Curvature Properties of Some Class of Minimal Hypersurfaces in Euclidean Spaces

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Dedicated to the birthday of Professor Mileva Prvanović

Abstract. We determine curvature properties of pseudosymmetry type of some class of minimal 2-quasi-umbilical hypersurfaces in Euclidean spaces $E_{n+1}^n$, $n \geq 4$. We present examples of such hypersurfaces. The obtained results are used to determine curvature properties of biharmonic hypersurfaces with three distinct principal curvatures in $E^5$. Those hypersurfaces were recently investigated by Y. Fu in [38].

1. Introduction

Let $M$ be a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature $N_n^s(\kappa)$ with signature $(s, n+1-s)$, $n \geq 4$, where $c = \frac{s}{n+1}$ and $\kappa$ are the sectional curvature and the scalar curvature of the ambient space, respectively. Let $\mathcal{U}_H \subset M$ be the set of all points at which the $(0,2)$-tensor $H^2$ is not expressed by a linear combination of the second fundamental tensor $H$ and the metric tensor $g$ of $M$. For precise definitions of the symbols used here, we refer to Section 2 of this paper (see also [19], [20] and [22]).

Curvature properties of pseudosymmetry type of hypersurfaces in semi-Riemannian spaces of constant curvature were investigated in several papers. In particular, hypersurfaces $M$ in $N_{n+1}^c$, $n \geq 4$, with the tensor $H$ satisfying on $\mathcal{U}_H$

\[ H^2 = \phi H^2 + \psi H + \rho g, \]

(1)
for some functions $\phi$, $\psi$ and $\rho$, were investigated in the following papers: [9]–[13], [17]–[18], [21]–[23], [25], [28]–[31], [33], [36], [40], [48]–[52].

The main results of Section 3 are presented in Proposition 3.1 and Theorem 3.2. In Proposition 3.1 we present curvature properties of minimal hypersurfaces $M$ in $N^{n+1}(c)$, $n \geq 4$, satisfying (1). In Theorem 3.2 we present curvature properties of minimal hypersurfaces $M$ in semi-Euclidean spaces $E^{n+1}_c$, $n \geq 4$, satisfying (1) with $\rho = 0$, i.e.

$$H^2 = \phi H^2 + \psi H.$$  \hfill (2)

We also present examples of hypersurfaces satisfying (1), see Example 3.1(iii) and Example 3.2(ii).

In Section 4 we consider hypersurfaces $M$ in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having at every point of $\mathcal{U}_H \subset M$ exactly three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ such that

$$\lambda_1 = 0, \quad \lambda_2 = -(n-2)\lambda, \quad \lambda_3 = \lambda_4 = \ldots = \lambda_n = \lambda \neq 0,$$  \hfill (3)

where $\lambda$ is a function on $\mathcal{U}_H$. Evidently, we have on $\mathcal{U}_H$: $tr(H) = 0$ and

$$H^2 = \phi H^2 + \psi H, \quad \phi = -(n-3)\lambda, \quad \psi = (n-2)\lambda^2, \quad \rho = 0.$$  \hfill (4)

In Proposition 4.1 we present curvature properties of hypersurfaces $M$ in $N^{n+1}(c)$, $n \geq 4$, satisfying (3). Using results of that proposition we obtain curvature properties of hypersurfaces $M$ in Euclidean spaces $E^{n+1}$, $n \geq 4$, satisfying (3). We also present examples of hypersurfaces satisfying (3), see Example 4.1 and Example 4.2(ii). We recall that a Riemannian manifold $(M, g)$, $n = \text{dim} M$, isometrically immersed in an $m$-dimensional Euclidean space $E^m$ is said to be biharmonic submanifold ([6]) if its mean curvature vector field $\vec{H}$ satisfies $\Delta \vec{H} = 0$, where $\Delta$ is the Laplace operator of $M$. For recent survey on biharmonic submanifolds we refer to the book of B.-Y. Chen [6]. It is clear that any minimal submanifold in $E^m$ is trivially biharmonic. A biharmonic submanifold in $E^m$ is called proper biharmonic if it is not minimal. Very recently, biharmonic hypersurfaces with three distinct principal curvatures in $E^5$ were investigated in [38]. In Theorem 3.2 of [38] it was stated that every biharmonic hypersurface $M$ with three distinct principal curvatures in $E^5$ is minimal. The principal curvatures: $\lambda_1$, $\lambda_2$ and $\lambda_3$ of $M$ satisfy (3) with $n = 4$. In Theorem 4.3 we present curvature properties of those hypersurfaces.

2. Preliminaries

Throughout the paper all manifolds are assumed to be connected paracompact manifolds of class $C^\infty$. Let $(M, g)$ be an $n$-dimensional, $n \geq 3$, semi-Riemannian manifold and let $\nabla$ be its Levi-Civita connection and $\Xi(M)$ the Lie algebra of vector fields on $M$.

We define on $M$ the endomorphisms $X \wedge A Y$ and $\mathcal{R}(X, Y)$ of $\Xi(M)$, respectively, by

$$(X \wedge A Y)Z = A(Y, Z)X - A(X, Z)Y, \quad \mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_{[X, Y]} Z,$$

where $A$ is a symmetric $(0, 2)$-tensor on $M$ and $X, Y, Z \in \Xi(M)$. The Ricci tensor $S$, the Ricci operator $\mathcal{S}$, the tensors $S^2$ and $S^3$ and the scalar curvature $\kappa$ of $(M, g)$ are defined by $S(X, Y) = tr[Z \rightarrow \mathcal{R}(Z, X)Y]$, $g(SX, Y) = S(X, Y)$, $S^2(X, Y) = S(SX, Y)$, $S^3(X, Y) = S^2(SX, Y)$ and $\kappa = tr S$, respectively. The endomorphisms $C(X, Y)$ and $\text{conh}(\mathcal{R})(X, Y)$ are defined by

$$C(X, Y)Z = \mathcal{R}(X, Y)Z - \frac{1}{n-2}(X \wedge g SY + SX \wedge g Y - \frac{\kappa}{n-1} X \wedge g Y)Z, \quad \text{conh}(\mathcal{R})(X, Y)Z = \mathcal{R}(X, Y)Z - \frac{1}{n-2}(X \wedge g SY + SX \wedge g Y),$$
respectively. Now the (0,4)-tensor G, the Riemann-Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the conharmonic tensor $\text{conh}(R)$ of $(M, g)$ are defined by

\[
G(X_1, X_2, X_3, X_4) = g((X_1 \land g X_2)X_3, X_4),
\]

\[
R(X_1, X_2, X_3, X_4) = g(R(X_1, X_2)X_3, X_4),
\]

\[
C(X_1, X_2, X_3, X_4) = g(C(X_1, X_2)X_3, X_4),
\]

\[
\text{conh}(R)(X_1, X_2, X_3, X_4) = g(\text{conh}(R)(X_1, X_2)X_3, X_4),
\]

respectively, where $X_1, X_2, \ldots \in \Xi(M)$. We define the following subsets of $M$: $\mathcal{U}_R = \{x \in M | R - \frac{x}{(n-1)n} G \neq 0 \text{ at } x\}, \mathcal{U}_G = \{x \in M | S - \frac{2}{n} g \neq 0 \text{ at } x\}$ and $\mathcal{U}_C = \{x \in M | C \neq 0 \text{ at } x\}$. We note that $\mathcal{U}_G \cup \mathcal{U}_C = \mathcal{U}_R$.

