \|g + L_s g\|_{(p,q;b)}^q &= \int_0^\infty t^{\frac{q}{p}-1} \gamma_{b^q}(t) ((g + L_s g)^*(t))^q dt \\ &\geq c_1^q \tilde{d}_1^{\frac{q}{p}} \gamma_{b^q}(\tilde{d}_1) \frac{p}{p+q} + \sum_{i=1}^{n-1} c_{i+1}^q \left(\tilde{d}_{i+1}^{\frac{q}{p}} \gamma_{b^q}(\tilde{d}_{i+1}) \frac{p}{p+q} - \tilde{d}_i^{\frac{q}{p}} \gamma_{b^q}(\tilde{d}_i) \frac{2p}{2p+q} \right) \\ &\geq 2^{\frac{q}{p}-\varepsilon q} \|g\|_{(p,q;b)}^q \end{aligned}$$

for some $\varepsilon > 0$ by Proposition 2.2 in [18]. This implies that $2^{\frac{1}{p}-\varepsilon} \|g\|_{(p,q;b)} \leq \|g + L_s g\|_{(p,q;b)} \leq 2^{\frac{1}{p}+\varepsilon} \|g\|_{(p,q;b)}$. If we use the density of simple functions in Lorentz-Karamata spaces, then we get the result. \square

Definition 1.7. Let G be a locally compact abelian group, and $(B(G), \|\cdot\|_B)$ be a Banach space of complex-valued measurable functions on G . $B(G)$ is called a homogeneous Banach space if the following are satisfied:

- H1.** $L_s f \in B(G)$ and $\|L_s f\|_B = \|f\|_B$ for all $f \in B(G)$ and $s \in G$ where $L_s f(x) = f(x - s)$.
- H2.** $s \rightarrow L_s f$ is a continuous map from G into $(B(G), \|\cdot\|_B)$.

Definition 1.8. A homogeneous Banach algebra on G is a subalgebra $B(G)$ of $L^1(G)$ such that $B(G)$ is itself a Banach algebra with respect to a norm $\|\cdot\|_B \geq \|\cdot\|_1$ and satisfies H1 and H2.

Definition 1.9. A homogeneous Banach algebra $B(G)$ is called a Segal algebra if it is dense in $L^1(G)$.

Definition 1.10. Let G be a locally compact abelian group with character group Γ . A Segal algebra $B(G)$ is called isometrically character-invariant if for every character χ and every $f \in B(G)$ one has $\chi f \in B(G)$ and $\|\chi f\|_B = \|f\|_B$. In other words, if $f \rightarrow \chi f$ is an isometry of $B(G)$, for all $\chi \in \Gamma$.

Definition 1.11. Let G be a locally compact abelian group with character group Γ , and μ be a positive Radon measure on Γ . A Banach algebra $(B(G), \|\cdot\|_B)$ in $L^1(G)$ is an F^μ -algebra if $\widehat{B(G)} \subset L^p(G)$ for some $p \in (0, \infty)$ where " $\widehat{\cdot}$ " denotes the Fourier transform.

Definition 1.12. Let G be a locally compact abelian group with character group Γ , and μ be a positive Radon measure on Γ . A Banach algebra $(B(G), \|\cdot\|_B)$ in $L^1(G)$ is a P^μ -algebra if there exist two sequences (Δ_n) and (θ_n) in Γ , a sequence (f_n) in $B(G)$ and a sequence $c_n \geq 1$ satisfying

- p1. $\Delta_i \cap \Delta_j = \emptyset$ if $i \neq j$, $\theta_n \subset \text{Int}(\Delta_n)$, $\mu(\theta_n) = \alpha > 0$, $\mu(\Delta_n) = \beta < \infty$ for $n = 1, 2, \dots$ (Int:Interior)
- p2. $0 \leq \widehat{f_n} \leq 1$, $\text{Supp } \widehat{f_n} \subset \Delta_n$, $\widehat{f_n}(\theta_n) = 1$ for $n = 1, 2, \dots$.
- p3. $\|f_n\|_B \leq c_n$, $\sum_{n=1}^\infty \left(\frac{1}{c_n}\right) < \infty$, $\sum_{n=1}^\infty \left(\frac{1}{c_n}\right) = \infty$ for some $a, b \in (0, \infty)$.

