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# Multivalued Operator with Respect Generalized Distance on Menger Probabilistic Metric Spaces

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**Abstract.** In this paper, we recall the concept of $r$-distance on a Menger probabilistic metric space. Further we prove a fixed point theorem for contractive type multi-valued operators in terms of a $r$-distance on a complete Menger probabilistic metric space.

## 1. Introduction and Preliminaries

K. Menger introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [19]. The idea of K. Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis. Probabilistic normed spaces were introduced by Šerstnev in 1962 [21] by means of a definition that was closely modelled on the theory of (classical) normed spaces, and used to study the problem of best approximation in statistics. In the sequel, we shall adopt the usual terminologies, notations and conventions of the theory of probabilistic normed spaces, as in [2–6, 10, 11, 13, 14, 16, 18, 19, 23].

Throughout this paper, the space of all probability distribution functions (briefly, d.f.'s) is denoted by $\Delta^+ = \{F : \mathbb{R} \cup \{-\infty, +\infty\} \longrightarrow [0,1] : F$ is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$. Here $l^- f(x)$ denotes the left limit of the function $f$ at the point $x$, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space $\Delta^+$ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t$ in $\mathbb{R}$. The maximal element for $\Delta^+$ in this order is the d.f. given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

**Definition 1.1.** ([19]) A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous $t$–norm if $T$ satisfies the following conditions:

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(a) $T$ is commutative and associative;
(b) $T$ is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$;
(d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $c \leq d$, and $a, b, c, d \in [0, 1]$.

Two typical examples of continuous $t$–norm are $T(a, b) = ab$ and $T(a, b) = \min(a, b)$.
Now $t$–norms are recursively defined by $T^1 = T$ and

$$T^n(x_1, \cdots, x_{n+1}) = T(T^{n-1}(x_1, \cdots, x_n), x_{n+1})$$

for $n \geq 2$ and $x_i \in [0, 1]$, for all $i \in \{1, 2, \ldots, n + 1\}$.

**Definition 1.2.** ([19]) A mapping $S : [0, 1]\times[0, 1] \longrightarrow [0, 1]$ is a continuous $s$–norm if $S$ satisfies the following conditions:

(a) $S$ is associative and commutative;
(b) $S$ is continuous;
(c) $S(a, 0) = a$ for all $a \in [0, 1]$;
(d) $S(a, b) \leq S(c, d)$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$.

Two typical examples of continuous $s$–norm are $S(a, b) = \min(a + b, 1)$ and $S(a, b) = \max(a, b)$.

**Definition 1.3.** A *Menger Probabilistic Metric space* (briefly, Menger PM-space) is a triple $(X, \mathcal{F}, T)$, where $X$ is a nonempty set, $T$ is a continuous $t$–norm, and $\mathcal{F}$ is a mapping from $X \times X$ into $D^+$ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair $(x, y)$, the following conditions hold: for all $x, y, z$ in $X$,

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t + s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

**Definition 1.4.** A *Menger Probabilistic Normed space* (briefly, Menger PN-space) is a triple $(X, \mu, T)$, where $X$ is a vector space, $T$ is a continuous $t$–norm, and $\mu$ is a mapping from $X$ into $D^+$ such that, the following conditions hold: for all $x, y$ in $X$,

(PN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(PN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for $\alpha \neq 0$;
(PN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

**Definition 1.5.** Let $(X, \mathcal{F}, T)$ be a Menger PM-space.

(1) A sequence $\{x_n\}_n$ in $X$ is said to be *convergent* to $x$ in $X$ if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer $N$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}_n$ in $X$ is called *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer $N$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geq N$.

(3) A Menger PM-space $(X, \mathcal{F}, T)$ is said to be *complete* if and only if every Cauchy sequence in $X$ is convergent to a point in $X$.

A subset $U$ of $X$ is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $F_{x,y}(t) > 1 - r$ for all $x, y \in U$.

**Definition 1.6.** Let $(X, \mathcal{F}, T)$ be a Menger PM space. For each $p$ in $X$ and $\lambda > 0$, the strong $\lambda -$ *neighborhood* of $p$ is the set

$$N_p(\lambda) \quad = \quad \{q \in X : F_{p,q}(\lambda) > 1 - \lambda\},$$

and the strong neighborhood system for $X$ is the union $\bigcup_{p\in V} \mathcal{N}_p$ where $\mathcal{N}_p = \{N_p(\lambda) : \lambda > 0\}$.