Let $\mathcal{B}$ be a tensor field sending any $X, Y \in \Xi(M)$ to a skew-symmetric endomorphism $\mathcal{B}(X, Y)$, and let $B$ be a (0,4)-tensor associated with $\mathcal{B}$ by

\[
B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4). \tag{5}
\]

The tensor $B$ is said to be a generalized curvature tensor if the following conditions are satisfied

\[
B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2),
\]

\[
B(X_1, X_2, X_3, X_4) + B(X_3, X_1, X_2, X_4) + B(X_2, X_3, X_1, X_4) = 0.
\]

For $\mathcal{B}$ as above, let $B$ be again defined by (5). We extend the endomorphism $\mathcal{B}(X, Y)$ to a derivation $\mathcal{B}(X, Y)$ of the algebra of tensor fields on $M$, assuming that it commutes with contractions and $\mathcal{B}(X, Y) \cdot f = 0$, for any smooth function $f$ on $M$. For a (0,k)-tensor field $T$, $k \geq 1$, we can define the $(0, k + 2)$-tensor $B \cdot T$ by

\[
(B \cdot T)(X_1, \ldots, X_k, X, Y) = (\mathcal{B}(X, Y) \cdot T)(X_1, \ldots, X_k)
\]

\[
= -T(\mathcal{B}(X, Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{B}(X, Y)X_k).
\]

In addition, if $A$ is a symmetric (0,2)-tensor then we define the $(0, k + 2)$-tensor $Q(A, T)$ by

\[
Q(A, T)(X_1, \ldots, X_k, X, Y) = (X \land A Y \cdot T)(X_1, \ldots, X_k)
\]

\[
= -T((X \land A Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, (X \land A Y)X_k).
\]

The tensor $Q(A, T)$ is called the Tachibana tensor of the tensors $A$ and $T$, or shortly the Tachibana tensor (see, e.g., [23]). We mention that in some papers the tensor $Q(g, R)$ is called the Tachibana tensor ([41], [42], [43], [47]).

For a symmetric (0,2)-tensor $E$ and a (0,k)-tensor $T$, $k \geq 2$, we define their Kulkarni-Nomizu product $E \land T$ by ([18])

\[
(E \land T)(X_1, X_2, X_3, X_4; Y_3, Y_4, \ldots, Y_k)
\]

\[
= E(X_1, X_4)T(X_2, X_3, Y_3, \ldots, Y_k) + E(X_2, X_3)T(X_1, X_4, Y_3, \ldots, Y_k)
\]

\[
- E(X_1, X_3)T(X_2, X_4, Y_3, \ldots, Y_k) - E(X_2, X_4)T(X_1, X_3, Y_3, \ldots, Y_k).
\]

For instance, the following tensors are generalized curvature tensors: $R, C, G, \text{conh}(R)$ and $E \land F$, where $E$ and $F$ are symmetric (0,2)-tensors. For a symmetric (0,2)-tensor $E$ we define the (0,4)-tensor $\overline{E}$ by $\overline{E} = \frac{1}{2} E \land E$. In particular, we have $\overline{G} = G = \frac{1}{2} g \land g$ and

\[
C = R - \frac{1}{n-2} g \land S + \frac{k}{(n-2)(n-1)} G. \tag{6}
\]

From (6) and the identity $Q(g, G) = 0$ we get immediately

\[
Q(g, C) = Q(g, R - \frac{1}{n-2} g \land S) = Q(g, \text{conh}(R)). \tag{7}
\]

We also have
Lemma 2.1. (cf. [27], Proposition 1) For any semi-Riemannian manifold \((M, g)\), \(n \geq 4\), the following identities hold good

\[
\begin{align*}
\text{conh}(R) \cdot S &= C \cdot S - \frac{\kappa}{(n-2)(n-1)} Q(g, S), \\
R \cdot \text{conh}(R) &= R \cdot C, \\
\text{conh}(R) \cdot R &= C \cdot R - \frac{\kappa}{(n-2)(n-1)} Q(g, R), \\
\text{conh}(R) \cdot \text{conh}(R) &= C \cdot C - \frac{\kappa}{(n-2)(n-1)} Q(g, C).
\end{align*}
\]  

(8)

For a symmetric \((0, 2)\)-tensor \(A\) we define the endomorphism \(\mathcal{A}\) and the tensors \(A^2\) and \(A^3\) by \(g(\mathcal{A}X, Y) = A(X, Y)\), \(A^2(X, Y) = A(\mathcal{A}X, Y)\) and \(A^3(X, Y) = A^2(\mathcal{A}X, Y)\), respectively.

Lemma 2.2. Let \(E_1, E_2\) and \(F\) be symmetric \((0, 2)\)-tensors at a point \(x\) of a semi-Riemannian manifold \((M, g)\), \(n \geq 3\). (i) ([17], [18]) At \(x\) we have

\[
E_1 \wedge Q(E_2, F) + E_2 \wedge Q(E_1, F) + Q(F, E_1 \wedge E_2) = 0.
\]

In particular, if \(E = E_1 = E_2\) then at \(x\) we have

\[
E \wedge Q(E, F) = -Q(F, E).
\]

Moreover (see, e.g., [21], Section 3)

\[
Q(E, E \wedge F) = -Q(F, E).
\]

(ii) ([44], Lemma 3.2) At \(x\) we have

\[
G \cdot F = Q(g, F), \quad (g \wedge F) \cdot F = Q(g, F^2), \\
- (g \wedge F) \cdot (g \wedge F) = Q(F^2, G).
\]

Moreover, if \(A\) is a symmetric \((0, 2)\)-tensor and \(B\) a generalized curvature tensor then

\[
G \cdot A = Q(g, A), \quad G \cdot B = Q(g, B).
\]

(iii) (see, e.g., [37], Lemma 2.4 (iii)) At \(x\) we have

\[
Q(E_1, E_2 \wedge F) + Q(E_2, F \wedge E_1) + Q(F, E_1 \wedge E_2) = 0.
\]

As an immediate consequence of (6) and Lemma 2.2(ii) we get (also see [28], p. 217)