An algebra is an $F^\mu P^\mu$ -algebra if it is both F^μ and P^μ -algebra. It is simply called FP -algebra if μ is the Haar measure on Γ .

Definition 1.13. Let B be a Banach algebra. B is said to have weak factorization if, given $f \in B$, there are $f_1, \dots, f_n, g_1, \dots, g_n \in B$ such that $f = \sum_{i=1}^n f_i g_i$.

Definition 1.14. Let G be a non-compact locally compact abelian group. The translation coefficient K_E of a homogeneous Banach space E on G is the infimum of the constants K such that

$$\limsup_{s \rightarrow \infty} \|f + L_s f\|_E \leq K \|f\|_E, \quad \forall f \in E.$$

2. The $W_{L(p,q;b)}^k(\mathbb{R}^n)$ and $A_{L(p,q;b)}^k(\mathbb{R}^n)$ Spaces

If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple of nonnegative integers α_j , then we call α a multi-index and denote by x^α the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Similarly, if $D_j = \frac{\partial}{\partial x_j}$ for $1 \leq j \leq n$, then

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}$$

denotes a differential operator of order $|\alpha|$. For given two locally integrable functions f and g on \mathbb{R}^n , we say that $\frac{\partial^\alpha f}{\partial x^\alpha} = g$ (weak derivative of f) if

$$\int_{\mathbb{R}^n} f(x) \frac{\partial^{|\alpha|} \varphi}{\partial x^\alpha}(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} g(x) \varphi(x) dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Definition 2.1. Lorentz-Karamata-Sobolev spaces are defined by

$$W_{L(p,q;b)}^k(\mathbb{R}^n) = \{f \in L(p, q; b)(\mathbb{R}^n) : D^\alpha f \in L(p, q; b)(\mathbb{R}^n)\} \tag{3}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ where k is a nonnegative integer, $p \in (1, \infty)$ and $q \in [1, \infty)$. Also they are equipped with the norm

$$\|f\|_{W_{L(p,q;b)}^k(\mathbb{R}^n)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{p,q;b}. \tag{4}$$

Clearly, if $k = 0$, then $W_{L(p,q;b)}^k(\mathbb{R}^n) = L(p, q; b)(\mathbb{R}^n)$. Besides this, if we define $W_{L(p,q;b)}^{k,0}(\mathbb{R}^n)$ as the space of the closure of $C_0^\infty(\mathbb{R}^n)$ in the space $W_{L(p,q;b)}^k(\mathbb{R}^n)$, then it is easy to see that $W_{L(p,q;b)}^{0,0}(\mathbb{R}^n) = L(p, q; b)(\mathbb{R}^n)$ where $p \in (1, \infty)$ and $q \in [1, \infty)$. For any k , the chain of imbeddings

$$W_{L(p,q;b)}^{k,0}(\mathbb{R}^n) \hookrightarrow W_{L(p,q;b)}^k(\mathbb{R}^n) \hookrightarrow L(p, q; b)(\mathbb{R}^n) \tag{5}$$

is also clear. Instead of dealing with Lorentz-Karamata-Sobolev spaces $W_{L(p,q;b)}^k(\mathbb{R}^n)$, we can pay attention to the completion of the set

$$\left\{ f \in C^k(\mathbb{R}^n) : \|f\|_{W_{L(p,q;b)}^k(\mathbb{R}^n)} < \infty \right\}$$

with respect to the norm in (4). Because, it is easy to show that these spaces are equal.

Now, we are going to give two propositions without their proofs. One can prove these by using the same methods used for abstract Sobolev spaces and Propositions 3.1 and 3.2 in [14].