The strong neighborhood system for $X$ determines a Hausdorff topology for $X$.

**Theorem 1.7.** ([10, 20]) *If $(X, \mathcal{F}, T)$ is a PM-space and $\{p_n\}$ and $\{q_n\}$ are sequences such that $p_n \to p$ and $q_n \to q$, then* $\lim_{n\to\infty} F_{p_n,q_n}(t) = F_{p,q}(t)$.

**Remark 1.8.** *We say the t-norm $T$ has $\Sigma$ property and write $T \in \Sigma$ whenever, Suppose for every $\alpha \in ]0,1[$ there exists a $\beta \in ]0,1[$ (which does not depend on n) with*

$$T^{n-1}(1-\beta, .., 1-\beta) > 1-\alpha \text{ for each } n \in \{1, 2, ...\}. \tag{1}$$

## 2. *r*–distance

Recently, Kada, Suzuki and Takahashi [9] introduced the concept of $w$–distance on a metric space and proved some fixed point theorems. In this section, using the concept of $w$–distance, we define the concept of $r$-distance on a Menger PM-space.

**Definition 2.1.** Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then the function $f : X^2 \times [0,\infty] \longrightarrow [0,1]$ is called a $r$-distance on $X$ if the following are satisfied:

(r1) $f_{x,z}(t+s) \geq T(f_{x,y}(t), f_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$;

(r2) for any $x \in X$ and $t \geq 0$, $f_{x,.} : X \times [0,\infty] \longrightarrow [0,1]$ is continuous;

(r3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $f_{z,x}(t) \geq 1-\delta$ and $f_{z,y}(s) \geq 1-\delta$ imply $F_{x,y}(t+s) \geq 1-\varepsilon$.

Let us give some examples of $r$-distance.

**Example 2.2.** *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then $f = F$ is a r-distance on X.*

*Proof.* Now (r1) and (r2) are obvious. We show (r3). Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$T(1-\delta, 1-\delta) \geq 1-\varepsilon.$$

Then, for $F_{z,x}(t) \geq 1-\delta$ and $F_{z,y}(s) \geq 1-\delta$ we have

$$\begin{aligned} F_{x,y}(t+s) &\geq T(F_{z,x}(t), F_{z,y}(s)) \\ &\geq T(1-\delta, 1-\delta) \geq 1-\varepsilon. \end{aligned}$$

□

**Example 2.3.** *Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then the function $f : X^2 \times [0,\infty) \longrightarrow [0,1]$ defined by $f_{x,y}(t) = 1-c$ for every $x, y \in X$ and $t > 0$ is a r-distance on X, where $c \in ]0,1[$.*

*Proof.* Now (r1) and (r2) are obvious. To show (r3), for any $\varepsilon > 0$, put $\delta = 1 - c/2$. Then we have that $f_{z,x}(t) \geq 1-c/2$ and $f_{z,y}(s) \geq 1-c/2$ imply $F_{x,y}(t+s) \geq 1-\varepsilon$. □

**Example 2.4.** *Let $(X, \mu, T)$ be a Menger PN-space. Then the function $f : X^2 \times [0,\infty) \longrightarrow [0,1]$ defined by $f_{x,y}(t+s) = T(\mu_x(t), \mu_y(s))$ for every $x, y \in X$ and $t, s > 0$ is a r-distance on X.*

*Proof.* Let $x, y, z \in X$ and $t, s > 0$. Then we have

$$\begin{aligned} f_{x,z}(t+s) &= T(\mu_x(t), \mu_z(s)) \\ &\geq T(T(\mu_x(t/2), \mu_y(t/2)), T(\mu_y(s/2), \mu_z(s/2))) \\ &= T(f_{x,y}(t), f_{y,z}(s)). \end{aligned}$$

Hence (r1) holds. Also (r2) is obvious. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$T(1-\delta, 1-\delta) \geq 1-\varepsilon.$$

Then, for $f_{z,x}(t) \geq 1-\delta$ and $f_{z,y}(s) \geq 1-\delta$ we have

$$\begin{aligned} F_{x,y}(t+s) &= \mu_{x-y}(t+s) \geq T(\mu_x(t), \mu_y(s)) \\ &\geq T(T(\mu_x(t/2), \mu_z(t/2)), T(\mu_y(s/2), \mu_z(s/2))) \\ &= T(f_{z,x}(t), f_{z,y}(s)) \\ &\geq T(1-\delta, 1-\delta) \geq 1-\varepsilon. \end{aligned}$$