Lemma 2.3. On any semi-Riemannian manifold \((M, g)\), \(n \geq 4\), we have the following identity

\[
C \cdot S = R \cdot S - \frac{1}{n-2} Q(g, S^2 - \frac{\kappa}{n-1} S).
\]  

(9)

Let \(B_{ij}, T_{ij},\) and \(A_{ij}\) be the local components of generalized curvature tensors \(B\) and \(T\) and a symmetric \((0, 2)\)-tensor \(A\) on \(M\), respectively, where \(i, j, k, l, m, p, q \in [1, 2, \ldots, n]\). The local components \((B \cdot T)_{ijklm}\) and \(Q(A, T)_{ijklm}\) of the tensors \(B \cdot T\), \(Q(A, T)\) are the following

\[
\begin{align*}
(B \cdot T)_{ijklm} &= g^{pq}(T_{pik}B_{qlm} + T_{pqj}B_{ilkm} + T_{qik}B_{pjlm} + T_{qik}B_{qjlm}), \\
Q(A, T)_{ijklm} &= A_{hl}T_{mjik} + A_{jl}T_{himk} + A_{jl}T_{hikm} \\
&- A_{km}T_{lij} - A_{km}T_{hijk} - A_{km}T_{lij} + A_{km}T_{lij}, \\
(B \cdot A)_{lk} &= g^{pq}(A_{ph}B_{qk} \pm A_{ph}B_{kq}), \\
Q(g, A)_{lk} &= g_{ln}A_{lm} + g_{lm}A_{nl} - g_{ln}A_{kl} - g_{km}A_{lk}.
\end{align*}
\]

The manifold \((M, g)\), \(n \geq 3\), is said to be an Einstein manifold [1] if \(S = \frac{\kappa}{n} g\) on \(M\).
Einstein manifolds form a subclass of the class of quasi-Einstein manifolds. The semi-Riemannian manifold \((M, g)\), \(n \geq 3\), is called a quasi-Einstein manifold if rank \((S - \alpha g) = 1\) on \(\mathcal{U}_S\), where \(\alpha\) is some function on this set. Every warped product manifold \(\tilde{M} \times_F \tilde{N}\) of an 1-dimensional \((\tilde{M}, \tilde{g})\) base manifold and an 2-dimensional manifold \((\tilde{N}, \tilde{g})\) or an \((n - 1)\)-dimensional Einstein manifold \((\tilde{N}, \tilde{g})\), \(n \geq 4\), with a warping function \(F\), is a quasi-Einstein manifold. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and investigation of quasi-umbilical hypersurfaces of conformally flat spaces. Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied among others in: [17], [21], [25], [31] and [40], see also [20]. We refer to [8] and [27] for recent results on quasi-Einstein manifolds.

The semi-Riemannian manifold \((M, g)\), \(n \geq 3\), is called a 2-quasi-Einstein manifold if rank \((S - \alpha g) \leq 2\) on \(\mathcal{U}_S\) and rank \((S - \alpha g) = 2\) on some open non-empty subset of \(\mathcal{U}_S\), where \(\alpha\) is some function on \(\mathcal{U}_S\). It is clear that every warped product manifold \(\tilde{M} \times_F \tilde{N}\) of an 2-dimensional \((\tilde{M}, \tilde{g})\) base manifold and an 2-dimensional manifold \((\tilde{N}, \tilde{g})\) or an \((n)\)dimensional Einstein manifold \((\tilde{N}, \tilde{g})\), \(n \geq 5\), with a warping function \(F\), is a 2-quasi-Einstein manifold. Therefore some exact solutions of the Einstein field equations are 2-quasi-Einstein manifolds, e.g. the Reissner-Nordström-de Sitter type spacetimes are such manifolds (see, e.g., [44]). It seems that the Reissner-Nordström spacetime is the “oldest” example of a 2-quasi-Einstein warped product manifold. It is easy to see that every 2-quasi-umbilical hypersurface in a space of constant curvature is a 2-quasi-Einstein manifold (see Remark 3.1). We refer to [24] for recent results on 2-quasi-Einstein warped product manifolds.

3. Hypersurfaces in spaces of constant curvature

Let \(M\) be a connected hypersurface isometrically immersed in a semi-Riemannian manifold \((\tilde{N}, \tilde{g})\) of dimension \(n + 1\), \(n \geq 3\). Let \(g\) be the metric tensor induced on \(M\) from \(\tilde{g}\). Let \(\nabla\) and \(\tilde{\nabla}\) be the Levi-Civita connections corresponding to the metric tensors \(g\) and \(\tilde{g}\), respectively. We denote by \(\xi\) a local unit normal vector field on \(M\) in \(\tilde{N}\) and let \(\epsilon = \tilde{g}(\xi, \xi) = \pm 1\). We can write the Gauss formula and the Weingarten formula of \((M, g)\) in \((\tilde{N}, \tilde{g})\) in the form: \(\tilde{\nabla}_X Y = \nabla_X Y + \epsilon H(X, Y) \xi\) and \(\nabla_X \xi = -AX\), respectively, where \(X, Y\) are vector fields tangent to \(M\). \(H\) is the second fundamental tensor and \(A\) the shape operator of \((M, g)\) in \((\tilde{N}, \tilde{g})\). We have \(H(X, Y) = g(AX, Y)\), for any vectors fields \(X, Y\) tangent to \(M\). Further, we set \(H^p(X, Y) = g(A^pX, Y)\), \(p = 1, 2, \ldots, H^1 = H\) and \(H^1 = A\). We denote by \(H^p\) the local components of the tensor \(H^p\).

According to [4], [5], [7], [46], [53], a hypersurface \(M\) in an \((n + 1)\)-dimensional Riemannian manifold \(N\) is said to be quasi-umbilical, resp., 2-quasi-umbilical, at a point \(x \in M\) if it has a principal curvature with multiplicity \(n - 1\), resp., \(n - 2\), i.e. when the principal curvatures of \(M\) at \(x\) are given by \(\lambda_1, \lambda_2, \lambda_3 = \ldots = \lambda_n\), resp., \(\lambda_1, \lambda_2, \lambda_3, \lambda_4 = \ldots = \lambda_n\). If \(M\) is a hypersurface in an \((n + 1)\)-dimensional semi-Riemannian manifold \(N\) then \(M\) is called quasi-umbilical (see, e.g., [34], [40]), resp., 2-quasi-umbilical (see, e.g., [36], [40]), at a point \(x \in M\) when rank \((H - \alpha g) = 1\), resp., rank \((H - \alpha g) = 2\), holds at \(x\), for some \(\alpha \in \mathbb{R}\).

We recall that a hypersurface \(M\) in a semi-Riemannian conformally flat manifold \(N\) is quasi-umbilical at a point \(x \in M\) if and only if its Weyl conformal curvature tensor \(C\) vanishes at this point ([34], Theorem 4.1). Thus a point \(x \in M\) is a non-quasi-umbilical point of \(M\) if and only if the tensor \(C\) is non-zero at \(x\), i.e. \(x \in \mathcal{U}_C \subset M\).

