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Some Inequalities on Submanifolds in Statistical Manifolds of Constant Curvature

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Abstract. In this paper, we study the behaviour of submanifolds in statistical manifolds of constant curvature. We investigate curvature properties of such submanifolds. Some inequalities for submanifolds with any codimension and hypersurfaces of statistical manifolds of constant curvature are also established.

1. Introduction

Statistical manifolds introduced, in 1985, by Amari have been studied in terms of information geometry. Since the geometry of such manifolds includes the notion of dual connections, also called conjugate connections in affine geometry, it is closely related to affine differential geometry. Further, a statistical structure being a generalization of a Hessian one, it connects Hessian geometry.

Let $\left(\tilde{M}, \tilde{g}\right)$ be a Riemannian manifold and M a submanifold of $\tilde{M}$. If (M, ∇, g) is a statistical manifold, then we call (M, ∇, g) a statistical submanifold of $\left(\tilde{M}, \tilde{g}\right)$, where ∇ is an affine connection on M and g is the metric tensor on M induced from the Riemannian metric $\tilde{g}$ on $\tilde{M}$. Let $\tilde{\nabla}$ be an affine connection on $\tilde{M}$. If $\left(\tilde{M}, \tilde{g}, \tilde{\nabla}\right)$ is a statistical manifold and M a submanifold of $\tilde{M}$, then (M, ∇, g) is also a statistical manifold by induced connection ∇ and metric g. In the case that $\left(\tilde{M}, \tilde{g}\right)$ is a semi-Riemannian manifold, the induced metric g has to be non-degenerate. For details, see ([11], [12]).

In the geometry of submanifolds, Gauss formula, Weingarten formula and the equations of Gauss, Codazzi and Ricci are known as fundamental equations. Corresponding fundamental equations on statistical submanifolds were obtained in [12]. A condition for the curvature of a statistical manifold to admit a kind of standard hypersurface was given by H. Furuhata [6] and he introduced a complex version of the notion of statistical structures as well.

On the other hand, B.-Y. Chen [4] established basic inequalities for submanifolds in real space forms, well-known as Chen inequalities. In particular, a sharp relationship between the Ricci curvature and the squared mean curvature for any n-dimensional Riemannian submanifold of a real space form was proved

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in [5], which is known as the Chen-Ricci inequality. Morever, Chen's inequalities for submanifolds of real space forms endowed with a semi-symmetric metric connection were obtained in [9].

In this paper, we obtain some inequalities for submanifolds with any codimension and hypersurfaces of statistical manifolds.

2. Basics on statistical submanifolds

Let $\left(\tilde{M}, \tilde{g}\right)$ be a Riemannian manifold of dimension $(n+k)$ and $\tilde{\nabla}$ an affine connection on $\tilde{M}$. Let us denote the set of sections of a vector bundle $E \longrightarrow \tilde{M}$ by $\Gamma(E)$. Thus, the set of tensor fields of type (p,q) on $\tilde{M}$ is denoted by $\Gamma\left(T\tilde{M}^{(p,q)}\right)$.

Definition 2.1. *[6] Let $\tilde{T} \in \Gamma\left(T\tilde{M}^{(1,2)}\right)$ be the torsion tensor field of $\tilde{\nabla}$. Then a pair $\left(\tilde{\nabla}, \tilde{g}\right)$ is called a statistical structure on $\tilde{M}$ if* (1) $\left(\tilde{\nabla}_X \tilde{g}\right)(Y,Z) - \left(\tilde{\nabla}_Y \tilde{g}\right)(X,Z) = \tilde{g}\left(\tilde{T}(X,Y), Z\right)$ *holds for $X, Y, Z \in \Gamma\left(T\tilde{M}\right)$, and* (2) $\tilde{T} = 0$.