Proposition 2.2. While $W_{L(p,q;b)}^k(\mathbb{R}^n)$ is not a Banach function space, it is a (homogeneous) Banach space with $\|\cdot\|_{W_{L(p,q;b)}^k(\mathbb{R}^n)}$.

Proposition 2.3. If $p, q \in (1, \infty)$, then $W_{L(p,q;b)}^k(\mathbb{R}^n)$ spaces are reflexive. In other words, the associate space of $W_{L(p,q;b)}^k(\mathbb{R}^n)$ is $W_{L(p',q';b^{-1})}^{-k}(\mathbb{R}^n)$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

After this point, we are going to deal with the algebraic structures of $L^1(\mathbb{R}^n) \cap W_{L(p,q;b)}^k(\mathbb{R}^n)$ spaces. For this reason, we will call this intersection space as $A_{L(p,q;b)}^k(\mathbb{R}^n)$ and endow it with the sum norm

$$\|f\|_A := \|f\|_1 + \|f\|_{W_{L(p,q;b)}^k(\mathbb{R}^n)}$$

for all $f \in A_{L(p,q;b)}^k(\mathbb{R}^n)$.

Proposition 2.4. $(A_{L(p,q;b)}^k(\mathbb{R}^n), \|\cdot\|_A)$ is a Segal algebra on \mathbb{R}^n if $p \in (1, \infty)$ and $q \in [1, \infty)$.

Proof. Let $p \in (1, \infty)$ and $q \in [1, \infty)$. Since $W_{L(p,q;b)}^k(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ are homogeneous Banach spaces, it is easy to see that $A_{L(p,q;b)}^k(\mathbb{R}^n)$ is also a homogeneous Banach Space under the sum norm $\|\cdot\|_A \geq \|\cdot\|_1$ by [11]. By a result of [[23], 3.2. Theorem], we get $A_{L(p,q;b)}^k(\mathbb{R}^n)$ is a homogeneous Banach algebra. By [[1], 2.19.Theorem], we know that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ and is contained in $W_{L(p,q;b)}^k(\mathbb{R}^n)$. Therefore, $A_{L(p,q;b)}^k(\mathbb{R}^n)$ is a Segal algebra on \mathbb{R}^n . \square

Theorem 2.5. $A_{L(p,q;b)}^k(\mathbb{R}^n)$ is an FP-algebra for $p \in (1, \infty)$ and $q \in [1, \infty)$.

Proof. Firstly, we are going to show the P -algebra property of $A_{L(p,q;b)}^k(\mathbb{R}^n)$ spaces.

(i) Let

$$\begin{aligned} \Delta_m &= \left[m - \frac{1}{4}, m + \frac{1}{4} \right] \times \cdots \times \left[m - \frac{1}{4}, m + \frac{1}{4} \right] \quad (n - \text{times}) \\ \Omega_m &= \left[m - \frac{1}{8}, m + \frac{1}{8} \right] \times \cdots \times \left[m - \frac{1}{8}, m + \frac{1}{8} \right] \quad (n - \text{times}) \end{aligned}$$

and

$$\Delta'_m = \left[m - \frac{1}{4}, m + \frac{1}{4} \right], \quad \Omega'_m = \left[m - \frac{1}{8}, m + \frac{1}{8} \right]$$