We denote by \(R\) and \(\tilde{R}\) the Riemann-Christoffel curvature tensors of \((M, g)\) and \((\tilde{N}, \tilde{g})\), respectively. Let \(x' = x'(y')\) be the local parametric expression of \((M, g)\) in \((\tilde{N}, \tilde{g})\), where \(y^k\) and \(x^r\) are local coordinates of \(M\) and \(N\), respectively, \(h, i, j, k \in \{1, 2, \ldots, n\}\) and \(p, r, t, u \in \{1, 2, \ldots, n + 1\}\). The Gauss equation of \((M, g)\) in \((\tilde{N}, \tilde{g})\) reads

\[
R_{hijk} = \tilde{R}_{sprtu} B^{sp}_{k} B^{tr}_{l} B^{ru}_{j} + \epsilon (H_{hkij} - H_{hijk}) + B^{tr}_{k} = \frac{\partial x^r}{\partial y^t},
\]  

(10)

where \(\tilde{R}_{sprtu}, R_{hijk}\) and \(H_{hkij}\) are the local components of the tensors \(\tilde{R}, R\) and \(H\), respectively.

Let \(M\) be a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature \(N^{n+1}_s\) with signature \((s, n + 1 - s)\), \(n \geq 4\), where \(c = \frac{s}{n(n+1)}\) and \(\bar{x}\) are the sectional curvature and the scalar...
curvature of the ambient space, respectively. Now (10) turns into
\[ R_{ijkl} = \varepsilon (H_{lh}H_{ij} - H_{lh}H_{ik}) + \frac{\kappa}{n(n+1)} G_{ijkl}, \quad \varepsilon = \pm 1. \] (11)

Contracting (11) with \( g^{ij} \) and \( g^{kh} \) we obtain
\[ S_{hk} = \varepsilon (tr(H) H_{hk} - H_{hk}^2) + \frac{(n-1)\kappa}{n(n+1)} g_{hk}, \] (12)
\[ \kappa = \varepsilon ((tr(H))^2 - tr(H^2)) + \frac{(n-1)\kappa}{n+1}, \] (13)
respectively, where \( tr(H^2) = g^{hk}H_{hk}^2 \) and \( S_{hk} \) are the local components of the Ricci tensor \( S \) of \( M \). It is known that on \( M \) we have ([34])
\[ R \cdot R - Q(S, R) = \frac{(n-2)\kappa}{n(n+1)} Q(g, C). \] (14)

In particular, if the ambient space is a semi-Euclidean space \( \mathbb{E}^{n+1}_s \) then (14) reduces to
\[ R \cdot R = Q(S, R). \] (15)

Let \( M \) be a hypersurface in \( N^{n+1}_c \), \( n \geq 4 \), satisfying (1) on \( U_H \). We define on \( U_H \) the following functions ([48], eq. (34)):
\[ \beta_1 = \varepsilon (\phi - tr(H)), \]
\[ \beta_2 = -\frac{e}{n-2} (\phi (2tr(H) - \phi) - (tr(H))^2 - \psi - (n-2)e\mu), \]
\[ \beta_3 = \varepsilon \mu tr(H) + \frac{1}{n-2} (\psi (2tr(H) - \phi) + (n-3)\rho), \]
\[ \beta_4 = \beta_3 - \varepsilon \beta_2 tr(H) + \frac{(n-1)\kappa \beta_1}{n(n+1)}, \] (16)
where the functions: \( \phi, \psi, \rho, \mu \) are defined by (1) and
\[ \mu = \frac{1}{n-2} \left( \frac{\kappa}{n-1} - \frac{\kappa}{n+1} \right), \] (17)
respectively. We also have on \( U_H \) ([48], eqs. (43), (52), (45), (46)):
\[ R \cdot S = \frac{\kappa}{n(n+1)} Q(g, S) + \rho Q(g, H) - \varepsilon \beta_1 Q(H, H^2), \] (18)
\[ C \cdot S = \beta_1 Q(H, S) + \beta_2 Q(g, S) + \beta_4 Q(g, H), \]
\[ (n-2) R \cdot C = (n-2) Q(S, R) - \frac{(n-2)^2 \kappa}{n(n+1)} Q(g, R) - \frac{(n-3)\kappa}{n(n+1)} Q(S, G) \]
\[ + \rho Q(H, G) + (\phi - tr(H)) g \wedge Q(H, H^2), \] (20)
\[ (n-2) C \cdot R = (n-3) Q(S, R) \]
\[ + \left( \frac{\kappa}{n-1} + \varepsilon \psi - \frac{(n^2 - 3n + 3)\kappa}{n(n+1)} \right) Q(g, R) \]
\[ - \frac{(n-3)\kappa}{n(n+1)} Q(S, G) + (\phi - tr(H)) H \wedge Q(g, H^2), \] (21)
where \( \beta_1, \ldots, \beta_4 \) are defined by (16).
Example 3.1. (i) (Example 1.1, [54]) The Clifford hypersurfaces in $N^n(c)$, $c \neq 0$, $n \geq 4$. (a) For $c > 0$ we set $N^n(c) = S^n(c) = \{ x \in R^{n+1} : < x, x > = \frac{1}{c} \}$, where $< \cdot, \cdot >$ is the standard inner product on $R^{n+1}$. For $1 \leq m \leq n-2$, $t \in (0, \frac{2}{\sqrt{m}})$, let $M_{m,n-m-1}(c,t) = S^m(\frac{c}{\sinh^2 t}) \times S^{m-1}(\frac{c}{\cosh^2 t})$. We view $x = (x_1, x_2) \in M_{m,n-m-1}(c,t)$ as a vector in $R^{n+1} = R^{n+1} \times R^{m-1}$, then $x \in S^n(c)$. This is the standard isometric embedding of $M_{m,n-m-1}(c,t)$ into $S^n(c)$. In this situation, for suitably chosen unit normal vector field, $M_{m,n-m-1}(c,t)$ has two distinct principal curvatures $\rho_1 = \sqrt{c} \cot t$ of the multiplicity $m$ and $\rho_2 = -\sqrt{c} \tan t$ of the multiplicity $n-m-1$.

(b) For $c < 0$ we set $N^n(c) = H^n(c) = \{ x \in R^{n+1} : < x, x > = \frac{1}{c} < 1 > x^n > 0 \}$. Here $x, y > 1 = x^y y^1 + \cdots + x^n y^n - x^n t^n y^n$ is the standard Lorentzian inner product on $R^{n+1}$. For $1 \leq m \leq n-2$, $t \in (0, +\infty)$, let $M_{m,n-m-1}(c,t) = S^m(\frac{c}{\sinh^2 t}) \times H^{m-1}(\frac{c}{\cosh^2 t})$. Then $M_{m,n-m-1}(c,t)$ is an embedded hypersurface in $H^n(c)$, and for suitably chosen unit normal vector field, it has two distinct principal curvatures $\rho_1 = -c \coth t$ of the multiplicity $m$ and $\rho_2 = -c \tanh t$ of the multiplicity $n-m-1$.