A *statistical manifold* is a Riemannian manifold $\left(\tilde{M}, \tilde{g}\right)$ of dimension $(n+k)$, endowed with a pair of torsion-free affine connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ satisfying

$$Z\tilde{g}(X,Y) = \tilde{g}\left(\tilde{\nabla}_Z X, Y\right) + \tilde{g}\left(X, \tilde{\nabla}^*_Z Y\right) \tag{2.1}$$

for any X, Y and $Z \in \Gamma\left(T\tilde{M}\right)$. It is denoted by $\left(\tilde{M}, \tilde{g}, \tilde{\nabla}\right)$. The connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are called *dual connections*, and it is easily shown that $\left(\tilde{\nabla}^*\right)^* = \tilde{\nabla}$. If $\left(\tilde{\nabla}, \tilde{g}\right)$ is a statistical structure on $\tilde{M}$, then $\left(\tilde{\nabla}^*, \tilde{g}\right)$ is also a statistical structure ([1], [12]).

On the other hand, any torsion-free affine connection $\tilde{\nabla}$ always has a dual connection given by

$$\tilde{\nabla} + \tilde{\nabla}^* = 2\tilde{\nabla}^0, \tag{2.2}$$

where $\tilde{\nabla}^0$ is Levi-Civita connection for $\tilde{M}$.

Denote by $\tilde{R}$ and $\tilde{R}^*$ the curvature tensor fields of $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively.

A statistical structure $\left(\tilde{\nabla}, \tilde{g}\right)$ is said to be of constant curvature $c \in \mathbb{R}$ if

$$\tilde{R}(X,Y)Z = c\left\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y\right\}. \tag{2.3}$$

A statistical structure $\left(\tilde{\nabla}, \tilde{g}\right)$ of constant curvature 0 is called *a Hessian structure.*

The curvature tensor fields $\tilde{R}$ and $\tilde{R}^*$ of dual connections satisfy

$$\tilde{g}\left(\tilde{R}^*(X,Y)Z, W\right) = -\tilde{g}\left(Z, \tilde{R}(X,Y)W\right). \tag{2.4}$$

From (2.4) it follows immediately that if $\left(\tilde{\nabla}, \tilde{g}\right)$ is a statistical structure of constant structure c, then $\left(\tilde{\nabla}^*, \tilde{g}\right)$ is also a statistical structure of constant c. In particular, if $\left(\tilde{\nabla}, \tilde{g}\right)$ is Hessian, so is $\left(\tilde{\nabla}^*, \tilde{g}\right)$ [6].

Let M be an n-dimensional submanifold of $\tilde{M}$. Then, for any $X, Y \in \Gamma(TM)$, according to [12], the corresponding Gauss formulas are:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y), \tag{2.6}$$

where h and $h^*$are symmetric and bilinear, called the imbedding curvature tensor of M in $\tilde{M}$ for $\tilde{\nabla}$ and the imbedding curvature tensor of M in $\tilde{M}$ for $\tilde{\nabla}^*$, respectively.

In [12], it is also proved that (∇, g) and (∇^*, g) are dual statistical structures on M, where g is induced metric on $\Gamma(TM)$ from the Riemannian metric $\tilde{g}$ on $\tilde{M}$.

Let us denote the normal bundle on M by $\Gamma(TM^{\perp})$. Since h and $h^{*}$are bilinear, we have the linear transformations A_{ξ} and A^{*}_{ξ} defined by

$$g(A_{\xi}X,Y)=\tilde{g}(h(X,Y),\xi), \tag{2.7}$$

$$g\left(A^{*}_{\xi}X,Y\right)=\tilde{g}(h^{*}(X,Y),\xi), \tag{2.8}$$

for any $\xi\in\Gamma(TM^{\perp})$ and $X,Y\in\Gamma(TM)$. Further, in [12], the corresponding Weingarten formulas are as follows

$$\tilde{\nabla}_{X}\xi=-A^{*}_{\xi}X+\nabla^{\perp}_{X}\xi, \tag{2.9}$$

$$\tilde{\nabla}^{*}_{X}\xi=-A_{\xi}X+\nabla^{*\perp}_{X}\xi, \tag{2.10}$$

for any $\xi\in\Gamma(TM^{\perp})$ and $X\in\Gamma(TM)$. The connections $\nabla^{\perp}_{X}$ and $\nabla^{*\perp}_{X}$ given by (2.9) and (2.10) are Riemannian dual connections with respect to the induced metric on $\Gamma(TM^{\perp})$.