Hence (r3)also holds. □

**Example 2.5.** *Let $(X, \mu, T)$ be a Menger PN-space. Then the function $f : X^2 \times [0, \infty] \longrightarrow [0, 1]$ defined by $f_{x,y}(t) = \mu_x(t)$ for every $x, y \in X$ and $t > 0$ is a r-distance on $X$.*

*Proof.* Let $x, y, z \in X$ and $t, s > 0$. Then we have

$$\begin{aligned} f_{x,z}(t+s) &= \mu_z(t+s) \\ &\geq T(\mu_y(t), \mu_z(s)) \\ &= T(f_{x,y}(t), f_{y,z}(s)). \end{aligned}$$

Hence (r1) holds. Also (r2) is obvious. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$T(1-\delta, 1-\delta) \geq 1-\varepsilon.$$

Then, for $f_{z,x}(t) \geq 1-\delta$ and $f_{z,y}(s) \geq 1-\delta$ we have

$$\begin{aligned} F_{x,y}(t+s) &= \mu_{x-y}(t+s) \\ &\geq T(\mu_x(t), \mu_y(s)) \\ &= T(f_{z,x}(t), f_{z,y}(s)) \\ &\geq T(1-\delta, 1-\delta) \geq 1-\varepsilon. \end{aligned}$$

Hence (r3) holds. □

**Example 2.6.** *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let $A$ be a continuous mapping from $X$ into $X$. Then the function $f : X^2 \times [0, \infty] \longrightarrow [0, 1]$ defined by*

$$f_{x,y}(t) = \min(F_{Ax,y}(t), F_{Ax,Ay}(s))$$

*for every $x, y \in X$ and $t, s > 0$ is a r-distance on $X$.*

*Proof.* Let $x, y, z \in X$ and $t, s > 0$. If $F_{Ax,z}(t) \leq F_{Ax,Ay}(t)$ then we have

$$\begin{aligned} f_{x,z}(t+s) &= F_{Ax,z}(t+s) \geq T(F_{Ax,Ay}(t), F_{Ay,z}(s)) \\ &\geq T(\min(F_{Ax,y}(t), F_{Ax,Ay}(t)), \min(F_{Ay,z}(s), F_{Ax,Ay}(s)) \\ &= T(f_{x,y}(t), f_{y,z}(s)). \end{aligned}$$

With this inequality, we have

$$\begin{aligned} f_{x,z}(t+s) &= F_{Ax,Az}(t+s) \geq T(F_{Ax,Ay}(t), F_{Ay,Az}(s)) \\ &\geq T(\min(F_{Ax,y}(t), F_{Ax,Ay}(t)), \min(F_{Ay,z}(s), F_{Ax,Ay}(s)) \\ &= T(f_{x,y}(t), f_{y,z}(s)). \end{aligned}$$

Hence (r1) holds. Since $A$ is continuous, (r2) is obvious . Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$T(1-\delta, 1-\delta) \geq 1-\varepsilon.$$

Then, from $f_{z,x}(t) \geq 1-\delta$ and $f_{z,y}(s) \geq 1-\delta$ we have $F_{Az,x}(t) \geq 1-\delta$ and $F_{Az,y}(s) \geq 1-\delta$. Therefore

$$\begin{aligned} F_{x,y}(t+s) &\geq T(F_{Az,x}(t), F_{Az,y}(s)) \\ &\geq T(1-\delta, 1-\delta) \geq 1-\varepsilon. \end{aligned}$$

Hence (r3) holds. □

Next, we discuss some properties of $r$-distance.

**Lemma 2.7.** *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let $f$ be a r-distance on it. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $X$, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero, and let $x, y, z \in X$ and $t, s > 0$. Then the following hold:*

(1) *if $f_{x_n,y}(t) \geq 1-\alpha_n$ and $f_{x_n,z}(s) \geq 1-\beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $f_{x,y}(t) = 1$ and $f_{x,z}(s) = 1$, then $y = z$;*

(2) *if $f_{x_n,y_n}(t) \geq 1-\alpha_n$ and $f_{x_n,z}(s) \geq 1-\beta_n$ for any $n \in \mathbb{N}$, then $F_{y_n,z}(t+s) \to 1$;*

(3)*if $f_{x_n,x_m}(t) \geq 1-\alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*

(4) *if $f_{y,x_n}(t) \geq 1-\alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

