



## Ulam's Type Stability of Hadamard Type Fractional Integral Equations

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**Abstract.** In this paper, we further investigate Ulam's type stability of Hadamard type fractional integral equations on a compact interval. We explore new conditions and develop valuable techniques to overcome the difficulty from the Hadamard type singular kernel and extend the previous Ulam's type stability results in [27] from  $[1, b]$  to  $[a, b]$  with  $a > 0$  via fixed point method. Finally, two examples are given to illustrate our results.

### 1. Introduction

The stability of functional equations originated from Ulam who posed this important question in 1940, concerning the stability of group homomorphisms. In 1941, Hyers gave a partial affirmative answer to the question of Ulam in the context of Banach spaces, that was the first significant breakthrough and a step toward more solutions in this area. Since then, a large number of papers have been published in connection with various generalizations of Ulam's type stability theory or the Ulam-Hyers stability theory. For the advanced contribution on Ulam's type stability, we refer to [1–3, 6, 7, 10, 11, 13, 18, 19, 21, 22, 26, 27] and other stability results [12, 17, 24, 25].

Fractional calculus has played a very important role in various fields such as mechanics, electricity, biology, economics, and signal and image processing. Recently, fractional differential and integral equations appear naturally in the fields such as viscoelasticity, electrical circuits, nonlinear oscillation of earthquake and etc. There are some remarkable monographs provide the main theoretical tools for the qualitative analysis of this research field, and at the same time, show the interconnection as well as the contrast between classical differential and integral models and fractional differential and integral models, are [4, 5, 9, 15, 16, 20, 23].

In [27], the authors firstly offered Ulam's type stability of Hadamard fractional differential equations and derived the Ulam-Hyers stability results on  $[1, b]$  by using standard method provided in [21]. However, the corresponding Ulam-Hyers stability results on  $[a, b]$  where  $a > 0$  has not been studied. In order to fix this gap, we will apply another method, fixed point method, to investigate Ulam's type stability of the

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following Hadamard type fractional integral equations [15] in the space of continuous functions:

$$y(x) = \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha - j + 1)} \left(\ln \frac{x}{a}\right)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t, y(t)) \frac{dt}{t}, \tag{1}$$

where  $\alpha \in (n - 1, n]$ ,  $n = 1, 2, \dots$ ,  $\Gamma(\cdot)$  is the Gamma function,  $a, b$  and  $b_j$  are fixed real numbers such that  $0 < a \leq x \leq b < +\infty$  and  $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function.

To achieve our results, we will explore new conditions and develop valuable techniques to overcome the difficult from the Hadamard type singular kernel  $(\ln \frac{x}{t})^{\alpha-1}$  and extend the previous Ulam’s type stability results in [27] from  $[1, b]$  to  $[a, b]$ .

**Definition 1.1.** *If for each function  $y$  satisfying*

$$\left| y(x) - \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha - j + 1)} \left(\ln \frac{x}{a}\right)^{\alpha-j} - \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t, y(t)) \frac{dt}{t} \right| \leq \varphi(x),$$

where  $\varphi$  is a nonnegative function, there is a solution  $y_0$  of the equation (1) and a constant  $c > 0$  independent of  $y$  and  $y_0$  such that

$$|y(x) - y_0(x)| \leq c\varphi(x), \quad x \in [a, b],$$

then the equation (1) is called Hyers-Ulam-Rassias stable.

In the case where  $\varphi$  takes the form of a constant function, the equation (1) is called Hyers-Ulam stable.

For a nonempty set  $X$ , a function  $d : X \times X \rightarrow [0, +\infty]$  is called a generalized metric on  $X$  if and only if  $d$  satisfies  $d(x, y) = 0$  if and only if  $x = y$ ,  $d(x, y) = d(y, x)$  for all  $x, y \in X$  and  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .

**Theorem 1.2.** (see [8]) *Let  $(X, d)$  be a generalized complete metric space. Assume that  $T : X \rightarrow X$  is a strictly contractive operator with the Lipschitz constant  $L < 1$ . If there exists a nonnegative integer  $k$  such that  $d(T^{k+1}x, T^kx) < +\infty$  for some  $x \in X$ , then the followings are true:*

- (a) *The sequence  $\{T^n x\}$  converges to a fixed point  $x^*$  of  $T$ ;*
- (b)  *$x^*$  is the unique fixed point of  $T$  in*

$$X^* = \{y \in X \mid d(T^k x, y) < +\infty\};$$

- (c) *If  $y \in X^*$ , then*

$$d(y, x^*) \leq \frac{1}{1-L} d(Ty, y).$$

## 2. Ulam’s type stability results

In this section, we will study Hyers-Ulam-Rassias stability and Hyers-Ulam stability of the equation (1) on a compact interval  $[a, b]$ .