The corresponding Gauss, Codazzi and Ricci equations are given by the following result.

Proposition 2.2. *[12] Let $\tilde{\nabla}$ be a dual connection on $\tilde{M}$ and ∇ the induced connection on M. Let $\tilde{R}$ and R be the Riemannian curvature tensors of $\tilde{\nabla}$ and ∇, respectively. Then,*

$$\tilde{g}\left(\tilde{R}(X,Y)Z,W\right)=g(R(X,Y)Z,W)+\tilde{g}(h(X,Z),h^{*}(Y,W))-\tilde{g}(h^{*}(X,W),h(Y,Z)), \tag{2.11}$$

$$\left(\tilde{R}(X,Y)Z\right)^{\perp}=\nabla^{\perp}_{X}h(Y,Z)-h(\nabla_{X}Y,Z)-h(Y,\nabla_{X}Z)-\left\{\nabla^{\perp}_{Y}h(Y,Z)-h(\nabla_{Y}X,Z)-h(X,\nabla_{Y}Z)\right\}, \tag{2.12}$$

$$\tilde{g}(R^{\perp}(X,Y)\xi,\eta)=\tilde{g}\left(\tilde{R}(X,Y)\xi,\eta\right)+g\left(\left[A^{*}_{\xi},A_{\eta}\right]X,Y\right), \tag{2.13}$$

where $R^{\perp}$ is the Riemannian curvature tensor on $TM^{\perp}$, $\xi,\eta\in\Gamma(TM^{\perp})$ and $\left[A^{*}_{\xi},A_{\eta}\right]=A^{*}_{\xi}A_{\eta}-A_{\eta}A^{*}_{\xi}$.

For the equations of Gauss, Codazzi and Ricci with respect to the dual connection $\tilde{\nabla}^{*}$ on $\tilde{M}$, we have

Proposition 2.3. *Let $\tilde{\nabla}^{*}$ be a dual connection on $\tilde{M}$ and ∇^{*} the induced connection on M. Let $\tilde{R}^{*}$ and R^{*} be the Riemannian curvature tensors for $\tilde{\nabla}^{*}$ and ∇^{*}, respectively. Then,*

$$\tilde{g}\left(\tilde{R}^{*}(X,Y)Z,W\right)=g(R^{*}(X,Y)Z,W)+\tilde{g}(h^{*}(X,Z),h(Y,W))-\tilde{g}(h(X,W),h^{*}(Y,Z)), \tag{2.14}$$

$$\left(\tilde{R}^{*}(X,Y)Z\right)^{\perp}=\nabla^{*\perp}_{X}h^{*}(Y,Z)-h^{*}\left(\nabla^{*}_{X}Y,Z\right)-h^{*}\left(Y,\nabla^{*}_{X}Z\right)-\left\{\nabla^{*\perp}_{Y}h^{*}(Y,Z)-h^{*}\left(\nabla^{*}_{Y}X,Z\right)-h^{*}\left(X,\nabla^{*}_{Y}Z\right)\right\} \tag{2.15}$$

$$\tilde{g}(R^{*\perp}(X,Y)\xi,\eta)=\tilde{g}\left(\tilde{R}^{*}(X,Y)\xi,\eta\right)+g\left(\left[A_{\xi},A^{*}_{\eta}\right]X,Y\right), \tag{2.16}$$

where $R^{*\perp}$ is Riemannian curvature tensor for $\nabla^{\perp *}$ on $TM^{\perp}$,$\xi,\eta\in\Gamma(TM^{\perp})$ and $\left[A_{\xi},A^{*}_{\eta}\right]=A_{\xi}A^{*}_{\eta}-A^{*}_{\eta}A_{\xi}$.