*Proof.* We first prove (2). Let $\varepsilon > 0$ be given. From the definition of $r$-distance, there exists $\delta > 0$ such that $f_{u,v}(t) \geq 1-\delta$ and $f_{u,z}(s) \geq 1-\delta$ imply $F_{v,z}(t+s) \geq 1-\varepsilon$. Choose $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta$ and $\beta_n \leq \delta$ for every $n \geq n_0$. Then we have, for any $n \geq n_0$ $f_{x_n,y_n}(t) \geq 1-\alpha_n \geq 1-\delta$ and $f_{x_n,z}(t) \geq 1-\beta_n \geq 1-\delta$ and hence $F_{y_n,z}(t+s) \geq 1-\varepsilon$. This implies that $\{y_n\}$ converges to $z$. It follows from (2) that (1) holds. Let us prove (3). Let $\varepsilon > 0$ be given. As in the proof of (1), choose $\delta > 0$ and then $n_0 \in \mathbb{N}$. Then for any $n, m \geq n_0+1$

$$f_{x_{n_0},x_n}(t) \geq 1-\alpha_{n_0} \geq 1-\delta \quad \text{and} \quad f_{x_{n_0},x_m}(s) \geq 1-\alpha_{n_0} \geq 1-\delta$$

and hence $F_{x_n,x_m}(t+s) \geq 1-\varepsilon$. This implies that $\{x_n\}$ is a Cauchy sequence. □

**Lemma 2.8.** *Let $f : X^2 \times [0,\infty] \longrightarrow [0,1]$ be a r-distance on $(X, F, T)$ in which $T \in \Sigma$. If we define $E_{\lambda,f} : X^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,f}(x,y) = \inf\{t > 0 : f_{x,y}(t) > 1-\lambda\}$$

*for each $\lambda \in ]0,1[$ and $x, y \in X$, then we have the following:*

(1) *For any $\mu \in ]0,1[$, there exists $\lambda \in ]0,1[$ such that*

$$E_{\mu,f}(x_1,x_k) \leq E_{\lambda,f}(x_1,x_2) + E_{\lambda,f}(x_2,x_3) + \cdots + E_{\lambda,f}(x_{k-1},x_k)$$

*for any $x_1, ..., x_k \in X$;*

(2) *For any sequence $\{x_n\}$ in $X$, we have, $f_{x_n,x}(t) \longrightarrow 1$ if and only if $E_{\lambda,f}(x_n,x) \to 0$. Also the sequence $\{x_n\}$ is Cauchy w.r.t. $f$ if and only if it is Cauchy with $E_{\lambda,f}$.*

*Proof.* The proof is the same as in Lemma 1.6 of [15]. □

**Lemma 2.9.** *Let $(X, \mathcal{F}, T)$ be a Menger PM-space, let $f$ be a r-distance on it and let $A$ be a mapping from $X$ into itself. Let $\{u_n\}$ be a sequence in $X$. Suppose that there exists $k \in ]0,1[$ such that*

$$f_{u_n,u_{n+1}}(kt) \geq f_{u_{n-1},u_n}(t)$$

*for every $n \in \mathbb{N}$, $t > 0$. Then the sequence $\{u_n\}$ is Cauchy.*

*Proof.* See Theorem 3.1 of [17]. □

## 3. Main Results

Let $(X, \mathcal{F}, T)$ a Menger PM-space. We will use the following notations:
$P(X)$- the set of all nonempty subsets of $X$;
$P_{cl}(X)$- the set of all nonempty closed subsets of $X$;
$P_{b,cl}(X)$- the set of all nonempty bounded and closed subsets of $X$;
$\Phi : P(X) \times P(X) \to D^+$,

$$\Phi_{Z,Y}(t) = \sup\{F_{x,y}(t) : \ x \in Z, \ \ y \in Y\}$$

for $t > 0$ in which $Y, Z \subset X$.

**Definition 3.1.** Let $(X, \mathcal{F}, T)$ a Menger PM-space. Assume that $A : X \to P(X)$ be a multi-valued operator and $f : X^2 \times [0, \infty] \longrightarrow [0, 1]$ be a $r$-distance on $(X, \mathcal{F}, T)$. Define the function $h : X \times [0, \infty] \longrightarrow [0, 1]$ as $h_x(t) = \phi_{x,A(x)}(t)$ , where

$$\phi_{x,A(x)}(t) = \sup\{f_{x,y}(t) : y \in A(x)\},$$

for $t > 0$.