Let  $0 < a < b$ ,  $0 < p < 1$ ,  $n - 1 < \alpha \leq n$ ,  $p < \alpha$  and  $M = \frac{1}{\Gamma(\alpha)} \left(\frac{1-p}{\alpha-p}\right)^{1-p} \left(\ln \frac{b}{a}\right)^{\alpha-p}$ .

We introduce the following assumptions:

[H1]:  $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function and for any  $t \in [a, b]$  and  $y, z \in \mathbb{R}$ ,

$$|f(t, y) - f(t, z)| \leq Lt^p |y - z|. \tag{2}$$

[H2]: There exists a continuous function  $y : [a, b] \rightarrow \mathbb{R}$  satisfies

$$\left| y(x) - \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha - j + 1)} \left(\ln \frac{x}{a}\right)^{\alpha-j} - \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t, y(t)) \frac{dt}{t} \right| \leq \varphi(x) \tag{3}$$

for all  $x \in [a, b]$ , and  $\varphi : [a, b] \rightarrow (0, +\infty)$  satisfies

$$\left( \int_a^x (\varphi(t))^{\frac{1}{p}} dt \right)^p \leq K\varphi(x). \tag{4}$$

[H3]:  $0 < KLM < 1$ .

Now we are ready to state our first result.

**Theorem 2.1.** Assume that [H1], [H2] and [H3] are satisfied. Then there exists a unique continuous function  $y_0 : [a, b] \rightarrow \mathbb{R}$  such that

$$y_0(x) = \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha - j + 1)} \left(\ln \frac{x}{a}\right)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t, y_0(t)) \frac{dt}{t} \tag{5}$$

and

$$|y(x) - y_0(x)| \leq \frac{\varphi(x)}{1 - KLM} \tag{6}$$

for all  $x \in [a, b]$ .

**Proof.** We mimic the framework in [14] to consider the space of continuous functions

$$X = \{g : [a, b] \rightarrow \mathbb{R} \mid g \text{ is continuous}\}, \tag{7}$$

endowed with the generalized metric on  $X$  defined by

$$d(g, h) = \inf \{C \in [0, +\infty) \mid |g(x) - h(x)| \leq C\varphi(x) \text{ for all } x \in [a, b]\}. \tag{8}$$

It follows [14] that  $(X, d)$  is a complete generalized metric space.

Define an operator  $T : X \rightarrow X$  by

$$(Ty)(x) = \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha - j + 1)} \left(\ln \frac{x}{a}\right)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t, y(t)) \frac{dt}{t}, \tag{9}$$

for all  $y \in X$  and  $x \in [a, b]$ . Clearly,  $T$  is a well defined operator.

Next, we shall verify that  $T$  is strictly contractive on  $X$ . Note that the definition of  $(X, d)$ , for any  $g, h \in X$ , it is possible to find a  $C_{gh} \in [0, +\infty]$  such that

$$|g(x) - h(x)| \leq C_{gh}\varphi(x), \tag{10}$$

for any  $x \in [a, b]$ .