(ii) (a) If $2 \leq m \leq n-3$ and $(m-1)c_1 \neq (n-m-2)c_2$, where $c_1 = \frac{c}{\sinh^2 t}$, $c_2 = \frac{c}{\cosh^2 t}$, $t \in (0, \frac{2}{\sqrt{m}})$ then in view of Proposition 3.4 of [39] the Riemann-Christoffel curvature tensor $R$ of $M_{m,n-m-1}(c,t)$ is expressed at every point by a linear combination of the tensors $g \wedge g, g \wedge S$ and $S \wedge S$, i.e. $M_{m,n,m}(c,t)$ is a Roter type hypersurface.

(b) If $2 \leq m \leq n-3$ and $(m-1)c_1 \neq (n-m-2)c_2$, where $c_1 = \frac{c}{\sinh^2 t}$, $c_2 = \frac{c}{\cosh^2 t}$, $t \in (0, +\infty)$, then in view of Proposition 3.4 of [39] the Riemann-Christoffel curvature tensor $R$ of $M_{m,n,m-1}(c,t)$ is expressed at every point by a linear combination of the tensors $g \wedge g$, $g \wedge S$ and $S \wedge S$, i.e. $M_{m,n,m-1}(c,t)$ is a Roter type hypersurface. (c) The Roter type manifolds (and in particular, hypersurfaces in space forms) were studied among others in the papers: [19], [20], [22], [25], [26], [32], [39] and [44].

(iii) Let $M$ be a $n$-dimensional hypersurface in the Euclidean space $E^{n+1}$, $n \geq 4$. Precisely, let $M$ the cone over the Clifford hypersurface $M_{m,n,m-1}(c,t)$ defined in (i). We refer to Section 3 of [45] for precise definition and properties of cones. In particular, from Section 3 of [45] follows immediately that $M$ has at every point three distinct principal curvatures $\lambda_1 = 0$, $\lambda_2 = \frac{1}{c} \rho_1$, and $\lambda_3 = \frac{1}{c} \rho_2$, $t \in R^+$, of the multiplicity 1, $m$ and $n-m-1$, respectively. Thus we see that the cone over the Clifford hypersurface $M_{m,n,m-1}(c,t)$, presented in (i) is a hypersurface in $E^{n+1}$, $n \geq 4$, having exactly three distinct principal curvatures and satisfying at every point $U_H = M$ the equation (1) with $\rho = 0$, i.e. (2).

(iv) We mention that an example of a hypersurface $M$ in $E^{n+1}$, $n \geq 4$, satisfying (1) on $U_H \subset M$, with non-zero function $\rho$ and $\phi = tr(H)$, is presented in [52].

(v) The Cartan hypersurfaces of dimension 6, 12 or 24 satisfy (2), with $\phi = tr(H) = 0$. Curvature properties of these hypersurfaces are presented in [18] (Theorem 4.3).

Proposition 3.1. If $M$ is a minimal hypersurface in a semi-Riemannian space of constant curvature $N^n_{\rho+1}(c)$, $n \geq 4$,
satisfying (1) on $\mathcal{U}_H \subset M$ then the following conditions are satisfied on this set: (14) and

$$
\begin{align*}
\beta_1 &= \varepsilon \phi, \\
\beta_2 &= \frac{\varepsilon}{n-2} (\phi^2 + \psi + (n-2)\varepsilon \mu), \\
\beta_3 &= \frac{1}{n-2} ((n-3)\rho - \psi \phi), \\
\beta_4 &= \beta_3 + \frac{(n-1)\kappa \varepsilon \phi}{n(n+1)},
\end{align*}
$$

(22)

$$ \begin{align*}
R \cdot S &= \frac{\kappa}{n(n+1)} Q(g, S) + \rho Q(g, H) - \phi Q(H, H^2), \\
C \cdot S &= \varepsilon \phi Q(H, S) + \beta_2 Q(g, S) + \beta_4 Q(g, H), \\
(n-2)R \cdot C &= \frac{1}{n(n+1)} Q(g, R) - \frac{(n-3)\kappa}{n(n+1)} Q(S, G) \\
&\quad + \rho Q(H, G) + \phi g \wedge Q(H, H^2), \\
(n-2)C \cdot R &= \frac{1}{n(n+1)} Q(S, G) + \phi H \wedge Q(g, H^2), \\
(\phi \psi + \rho) H &= A^2 + \varepsilon (\phi^2 + \psi) A - \phi \rho g, \\
A^3 &= -\varepsilon (\phi^2 + 2\psi) A^2 + (2\phi \rho - \psi^2) A - \varepsilon \rho^2 g, \\
(\phi \psi + \rho)^2 R &= \frac{\varepsilon}{2} (A^2 + \varepsilon (\phi^2 + \psi) A - \phi \rho g) + (A^2 + \varepsilon (\phi^2 + \psi) A - \phi \rho g) \\
&\quad + \frac{(\phi \psi + \rho)^2 \kappa}{n(n+1)} G,
\end{align*}
$$

(23)-(29)

where $\beta_1, \ldots, \beta_4$ are defined by (22) and

$$
A = S - \frac{(n-1)\kappa}{n(n+1)} \rho.
$$

(30)

**Proof.** Since $M$ is a minimal hypersurface, (16) and (18)-(21) turn into (22)-(26), respectively. From (1), (12) and (30) we get easily

$$
\begin{align*}
A &= -\varepsilon H^2, \\
A^2 &= H^4, \\
A^3 &= -\varepsilon H^6, \\
H^4 &= (\phi^2 + \psi) H^2 + (\phi \psi + \rho) H + \phi \rho g, \\
H^6 &= (\phi^2 + \psi) H^4 + (\sqrt{n} \phi \psi + 2\rho) H^2 + \phi (\phi \psi + \rho) H + \rho (\phi \psi + \rho) g.
\end{align*}
$$

(31)-(33)

Now (27)-(29) are immediate consequences of (11) and (31)-(33). Our proposition is thus proved.