For a positive constant $b \in (0, 1)$ define the set $I^x_b \subset X$ as follows:

$$I^x_b = \{y \in A(x) : \ \ f_{x,y}(t) \geq \phi_{x,A(x)}(bt)\},$$

for all $t > 0$. We will present now a fixed point theorem for multi-valued operators on a complete Menger PM-space endowed with a $r$-distance. Our result generalized and extend some recent results presented at [1, 7, 8, 12, 22].

**Theorem 3.2.** *Let $(X, \mathcal{F}, T)$ a complete Menger PM-space, $A : X \to P_{cl}(X)$ a multi-valued operator, $f : X^2 \times [0, \infty] \longrightarrow [0, 1]$ be a $r$-distance on $X$ and $b \in (0, 1)$.*
*Suppose that:*
(i) *there exists $c \in (0, 1)$, with $c < b$, such that for any $x \in X$ there is $y \in I^x_b$ satisfying*

$$f_{y,x}(t) \leq \phi_{x,A(x)}(ct)$$

*for all $t > 0$.*
(ii) *the function $h_x(t) = \phi_{x,A(x)}(t)$ is continuous.*
*Then $A$ has a fixed point in $X$.*

*Proof.* Since $A(X) \subset P_{cl}(X)$, then, for any $x \in X$, $I^x_b$ is nonempty for any constant $b \in (0, 1)$. For any initial point $x_0 \in X$, there is $x_1 \in I^{x_0}_b$ such that

$$f_{x_0,x_1}(t) \leq \phi_{x_1,A(x_1)}(ct)$$

for all $t > 0$. For any $x_1 \in X$, there is $x_2 \in I^{x_1}_b$ such that

$$f_{x_1,x_2}(t) \leq \phi_{x_2,A(x_2)}(ct)$$

for all $t > 0$. We obtain an iterative sequence $\{x_n\}_{n=0}^{\infty}$ where $x_{n+1} \in I^{x_n}_b$ such that

$$f_{x_n,x_{n+1}}(t) \leq \phi_{x_{n+1},A(x_{n+1})}(ct) \tag{2}$$

for all $t > 0$ and for $n = 1, 2, \cdots$. Now, we show that the sequence $\{x_n\}_{n=0}^{\infty}$ is Cauchy. Since $x_{n+1} \in I^{x_n}_b$, we have

$$f_{x_n,x_{n+1}}(t) \geq \phi_{x_n,A(x_n)}(bt) \tag{3}$$

for all $t > 0$ and for $n = 1, 2, \cdots$. Form (2) and (3) and since $c < b$ we have

$$\phi_{x_n,Ax_n}(t) \leq \phi_{x_{n+1},A(x_{n+1})}\left(\frac{c}{b}t\right) \tag{4}$$

for all $t > 0$ and for $n = 1, 2, \cdots$. Then

$$\phi_{x_0, Ax_0}(t) \le \phi_{x_n, A(x_n)}\left(\left(\frac{c}{b}\right)^n t\right) \tag{5}$$

for all $t > 0$ and for $n = 1, 2, \cdots$, which implies that the sequence $\{h_{x_n}(t)\}_{n=0}^{\infty} = \{\phi_{x_n, A(x_n)}(t)\}_{n=0}^{\infty}$ converges to 1.

On the other hand, by (2) and (3), we have

$$\begin{aligned} f_{x_n, x_{n+1}}\left(\frac{b}{c}\frac{t}{b}\right) &\ge \phi_{x_n, Ax_n}\left(\frac{b}{c}t\right) \\ &\ge f_{x_{n-1}, x_n}\left(\frac{b}{c^2}t\right) \end{aligned} \tag{6}$$

for all $t > 0$ and for $n = 1, 2, \cdots$. Then,

$$f_{x_n, x_{n+1}}\left(\frac{c}{b}t\right) \ge f_{x_{n-1}, x_n}(t) \tag{7}$$

for all $t > 0$ and for $n = 1, 2, \cdots$. Hence, by Lemma 2.9 the sequence $\{x_n\}_{n=0}^{\infty}$ is Cauchy. Since $X$ is a complete Menger PM-space, there exists a $x \in X$ such that $x_n$ converges to $x$. Since $h$ is continuous, we have

$$1 = \lim_{n\to\infty} h_{x_n}(t) = h_x(t)$$

for all $t > 0$, then,

$$\phi_{x, A(x)}(t) = 1 \tag{8}$$

for all $t > 0$. From $A(x) \in P_{cl}(X)$ and (8) we have that $x \in A(x)$. Hence, $A$ has a fixed point in $X$. $\square$

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