From the definition of  $T$  in (9) and (2), (4), and (10), we obtain

$$\begin{aligned}
 & |(Tg)(x) - (Th)(x)| \\
 &= \frac{1}{\Gamma(\alpha)} \left| \int_a^x (\ln x - \ln t)^{\alpha-1} \frac{1}{t} [f(t, g(t)) - f(t, h(t))] dt \right| \\
 &\leq L \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} t^{-1} t^p |g(t) - h(t)| dt \\
 &\leq LC_{gh} \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} t^{p-1} \varphi(t) dt \\
 &= LC_{gh} \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} (t^{\frac{p-1}{\alpha-1}})^{\alpha-1} \varphi(t) dt \\
 &= LC_{gh} \frac{1}{\Gamma(\alpha)} \int_a^x [(\ln x - \ln t) t^{\frac{p-1}{\alpha-1}}]^{\alpha-1} \varphi(t) dt \\
 &\leq LC_{gh} \frac{1}{\Gamma(\alpha)} \left( \int_a^x [(\ln x - \ln t) t^{\frac{p-1}{\alpha-1}}]^{\frac{\alpha-1}{1-p}} dt \right)^{1-p} \left( \int_a^x (\varphi(t))^{\frac{1}{p}} dt \right)^p \\
 &\leq KLC_{gh} \varphi(x) \frac{1}{\Gamma(\alpha)} \left( \int_a^x (\ln x - \ln t)^{\frac{\alpha-1}{1-p}} t^{-1} dt \right)^{1-p} \\
 &= KLC_{gh} \varphi(x) \frac{1}{\Gamma(\alpha)} \left( - \int_a^x (\ln x - \ln t)^{\frac{\alpha-1}{1-p}} d(\ln x - \ln t) \right)^{1-p} \\
 &= KLC_{gh} \varphi(x) \frac{1}{\Gamma(\alpha)} \left[ \frac{1-p}{\alpha-p} (\ln x - \ln a)^{\frac{\alpha-p}{1-p}} \right]^{1-p} \\
 &\leq KLC_{gh} \varphi(x) \frac{1}{\Gamma(\alpha)} \left( \frac{1-p}{\alpha-p} \right)^{1-p} \left( \ln \frac{b}{a} \right)^{\alpha-p}.
 \end{aligned}$$

This yields that

$$|(Tg)(x) - (Th)(x)| \leq KLMC_{gh} \varphi(x),$$

for all  $x \in [a, b]$ . That is,  $d(Tg, Th) \leq KLMC_{gh}$ . Hence, we can conclude that  $d(Tg, Th) \leq KLMD(g, h)$  for any  $g, h \in X$ , and since  $0 < KLM < 1$ , the strictly continuous property is verified. Let us take  $g_0 \in X$ . From the continuous property of  $g_0$  and  $Tg_0$ , it follows that there exists a constant  $0 < C_1 < +\infty$  such that

$$\begin{aligned}
 & |(Tg_0)(x) - g_0(x)| \\
 &= \left| \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha - j + 1)} (\ln \frac{x}{a})^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^x (\ln \frac{x}{t})^{\alpha-1} f(t, g_0(t)) \frac{dt}{t} - g_0(x) \right| \leq C_1 \varphi(x)
 \end{aligned}$$

for all  $x \in [a, b]$ , since  $f$  and  $g_0$  are bounded on  $[a, b]$  and  $\varphi(x) > 0$ . Thus, (8) implies that

$$d(Tg_0, g_0) < +\infty.$$

Now, we can use the Banach Fixed Point Theorem and conclude that there exists a continuous function  $y_0 : [a, b] \rightarrow \mathbb{R}$  such that  $T^n g_0 \rightarrow y_0$  in  $(X, d)$  as  $n \rightarrow +\infty$  and  $Ty_0 = y_0$ , that is,  $y_0$  satisfies equation (5) for every  $x \in [a, b]$ . We will now verify that  $\{g \in X \mid d(g_0, g) < +\infty\} = X$ . For any  $g \in X$ , since  $g$  and  $g_0$  are bounded on  $[a, b]$  and  $\min_{x \in [a, b]} \varphi(x) > 0$ , there exists a constant  $0 < C_g < +\infty$  such that

$$|g_0(x) - g(x)| \leq C_g \varphi(x)$$

for any  $x \in [a, b]$ . Hence, we have  $d(g_0, g) < +\infty$  for all  $g \in X$ , that is,  $\{g \in X \mid d(g_0, g) < +\infty\} = X$ . Hence, we conclude that  $y_0$  is the unique continuous function with the property (5). On the other hand, from (3) it follows that

$$d(y, Ty) \leq 1. \tag{11}$$

At last,

$$d(y, y_0) \leq \frac{1}{1 - KLM} d(Ty, y) \leq \frac{1}{1 - KLM},$$

which means that the inequality (6) holds true for all  $x \in [a, b]$ .  $\square$

Next, we try to reduce [H1] to the following assumption:

[H1\*]:  $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is Carathéodory function and there exists  $M_* > 0$  such that

$$|f(t, y)| \leq M_* t^p (\ln b - \ln t)^q, \quad p - \alpha \leq q \leq 0.$$

Moreover, for any  $t \in [a, b]$  and  $y, z \in \mathbb{R}$ ,

$$|f(t, y) - f(t, z)| \leq L t^p (\ln b - \ln t)^q |y - z|, \quad p - \alpha \leq q \leq 0.$$