**Remark 3.1.** (i) Let $M$ be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. If at every point of $\mathcal{U}_H \subset M$ we have exactly three distinct principal curvatures $\lambda_1, \lambda_2$ and $\lambda_3$, then (18)-(21) hold on $\mathcal{U}_H$ with $\varepsilon = 1$ and

$$
\phi = \lambda_1 + \lambda_2 + \lambda_3, \quad \psi = -\lambda_1 \lambda_2 - \lambda_2 \lambda_3 - \lambda_2 \lambda_3, \quad \rho = \lambda_1 \lambda_2 \lambda_3.
$$

(34)

(ii) Let $M$ be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. If at every point of $\mathcal{U}_H \subset M$ we have exactly three distinct principal curvatures $\lambda_1, \lambda_2$ and $\lambda_3 = \lambda_4 = \ldots = \lambda_n = \lambda$, then from (12) it follows that

$$
\text{rank} \left( S - \frac{(n-1)\kappa}{n(n+1)} + \lambda (\text{tr}(H) - \lambda) \right) g = 2
$$

(35)
on $\mathcal{U}_H$. Moreover, the following condition holds on $\mathcal{U}_H$ (see [36], p. 53)

$$C \cdot C = -\frac{(n-3)\lambda_1 \lambda_2}{(n-1)(n-2)} Q(g, C).$$

(36)

We refer to [13], [35], [19], [27] and [32] for results on semi-Riemannian manifolds $(M, g)$, dim $M \geq 4$, and in particular, on hypersurfaces $M$ in $N^{n+1}_c$, $n \geq 4$, satisfying on $\mathcal{U}_C \subset M$

$$C \cdot C = L Q(g, C),$$

(37)

where $L$ is some function on this set. We mention that the warped product manifold $\overline{M} \times_F \overline{N}$, of manifolds $(\overline{M}, \overline{g})$, dim $\overline{M} = 2$, and $(\overline{N}, \overline{g})$, dim $\overline{N} = 2$, and the warping function $F$ satisfies (37) on $\mathcal{U}_C \subset \overline{M} \times_F \overline{N}$. Here we also mention that the warped product manifold $\overline{M} \times_F \overline{N}$, of manifolds $(\overline{M}, \overline{g})$, dim $\overline{M} = 1$, and $(\overline{N}, \overline{g})$, dim $\overline{N} = 3$, and the warping function $F$ satisfies on $\mathcal{U}_C \subset \overline{M} \times_F \overline{N}$

$$R \cdot R - Q(S, R) = L Q(g, C),$$

where $L$ is some function on this set ([11]).

Proposition 3.1 leads to the following

**Theorem 3.2.** If $M$ is a minimal hypersurface in a semi-Euclidean space $E_a^{n+1}$, $n \geq 4$, satisfying (2) on $\mathcal{U}_H \subset M$ then the following conditions are satisfied on this set: (15) and

$$\begin{align*}
\phi \psi H &= S^2 + \varepsilon(\phi^2 + \psi) S, \\
S^3 &= -\varepsilon(\phi^2 + 2\psi) S^2 - \psi^2 S, \\
(\phi \psi)^2 R &= \frac{\varepsilon}{2} (S^2 + \phi^2 + \psi) S \wedge (S^2 + \phi^2 + \psi) S, \\
R \cdot S &= \varepsilon \phi Q(H, S), \\
C \cdot S &= \varepsilon \phi Q(H, S) - \frac{\psi \phi}{n-2} Q(g, H) + \frac{\varepsilon}{n-2} (\phi^2 + \psi + \frac{\varepsilon}{n-1}) Q(g, S), \\
(n-2) R \cdot C &= (n-2) Q(S, R) - \varepsilon \phi g \wedge Q(H, S), \\
(n-2) C \cdot R &= (n-3) Q(S, R) + (\varepsilon \psi + \frac{\varepsilon}{n-1}) Q(g, R) - \varepsilon \phi H \wedge Q(g, S).
\end{align*}$$

Example 3.2. (i) Let $M$ be a $(n-1)$-dimensional hypersurface in $n$-dimensional standard unit sphere $S^n(1)$ in the Euclidean space $E^{n+1}$, $n \geq 4$. Precisely, let $M$ be the Clifford torus $S^p(c_1) \times S^{n-p-1}(c_2)$, $c_1 = r_1^{-1}$, $c_2 = r_2^{-1}$, $r_1 = \sqrt{\frac{n-1}{n-2}}$, $r_2 = \sqrt{\frac{n-p-1}{n-p-3}}$, $1 \leq p \leq n-2$. It is well-known that $M$ is a minimal hypersurface of $S^n(1)$ having at every point exactly two principal curvatures $\rho_1$ and $\rho_2$ of the multiplicity $p$ and $n-p-1$, respectively, satisfying

$$\rho_1 \rho_2 + 1 = 0, \quad \rho_i^2 = r_i^{-2} - 1, \quad i = 1, 2.$$

(38)

(ii) Let $M$ be a $n$-dimensional hypersurface in the Euclidean space $E^{n+1}$, $n \geq 4$. Precisely, let $M$ be the cone over $\overline{M}$. We refer to Section 3 of [45] for precise definition and properties of such hypersurfaces. In particular, from Section 3 of [45] it follows immediately that $M$ has at every point three distinct principal curvatures $\lambda_1 = 0$, $\lambda_2 = \frac{1}{p} \rho_1$ and $\lambda_3 = \frac{1}{p} \rho_2$, $t \in \mathbb{R}^*$, of the multiplicity 1, $p$ and $n-p-1$, respectively. Thus we see that the cone $M$ over the Clifford torus $S^p(c_1) \times S^{n-p-1}(c_2)$ is a hypersurface in $E^{n+1}$, $n \geq 4$, having exactly three distinct principal curvatures satisfying at every point (2). Using (38) we can check that

$$\psi = -\lambda_2 \lambda_3 = t^2$$

and

$$\begin{align*}
\phi^2 &= (\lambda_2 + \lambda_3)^2 = \frac{1}{t^2} (\rho_1 + \rho_2)^2 = \frac{1}{t^2} (\rho_1^2 + \rho_2^2 - 2) = \frac{1}{t^2} \left( \frac{1}{r_1^2} + \frac{1}{r_2^2} - 4 \right) \\
&= \frac{1}{t^2} \left( \frac{(n-1)^2}{p(n-p-1)} - 4 \right) = \frac{(n-p-1)^2 - 4p(n-p-1) + 4p(n-p-1)t^2}{p(n-p-1)t^2} = \frac{(n-2p-1)^2}{p(n-p-1)t^2}.
\end{align*}$$
If \( p \neq n - p - 1 \) then in view of Theorem 3.2 the Riemann–Christoffel curvature tensor \( R \) of the cone \( M \) is expressed at every point by a linear combination of the tensors \( g \wedge g, g \wedge S \) and \( S \wedge S, g \wedge S^2, S \wedge S^2 \) and \( S^2 \wedge S^2 \). We refer to [50] and [52] for further results on hypersurfaces with the curvature tensor having the above presented property.