for $m \geq 1$. By [[23], 1.8.Theorem], there exists a generalized trapezium function $f_1 \in L^1(\mathbb{R})$ such that $0 \leq \widehat{f_1} \leq 1$, $\text{supp} \widehat{f_1} \subset \Delta'_1$ and $\widehat{f_1}(\Omega'_1) = 1$. If we let $f_m(t) = e^{i(m-1)t} f_1(t)$, then it is easy to see that $0 \leq \widehat{f_m} \leq 1$, $\text{supp} \widehat{f_m} \subset \Delta'_m$ and $\widehat{f_m}(\Omega'_m) = 1$ for $m \geq 2$. If we define F_m by $F_m(x_1, \dots, x_n) = f_m(x_1) \cdots f_m(x_n)$ for $m = 1, 2, \dots$, then $F_m \in L^1(\mathbb{R}^n)$, $\widehat{F_m}(t_1, \dots, t_n) = \widehat{f_m}(t_1) \cdots \widehat{f_m}(t_n)$ and $0 \leq \widehat{F_m} \leq 1$, $\text{supp} \widehat{F_m} \subset \Delta_m$, $\widehat{F_m}(\Omega_m) = 1$. If $P(L^1(\mathbb{R}^n))$ is the set of all f in $L^1(\mathbb{R}^n)$ whose Fourier transform \widehat{f} has compact support, then it is seen that $F_m \in P(L^1(\mathbb{R}^n))$. Since $P(L^1(\mathbb{R}^n))$ is dense in every homogeneous Banach algebra [[23], 3.7.Theorem], we have $F_m \in A_{L(p,q;b)}^k(\mathbb{R}^n)$. For $1 \leq j \leq k$ and $m \geq 2$, the equality

$$\begin{aligned} f_m^{(j)}(t) &= (e^{i(m-1)t})^{(j)} f_1(t) + \binom{j}{1} (e^{i(m-1)t})^{(j-1)} f_1'(t) + \\ &\quad + \binom{j}{2} (e^{i(m-1)t})^{(j-2)} f_1''(t) + \dots + \binom{j}{j} (e^{i(m-1)t}) f_1^{(j)}(t) \\ &= i^j (m-1)^j e^{i(m-1)t} f_1(t) + \binom{j}{1} i^{j-1} (m-1)^{j-1} e^{i(m-1)t} f_1'(t) + \\ &\quad + \binom{j}{2} i^{j-2} (m-1)^{j-2} e^{i(m-1)t} f_1''(t) + \dots + \binom{j}{j} (e^{i(m-1)t}) f_1^{(j)}(t) \end{aligned}$$

is written. Since $f_m \in P(L^1(\mathbb{R})) \subset A_{L(p,q;b)}^k(\mathbb{R})$, if

$$M = \max \left\{ \|f_1\|_{p,q;b}, \|f_1'\|_{p,q;b}, \dots, \|f_1^{(j)}\|_{p,q;b} \right\}$$

then, we get

$$\begin{aligned} \left\| f_m^{(j)} \right\|_{p,q;b} &= \left\| i^j (m-1)^j e^{i(m-1)t} f_1(t) + \dots + (e^{i(m-1)t}) f_1^{(j)}(t) \right\|_{p,q;b} \\ &\leq (m-1)^j \|f_1\|_{p,q;b} + (m-1)^{j-1} \binom{j}{1} \|f_1'(t)\|_{p,q;b} + \dots + \|f_1^{(j)}(t)\|_{p,q;b} \\ &\leq 2^j (m-1)^j M. \end{aligned} \tag{6}$$

Again, for $1 \leq |\alpha| = j \leq k$ and $0 \leq j_i \leq j$, $j_1 + \dots + j_n = j$, it can be written by (6) that

$$\begin{aligned} \|D^\alpha F_m(x_1, \dots, x_n)\|_{p,q;b} &= \left\| f_m^{(j_1)}(x_1) f_m^{(j_2)}(x_2) \cdots f_m^{(j_n)}(x_n) \right\|_{p,q;b} \\ &\leq (2^j (m-1)^j M)^n \leq (2^k (m-1)^k M)^n \end{aligned}$$

and so

$$\begin{aligned}
 \|F_m\|_A &= \|F_m\|_1 + \|F_m\|_{W^k_{L(p,q;b)}(\mathbb{R}^n)} = \|F_m\|_1 + \sum_{|\alpha| \leq k} \|D^\alpha F_m\|_{p,q;b} \\
 &= \|F_m\|_1 + \|F_m\|_{p,q;b} + \sum_{|\alpha|=1} \|D^\alpha F_m\|_{p,q;b} + \sum_{|\alpha|=2} \|D^\alpha F_m\|_{p,q;b} + \\
 &\quad + \dots + \sum_{|\alpha|=k} \|D^\alpha F_m\|_{p,q;b} \\
 &\leq \|F_m\|_1 + \|F_m\|_{p,q;b} + \left[\binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k} \right] (2^k (m-1)^k M)^n \\
 &\leq B(m-1)^{kn}
 \end{aligned} \tag{7}$$