**Theorem 2.2.** Let  $M^* = \frac{1}{\Gamma(\alpha)} \left(\frac{1-p}{\alpha+q-p}\right)^{1-p} \left(\ln \frac{b}{a}\right)^{\alpha+q-p}$ . Assume that [H1\*], [H2] and  $0 < KLM^* < 1$  are satisfied. Then there exists a unique continuous function  $y_0 : [a, b] \rightarrow \mathbb{R}$  such that

$$|y(x) - y_0(x)| \leq \frac{\varphi(x)}{1 - KLM^*}$$

for all  $x \in [a, b]$ .

**Proof.** Firstly, we show that the second integral term in (9) is bounded. In fact,

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t, y(t)) \frac{dt}{t} \\ & \leq \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} t^{-1} t^p (\ln b - \ln t)^q M_* dt \\ & \leq \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} t^{-1} t^p (\ln x - \ln t)^q M_* dt \\ & = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha+q-1} \left(t^{\frac{p-1}{\alpha+q-1}}\right)^{\alpha+q-1} M_* dt \\ & \leq \frac{1}{\Gamma(\alpha)} \int_a^x [(\ln x - \ln t) t^{\frac{p-1}{\alpha+q-1}}]^{\alpha+q-1} M_* dt \\ & \leq \frac{1}{\Gamma(\alpha)} \left( \int_a^x [(\ln x - \ln t) t^{\frac{p-1}{\alpha+q-1}}]^{\frac{\alpha+q-1}{1-p}} dt \right)^{1-p} \left( \int_a^x (M_*)^{\frac{1}{p}} dt \right)^p \\ & \leq \frac{M_*(b-a)^p}{\Gamma(\alpha)} \left( \int_a^x (\ln x - \ln t)^{\frac{\alpha+q-1}{1-p}} t^{-1} dt \right)^{1-p} \\ & = \frac{M_*(b-a)^p}{\Gamma(\alpha)} \left( - \int_a^x (\ln x - \ln t)^{\frac{\alpha+q-1}{1-p}} d(\ln x - \ln t) \right)^{1-p} \\ & \leq \frac{M_*(b-a)^p}{\Gamma(\alpha)} \left[ \frac{1-p}{\alpha+q-p} (\ln b - \ln a)^{\frac{\alpha+q-p}{1-p}} \right]^{1-p}. \end{aligned}$$

Secondly, we follow the framework in Theorem 2.1 to prove this result. Note that

$$\begin{aligned}
 & |(Tg)(x) - (Th)(x)| \\
 &= \frac{1}{\Gamma(\alpha)} \left| \int_a^x (\ln x - \ln t)^{\alpha-1} \frac{1}{t} [f(t, g(t)) - f(t, h(t))] dt \right| \\
 &\leq L \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} t^{-1} t^p (\ln b - \ln t)^q |g(t) - h(t)| dt \\
 &\leq L \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} t^{p-1} (\ln x - \ln t)^q |g(t) - h(t)| dt \\
 &\leq LC_{gh} \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha+q-1} t^{p-1} \varphi(t) dt \\
 &= LC_{gh} \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha+q-1} (t^{\frac{p-1}{\alpha+q-1}})^{\alpha+q-1} \varphi(t) dt \\
 &= LC_{gh} \frac{1}{\Gamma(\alpha)} \int_a^x [(\ln x - \ln t) t^{\frac{p-1}{\alpha+q-1}}]^{\alpha+q-1} \varphi(t) dt \\
 &\leq LC_{gh} \frac{1}{\Gamma(\alpha)} \left( \int_a^x [(\ln x - \ln t) t^{\frac{p-1}{\alpha+q-1}}]^{\frac{\alpha+q-1}{1-p}} dt \right)^{1-p} \left( \int_a^x (\varphi(t))^{\frac{1}{p}} dt \right)^p \\
 &\leq KLC_{gh} \varphi(x) \frac{1}{\Gamma(\alpha)} \left( \int_a^x (\ln x - \ln t)^{\frac{\alpha+q-1}{1-p}} t^{-1} dt \right)^{1-p} \\
 &= KLC_{gh} \varphi(x) \frac{1}{\Gamma(\alpha)} \left( - \int_a^x (\ln x - \ln t)^{\frac{\alpha+q-1}{1-p}} d(\ln x - \ln t) \right)^{1-p} \\
 &= KLC_{gh} \varphi(x) \frac{1}{\Gamma(\alpha)} \left[ \frac{1-p}{\alpha+q-p} (\ln x - \ln a)^{\frac{\alpha+q-p}{1-p}} \right]^{1-p} \\
 &\leq KLC_{gh} \varphi(x) \frac{1}{\Gamma(\alpha)} \left( \frac{1-p}{\alpha+q-p} \right)^{1-p} \left( \ln \frac{b}{a} \right)^{\alpha+q-p} \\
 &= KLM^* C_{gh} \varphi(x).
 \end{aligned}$$