**Remark 3.2.** (i) Let \( M \) be a hypersurface in \( N^{n+1}_c \), \( n \geq 4 \), and let the condition

\[
H^3 = tr(H)H^2 + \psi H + \rho g,
\]

be satisfied on \( \mathcal{U}_H \subset M \), where \( \psi \) and \( \rho \) are some functions on this set. Using the identity (9), and (3.6) and (3.7) of [23] we get on \( \mathcal{U}_H \)

\[
C \cdot S = \left( \psi + \frac{\kappa}{(n - 2)(n - 1)} - \frac{(2n - 3)\kappa}{n(n + 1)} \right) Q(g, S) + \frac{n - 3}{n - 2} Q(g, S^2). \tag{39}
\]

(ii) (cf., [29], Lemma 4.2) Let \( M \) be a hypersurface in a semi-Euclidean space \( \mathbb{E}^{n+1}_c \), \( n \geq 4 \), satisfying on \( \mathcal{U}_H \subset M \) the relation

\[
H^3 = tr(H)H^2 - \frac{\kappa}{n - 1} H.
\]

Now (39), by (3.9) of [23], and the conditions \( \kappa = 0 \) and \( \psi = -\frac{\kappa}{n - 1} \), reduces on \( \mathcal{U}_H \) to

\[
C \cdot S = 0. \tag{40}
\]

Hypersurfaces satisfying (40) were investigated among others in: [2], [9]–[12], [21], [28]–[30].

(iii) Let \( (M, g) \), \( n \geq 4 \), be a non-Riemannian semi-Riemannian manifolds with parallel Weyl tensor \( (FC = 0) \), which are in addition non-locally symmetric \( (VR \neq 0) \) and non-conformally flat \( (C \neq 0) \). Such manifolds are called essentially conformally symmetric manifolds, e.c.s. manifolds, in short (see, e.g., [14]). Certain e.c.s. metrics are realized on compact manifolds ([15], [16]). As it was stated in [14], e.c.s. manifolds are semisymmetric manifolds \( (R \cdot R = 0) \) satisfying: \( \kappa = 0 \), \( S^2 = 0 \) and \( C(SX_1, X_2, X_3, X_4) = 0 \), for any \( X_1, \ldots, X_4 \in \Xi(M) \). Thus, in view of Lemma 2.3, we see that (40) holds on every e.c.s. manifold.

4. Some special minimal 2-quasi-umbilical hypersurfaces

In this section we consider hypersurfaces \( M \) in a Riemannian space of constant curvature \( N^{n+1}_c \), \( n \geq 4 \), having at every point of \( \mathcal{U}_H \subset M \) exactly three distinct principal curvatures \( \lambda_1, \lambda_2 \) and \( \lambda_3 = \lambda \) such that (3) is satisfied. Thus at every point of \( \mathcal{U}_H \) we have: \( \lambda \neq 0, tr(H) = 0 \) and

\[
\begin{align*}
\text{rank}(H - \lambda g) &= 2, \tag{41} \\
\text{rank}\left(S - \frac{(n - 1)\kappa}{n(n + 1)} \cdot \lambda^2\right)g &= 2. \tag{42}
\end{align*}
\]

The last condition follows immediately from (35). Therefore \( \mathcal{U}_H \) is a minimal, 2-quasi-umbilical and 2-quasi Einstein open submanifold of \( M \). Evidently, (36) reduces to

\[
C \cdot C = 0. \tag{43}
\]

This, together with (7) and (8), yields

\[
\text{conh}(R) \cdot \text{conh}(R) = -\frac{\kappa}{(n - 2)(n - 1)} Q(g, \text{conh}(R)). \tag{44}
\]
Furthermore (1), (12), (13), (16) and (18)-(34) give (4) and
\[ S = -H^2 + \frac{(n-1)K}{n(n+1)}g, \quad \kappa = -tr(H^2) + \frac{(n-1)K}{n+1}, \]  
(45)
\[ \beta_1 = \phi, \quad \beta_2 = \frac{1}{n-2} \left( \phi^2 + \psi + (n-2)\mu \right), \]
\[ \beta_3 = \frac{-\psi\phi}{n-2}, \quad \beta_4 = \left( \frac{(n-1)K}{n(n+1)} - \frac{\psi}{n-2} \right) \phi, \]
(46)
\[ R \cdot S = \frac{K}{n(n+1)} Q(g, S) - \phi Q(H, H^2), \]
(47)
\[ C \cdot S = \phi Q(H, S) + \frac{1}{n-2} \left( \phi^2 + \psi + (n-2)\mu \right) Q(g, S) \]
\[ + \left( \frac{(n-1)K\phi}{n(n+1)} - \frac{1}{n-2} \psi\phi \right) Q(g, H), \]
(48)
\[ (n-2) R \cdot C = (n-2) Q(S, R) + \phi g \wedge Q(H, H^2) \]
\[ - \frac{(n-2)^2 K}{n(n+1)} Q(g, R) - \frac{(n-3)K}{n(n+1)} Q(S, G), \]
(49)
\[ (n-2) C \cdot R = (n-3) Q(S, R) \]
\[ + \left( \frac{\kappa}{n-1} + \psi - \frac{(n^2-3n+3)K}{n(n+1)} \right) Q(g, R) \]
\[ - \frac{(n-3)K}{n(n+1)} Q(S, G) + \phi H \wedge Q(g, H^2), \]
(50)
Next, using (4) and (12), we find
\[ H^2 = -S + \frac{(n-1)K}{n(n+1)}g, \]
(51)
\[ H^4 = (\phi^2 + \psi) H^2 + \phi \psi H, \]
(52)
\[ \phi \psi H = S^2 - \left( \frac{2(n-1)K}{n(n+1)} - \phi^2 - \psi \right) S \]
\[ + \left( \frac{(n-1)K}{n(n+1)} \left( \phi^2 + \psi - \frac{(n-1)K}{n(n+1)} \right) \right) g. \]
(53)
Further, (28) turns into
\[ S^3 = \left( \frac{3(n-1)K}{n(n+1)} - \phi^2 - 2\psi \right) S^2 + \left( \phi \psi \left( \frac{2(n-1)K}{n(n+1)} - \psi \right) - \left( \frac{(n-1)K}{n(n+1)} \right)^2 \right) S \]
\[ + \frac{(n-1)K}{n(n+1)} \left( \frac{(n-1)K}{n(n+1)} \phi^2 + \psi - \frac{(n-1)K}{n(n+1)} \right) - \psi \left( 2\phi^2 + \psi - \frac{(n-1)K}{n(n+1)} \right) g. \]
(54)
We note that by the Gauss equation (11) and (53) we obtain on \( U_M \) the following relation
\[ 2(\phi \psi^2) \left( R - \frac{K}{n(n+1)} G \right) \]
\[ = \left( S^2 - \left( \frac{2(n-1)K}{n(n+1)} - \phi^2 - \psi \right) S \right) \wedge \left( S^2 - \left( \frac{2(n-1)K}{n(n+1)} - \phi^2 - \psi \right) S \right), \]
(55)
It is obvious that if the hypersurface \( M \) in \( N^{n+1}(C) \), \( n \geq 4 \), has at every point exactly three distinct principal curvatures then \( M = U_M \). In this case we also have \( M = U_C = U_C \).