for $m \geq 2$ and some constant $B > 0$. Since we can take B and C_1 large enough such that $C_m = B(m-1)^{kn} \geq 1$ for $m = 2, 3, \dots$, $C_1 > \|F_1\|_A$ and $C_1 > 1$, we have

$$\sum_{m=1}^{\infty} \frac{1}{C_m^{k+1}} < \infty \quad \text{but} \quad \sum_{m=1}^{\infty} \frac{1}{C_m^{1/kn}} = \infty, \quad \text{for } k \geq 1.$$

Thus we get the result.

Now let $k = 0$ and say $L^1(\mathbb{R}^n) \cap L(p, q; b)(\mathbb{R}^n) = B(p, q; b)(\mathbb{R}^n)$. Then $A^0_{L(p,q;b)}(\mathbb{R}^n) = L^1(\mathbb{R}^n) \cap W^0_{L(p,q;b)}(\mathbb{R}^n) = B(p, q; b)(\mathbb{R}^n)$. Since $B(p, q; b)(\mathbb{R}^n)$ is a character invariant Segal algebra by [13] and every character Segal algebra is a P -algebra by 4.9.Theorem of [23], we get that $A^0_{L(p,q;b)}(\mathbb{R}^n)$ is a P -algebra.

(ii) it is obvious from (5) that $A^k_{L(p,q;b)}(\mathbb{R}^n) \subset B(p, q; b)(\mathbb{R}^n)$. Since $B(p, q; b)(\mathbb{R}^n)$ is a Segal algebra with $B(\widehat{p, q; b})(\mathbb{R}^n) \subset L(p, q; b)(\mathbb{R}^n)$ for $p \in (1, \infty)$ and $q \in [1, \infty)$ by [13], we get $B(p, q; b)(\mathbb{R}^n)$ is an F -algebra for $p \in (1, \infty)$ and $q \in [1, \infty)$ by 4.5.Definition of [23]. It is known from 4.6.Theorem of [23] that F -algebra property is a going-down property. In other words, if B is an F -algebra and A is a subalgebra of B , then A is also an F -algebra. Therefore, $A^k_{L(p,q;b)}(\mathbb{R}^n)$ is an F -algebra due to $A^k_{L(p,q;b)}(\mathbb{R}^n) \subset B(p, q; b)(\mathbb{R}^n)$.

(i) and (ii) give the result. \square

In 8.8.Theorem of [23], it is proved that an FP -algebra doesn't admit the weak factorization property. So, we can write the following theorem.

Theorem 2.6. $A^k_{L(p,q;b)}(\mathbb{R}^n)$ doesn't admit the weak factorization property.

Remark 2.7. We know that a character invariant Segal algebra on the locally compact abelian group G has weak factorization if and only if it is equal to $L^1(G)$ by Theorem 2.2 of [16]. For $p \in (1, \infty)$ and $q \in [1, \infty)$, we have $A^k_{L(p,q;b)}(\mathbb{R}^n) \subsetneq L^1(\mathbb{R}^n)$. Therefore, an alternative proof for the preceding theorem may be done by showing character invariance of $A^k_{L(p,q;b)}(\mathbb{R}^n)$.

Theorem 2.8. [5] Suppose S is a Segal algebra in $L^1(G)$ of the form $L^1(G) \cap E$, where G is a noncompact locally compact abelian group, E is a homogeneous Banach space on G . If the translation coefficient K_E of E is less than 2, then the multipliers space of S is isometrically isomorphic to the space of all bounded regular Borel measures on G , $M(G)$.