Then, one can complete the rest proof by proceeding the standard process in Theorem 2.1. □

Now, we present Hyers-Ulam stability of the equation (1).

Let  $0 < a < b$ ,  $n - 1 < \alpha \leq n$ , set  $M' = \frac{1}{\Gamma(\alpha+1)} \left( \ln \frac{b}{a} \right)^\alpha$ .

We need the following assumptions:

[H1']:  $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function and for any  $t \in [a, b]$  and  $y, z \in \mathbb{R}$ ,

$$|f(t, y) - f(t, z)| \leq L|y - z|. \tag{12}$$

[H2']: There exists a continuous function  $y : [a, b] \rightarrow \mathbb{R}$  satisfies

$$\left| y(x) - \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha - j + 1)} \left( \ln \frac{x}{a} \right)^{\alpha-j} - \frac{1}{\Gamma(\alpha)} \int_a^x \left( \ln \frac{x}{t} \right)^{\alpha-1} f(t, y(t)) \frac{dt}{t} \right| \leq \theta \tag{13}$$

for all  $x \in [a, b]$ .

[H3']:  $0 < LM' < 1$ .

**Theorem 2.3.** Assume that [H1'], [H2'] and [H3'] are satisfied. Then there exists a unique continuous function  $y_0 : [a, b] \rightarrow \mathbb{R}$  such that

$$y_0(x) = \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha - j + 1)} \left( \ln \frac{x}{a} \right)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^x \left( \ln \frac{x}{t} \right)^{\alpha-1} f(t, y_0(t)) \frac{dt}{t} \tag{14}$$

and

$$|y(x) - y_0(x)| \leq \frac{\theta}{1 - LM'} \tag{15}$$

for all  $x \in [a, b]$ .

**Proof.** We consider the space of continuous functions presented in (7) again and endowed with the generalized metric defined by

$$d(g, h) = \inf \{C \in [0, +\infty] \mid |g(x) - h(x)| \leq C \text{ for all } x \in [a, b]\}. \tag{16}$$

Define the same operator  $T$  in (9), we shall verify that  $T$  is strictly contractive on  $X$ . Note that the definition of  $(X, d)$ , for any  $g, h \in X$ , it is possible to find a  $C_{gh} \in [0, +\infty]$  such that

$$|g(x) - h(x)| \leq C_{gh}, \tag{17}$$

for any  $x \in [a, b]$ . It follows the definition of  $T$  in (9) and our assumptions, we obtain

$$\begin{aligned} & |(Tg)(x) - (Th)(x)| \\ &= \frac{1}{\Gamma(\alpha)} \left| \int_a^x (\ln x - \ln t)^{\alpha-1} \frac{1}{t} [f(t, g(t)) - f(t, h(t))] dt \right| \\ &\leq L \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} \frac{1}{t} |g(t) - h(t)| dt \\ &\leq LC_{gh} \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} \frac{1}{t} dt \\ &= LC_{gh} \frac{1}{\Gamma(\alpha)} \left[ - \int_a^x (\ln x - \ln t)^{\alpha-1} d(\ln x - \ln t) \right] \\ &= LC_{gh} \frac{1}{\Gamma(\alpha)} \left[ \frac{1}{\alpha} (\ln x - \ln a)^\alpha \right] \\ &\leq LC_{gh} \frac{(\ln b - \ln a)^\alpha}{\Gamma(\alpha + 1)}. \end{aligned}$$

Therefore,  $d(Tg, Th) \leq LM' C_{gh}$ . Hence, we can conclude that  $d(Tg, Th) \leq LM' d(g, h)$  for any  $g, h \in X$ , and since  $0 < LM' < 1$ , the strictly continuous property is verified. Similarly as in the proof of Theorem 2.1, one can derive the results.  $\square$

### 3. Examples

In this section we give two examples to illustrate the usefulness of our main results.