The above presented results lead immediately to the following proposition.
Proposition 4.1. Let $M$ be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having exactly three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ satisfying at every point of $M$: $\lambda_1 = 0$, $\lambda_2 = -(n-2)\lambda$ and $\lambda_3 = \lambda_4 = \ldots = \lambda_n = \lambda \neq 0$. Then $M$ is a minimal, 2-quasi-umbilical and 2-quasi-Einstein hypersurface satisfying (14) and (41)-(55).

From the last proposition, (14) and (17) we immediately get the following.

Proposition 4.2. Let $M$ be a hypersurface in an Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$, having exactly three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ satisfying at every point of $M$: $\lambda_1 = 0$, $\lambda_2 = -(n-2)\lambda$ and $\lambda_3 = \lambda_4 = \ldots = \lambda_n = \lambda \neq 0$. Then $M$ is a minimal, 2-quasi-umbilical and 2-quasi-Einstein hypersurface satisfying (15), (43), (44) and

\[ S = H^2, \quad \kappa = -\text{tr}(H^2) = -(n-2)(n-1)\lambda^2, \]
\[ \phi \psi H = S^2 + (\phi^2 + \psi)S, \]
\[ S^3 = -(\phi^2 + 2\psi)S^2 - \psi^2S, \]
\[ \phi = -(n-3)\lambda, \quad \psi = (n-2)\lambda^2, \quad \mu = \frac{\kappa}{(n-2)(n-1)}, \]
\[ \text{rank} \left( S - \frac{\kappa}{(n-2)(n-1)} g \right) = 2, \]
\[ R = \frac{1}{2(\phi\psi)^2} \left( S^2 + (\phi^2 + \psi)S \right) \wedge \left( S^2 + (\phi^2 + \psi)S \right), \]
\[ R \cdot S = \phi Q(H, S) = \frac{n-1}{\kappa} Q(S, S^2), \]
\[ C \cdot S = \phi Q(H, S) + \frac{\phi^2}{n-2} Q(g, S) - \frac{\phi\psi}{n-2} Q(g, H) \]
\[ = \frac{n-1}{\kappa} Q(S - \frac{\kappa}{(n-2)(n-1)} g, S^2 - \frac{\kappa}{n-1} S), \]
\[ (n-2) R \cdot C = (n-2) Q(S, R) - \phi g \wedge Q(H, S), \]
\[ (n-2) C \cdot R = (n-3) Q(S, R) - \phi H \wedge Q(g, S). \]

Theorem 4.3. Let $M$ be a hypersurface in an Euclidean space $\mathbb{E}^n$ having exactly three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ satisfying at every point of $M$: $\lambda_1 = 0$, $\lambda_2 = -2\lambda$ and $\lambda_3 = \lambda_4 = \lambda \neq 0$. Then $M$ is a minimal, 2-quasi-umbilical and 2-quasi-Einstein hypersurface satisfying: (15), (43), (44) and

\[ \lambda^2 = \frac{-\kappa}{6}, \quad \lambda H = \frac{3}{2} S^2 - \frac{3}{2} S, \quad H^2 = -S, \]
\[ S^3 = \frac{5\kappa}{6} S^2 - \frac{\kappa}{2} S, \quad \text{rank} (S - \frac{\kappa}{6} g) = 2, \]
\[ R = \frac{27}{\kappa^3} \left( S^2 - \frac{\kappa}{2} S \right) \wedge \left( S^2 - \frac{\kappa}{2} S \right), \]
\[ R \cdot S = \frac{3}{\kappa} Q(S, S^2), \]
\[ C \cdot S = \frac{3}{\kappa} Q(S - \frac{\kappa}{6} g, S^2 - \frac{\kappa}{3} S), \]
\[ R \cdot C = Q(S, R) + \frac{3}{2\kappa} g \wedge Q(S^2, S), \]
\[ C \cdot R = \frac{1}{2} Q(S, R) + \frac{3}{2\kappa} S^2 \wedge Q(g, S) - \frac{3}{8} Q(g, S \wedge S). \]
Example 4.1. If $p = 1$ then the hypersurface $M$ defined in Example 3.2 (ii) has at every point three distinct principal curvatures $λ_1 = 0$, $λ_2 = \frac{1}{p_1}$ and $λ_3 = \frac{1}{p_2}$, of multiplicity 1, 1 and $n - 2$, respectively. Further, we set $λ = λ_3 = \frac{1}{p_2} = \frac{1}{\sqrt{n-1}}$. This by (38) yields $λ_2 = -(n - 2)λ$. Thus we see that the cone over the Clifford torus $S^1(c_1) × S^{n-2}(c_2)$, $c_1^2 = r_1 = \sqrt{\frac{1}{n-1}}$, $c_2^2 = r_2 = \sqrt{\frac{2}{n-1}}$, is a hypersurface in $E^{n+1}$, $n ≥ 4$, having exactly three distinct principal curvatures satisfying at every point (3).

Example 4.2. (i) Let $M$ be a surface in $E^{n+1}$, $n ≥ 4$, given by the immersion $f : M → E^{n+1}$ and $BM$ be the tangent bundle of the unit normals to $M$. The hypersurface $M$ defined by the map $Φ_t : BM → E^{n+1}$, $Φ_t(x, ξ) = f(x) + tξ$, $t > 0$, is called the tube of radius $t$ over $M$. If $μ_1$ and $μ_2$ are the principal curvatures of $M$ then the principal curvatures of the tube $M$ are the following ([3]): $λ_1 = \frac{μ_1}{1-tμ_1}$, $λ_2 = \frac{μ_2}{1-tμ_2}$, $λ_3 = λ_4 = \ldots = λ_t = -\frac{1}{t}$. Clearly, (37) holds on $M$ ([13], Example 2).

(ii) In addition, we assume that the principal curvatures $μ_1$ and $μ_2$ of $M$ are constant, and $μ_1 = 0$ and $μ > 0$. Moreover, let $t = \frac{n-2}{nμ^2}$. Now the principal curvatures of $M$ are the following: $λ_1 = 0$, $λ_2 = (n-1)μ$, $λ_3 = (\frac{1-nμ}{n})^\frac{1}{2}$ with multiplicity 1, 1 and $n - 2$, respectively. Finally, if we set $λ = -(\frac{(n-1)μ}{n-2})$ then $λ_2 = -(n-2)λ$, and $λ_3 = λ$. Thus we see that (3) holds at every point of $M$.

References


[38] Y. Fu, Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean 5-space, J. Geom. Phys. 75 (2014), 113–119.