Theorem 2.9. The multipliers space of $A^k_{L(p,q;b)}(\mathbb{R}^n)$ is isometrically isomorphic to $M(\mathbb{R}^n)$ for $p \in (1, \infty)$ and $q \in [1, \infty)$.

Proof. Let $f \in A^k_{L(p,q;b)}(\mathbb{R}^n)$. Since $\|L_s f\|_{p,q;b} = \|f\|_{p,q;b}$ for all $f \in L(p,q;b)(\mathbb{R}^n)$ and $s \in \mathbb{R}^n$ by Proposition 3.1 of [14], we have

$$\begin{aligned} \|f + L_s f\|_{W^k_{L(p,q;b)}(\mathbb{R}^n)} &= \sum_{|\alpha| \leq k} \|D^\alpha (f + L_s f)\|_{p,q;b} \\ &\leq \|f + L_s f\|_{p,q;b} + \sum_{1 \leq |\alpha| \leq k} \|D^\alpha f\|_{p,q;b} + \sum_{1 \leq |\alpha| \leq k} \|L_s D^\alpha f\|_{p,q;b} \\ &= \|f + L_s f\|_{p,q;b} + 2 \sum_{1 \leq |\alpha| \leq k} \|D^\alpha f\|_{p,q;b}. \end{aligned}$$

If $f = 0$ (a.e.), then it is trivial that

$$\limsup_{|s| \rightarrow \infty} \|f + L_s f\|_{W^k_{L(p,q;b)}(\mathbb{R}^n)} = 0.$$

Now let $f \neq 0$. By Lemma 1.6, we know that $K_{L(p,q;b)(G)} = 2^{\frac{1}{p}}$ for $p \in (1, \infty)$ and $q \in [1, \infty)$. Then, we get

$$\begin{aligned} \limsup_{|s| \rightarrow \infty} \|f + L_s f\|_{W^k_{L(p,q;b)}(\mathbb{R}^n)} &\leq \limsup_{|s| \rightarrow \infty} \|f + L_s f\|_{p,q;b} + 2 \sum_{1 \leq |\alpha| \leq k} \|D^\alpha f\|_{p,q;b} \\ &= 2^{\frac{1}{p}} \|f\|_{p,q;b} + 2 \sum_{1 \leq |\alpha| \leq k} \|D^\alpha f\|_{p,q;b} \\ &= 2^{\frac{1}{p}} \|f\|_{p,q;b} + 2 \|f\|_{p,q;b} - 2 \|f\|_{p,q;b} + 2 \sum_{1 \leq |\alpha| \leq k} \|D^\alpha f\|_{p,q;b} \\ &= (2^{\frac{1}{p}} - 2) \|f\|_{p,q;b} + 2 \sum_{|\alpha| \leq k} \|D^\alpha f\|_{p,q;b} \\ &= \|f\|_{W^k_{L(p,q;b)}(\mathbb{R}^n)} \left(2 - \frac{(2 - 2^{\frac{1}{p}}) \|f\|_{p,q;b}}{\|f\|_{W^k_{L(p,q;b)}(\mathbb{R}^n)}} \right). \end{aligned}$$

Since $0 < \|f\|_{p,q;b} \leq \|f\|_{W^k_{L(p,q;b)}(\mathbb{R}^n)}$, $0 < 2 - 2^{\frac{1}{p}} < 1$ and $0 < 2 - \frac{(2 - 2^{\frac{1}{p}}) \|f\|_{p,q;b}}{\|f\|_{W^k_{L(p,q;b)}(\mathbb{R}^n)}} < 2$ for all $p \in (1, \infty)$, we see that

$K_{W^k_{L(p,q;b)}(\mathbb{R}^n)} < 2$. Therefore, the multipliers space of $A^k_{L(p,q;b)}(\mathbb{R}^n)$ is isometrically isomorphic to $M(\mathbb{R}^n)$ by the preceding theorem for $p \in (1, \infty)$ and $q \in [1, \infty)$. \square

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