**Example 3.1.** Let  $a = 1$ ,  $p = \frac{1}{3}$ ,  $\alpha = \frac{1}{2}$ ,  $n = 1$ ,  $b = 1 - \frac{2}{3} \ln \frac{2}{5}$ ,  $K = 1$ ,  $M = 1.257 < \frac{3}{2}$ , set  $L = \frac{1}{2} < \min\{1, M^{-1}\}$ . Clearly,  $0 < KLM < 1$ .

We assume that a continuous function  $y : [1, 1 - \frac{2}{3} \ln \frac{2}{5}] \rightarrow \mathbb{R}$  satisfies

$$\left| y(x) - \frac{b_1}{\Gamma(\frac{1}{2})} (\ln x)^{-\frac{1}{2}} - \frac{1}{\Gamma(\frac{1}{2})} \int_1^x (\ln \frac{x}{t})^{-\frac{1}{2}} \frac{1}{2} t^{\frac{1}{3}} y(t) \frac{dt}{t} \right| \leq e^{-\frac{1}{2}x},$$

for all  $x \in [1, 1 - \frac{2}{3} \ln \frac{2}{5}]$ . Set  $f(t, y(t)) = \frac{1}{2} t^{\frac{1}{3}} y(t)$ ,  $\varphi(x) = e^{-\frac{1}{2}x}$ . We obtain

$$\left| \left( \int_1^x e^{-\frac{3}{2}t} dt \right)^{\frac{1}{3}} \right| = \left( \frac{2}{3} e^{-\frac{3}{2}} - \frac{2}{3} e^{-\frac{3}{2}x} \right)^{\frac{1}{3}} \leq e^{-\frac{1}{2}x},$$

for each  $x \in [1, 1 - \frac{2}{3} \ln \frac{2}{5}]$ , since  $(\frac{2}{3}e^{-\frac{3}{2}} - \frac{2}{3}e^{-\frac{3}{2}x})^{\frac{1}{3}} - e^{-\frac{1}{2}x} \leq 0$  for all  $x \in [1, 1 - \frac{2}{3} \ln \frac{2}{5}]$ .

According to Theorem 2.1, there exists a unique continuous function  $y_0 : [1, 1 - \frac{2}{3} \ln \frac{2}{5}] \rightarrow \mathbb{R}$  such that

$$y_0(x) = \frac{b_1}{\Gamma(\frac{1}{2})} (\ln x)^{-\frac{1}{2}} + \frac{1}{\Gamma(\frac{1}{2})} \int_1^x (\ln \frac{x}{t})^{-\frac{1}{2}} \frac{1}{2} t^{\frac{1}{3}} y_0(t) \frac{dt}{t}$$

and

$$|y(x) - y_0(x)| \leq 4e^{-\frac{1}{2}x},$$

for all  $x \in [1, 1 - \frac{2}{3} \ln \frac{2}{5}]$ .

**Example 3.2.** Let  $a = 1$ ,  $b = 2$ ,  $\alpha = \frac{1}{2}$ ,  $n = 1$ ,  $M' = 0.94$  and  $L = \frac{1}{2}$ . Clearly,  $LM' = 0.47 < 0.5$ .

Now, we assume that a continuous function  $y : [1, 2] \rightarrow \mathbb{R}$  satisfies

$$\left| y(x) - \frac{b_1}{\Gamma(\frac{1}{2})} (\ln x)^{-\frac{1}{2}} - \frac{1}{\Gamma(\frac{1}{2})} \int_1^x (\ln \frac{x}{t})^{-\frac{1}{2}} \frac{1}{2} y(t) \frac{dt}{t} \right| \leq \epsilon \quad (18)$$

for all  $x \in [1, 2]$  and some  $\epsilon > 0$ .

Then by Theorem 2.3, there exists a unique continuous function  $y_0 : [1, 2] \rightarrow \mathbb{R}$  such that

$$y_0(x) = \frac{b_1}{\Gamma(\frac{1}{2})} (\ln x)^{-\frac{1}{2}} + \frac{1}{\Gamma(\frac{1}{2})} \int_1^x (\ln \frac{x}{t})^{-\frac{1}{2}} \frac{1}{2} y_0(t) \frac{dt}{t}$$

and

$$|y(x) - y_0(x)| \leq 2\epsilon,$$

for all  $x \in [1, 2]$ .

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