3. Statistical hypersurfaces

Let $\left(\tilde{M},\tilde{g},\tilde{\nabla}\right)$ be a statistical manifold and $f:M\longrightarrow\tilde{M}$ be an immersion. We define a pair g and ∇ on M by

$$g=f^{*}\tilde{g},\quad g(\nabla_{X}Y,Z)=\tilde{g}\left(\tilde{\nabla}_{X}f_{*}Y,f_{*}Z\right) \tag{3.1}$$

for any $X,Y,Z\in\Gamma(TM)$, where the connection induced from $\tilde{\nabla}$ by f on the induced bundle $f^{*}T\tilde{M}\longrightarrow M$ is denoted by the same symbol $\tilde{\nabla}$. Then the pair (∇,g) is a statistical structure on M, which is called the one induced by f from $\left(\tilde{\nabla},\tilde{g}\right)$ (cf. [6]).

Definition 3.1. *[6] Let (M, g, ∇) and $\left(\tilde{M}, \tilde{g}, \tilde{\nabla}\right)$ be two statistical manifolds. An immersion $f : M \longrightarrow \tilde{M}$ is called a statistical immersion if (∇, g) coincides with the induced statistical structure, i.e., if* (3.1) *holds.*

Let us assume that $f : (M, g, \nabla) \longrightarrow \left(\tilde{M}, \tilde{g}, \tilde{\nabla}\right)$ is a statistical immersion of codimension one and $\xi \in \Gamma\left(f^{*}T\tilde{M}\right)$ is a unit normal vector field of f. Also we denote the dual connection of $\tilde{\nabla}$ with respect to $\tilde{g}$ by $\tilde{\nabla}^{*}$. Thus, from [6], we have the following Gauss and Weingarten formulas:

$$\tilde{\nabla}_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi, \tag{3.2}$$

$$\tilde{\nabla}_X \xi = -f_* A^* X + \tau^*(X)\xi, \tag{3.3}$$

$$\tilde{\nabla}^*_X f_* Y = f_* \nabla^*_X Y + h^*(X, Y)\xi, \tag{3.4}$$

$$\tilde{\nabla}^*_X \xi = -f_* A X + \tau(X)\xi, \tag{3.5}$$

where $h, h^* \in \Gamma\left(TM^{(0,2)}\right)$, $A, A^* \in \Gamma\left(TM^{(1,1)}\right)$ and $\tau, \tau^* \in \Gamma(TM^*)$ satisfy

$$h(X, Y) = g(AX, Y) \quad h^*(X, Y) = g(A^*X, Y), \tag{3.6}$$

$$\tau(X) + \tau^*(X) = 0, \tag{3.7}$$

for any $X, Y \in \Gamma(TM)$.

Denote by $\tilde{R}, \tilde{R}^*, R$ and R^* the curvature tensor fields of the connections $\tilde{\nabla}, \tilde{\nabla}^*, \nabla$ and ∇^*, respectively. Then, for the Gauss equation of a statistical hypersurface, we calculate

$$\begin{aligned}\tilde{R}(X, Y)Z = {} & R(X, Y)Z - h(Y, Z)A^*X + h(X, Z)A^*Y + (\nabla_X h)(Y, Z)\xi \\ & - (\nabla_Y h)(X, Z)\xi + \tau^*(X)h(Y, Z)\xi - \tau^*(Y)h(X, Z)\xi.\end{aligned} \tag{3.8}$$

From (3.8), the normal component of $\tilde{R}(X, Y)Z$ is

$$\left(\tilde{R}(X, Y)Z\right)^{\perp} = (\nabla_X h)(Y, Z)\xi - (\nabla_Y h)(X, Z)\xi + \tau^*(X)h(Y, Z)\xi - \tau^*(Y)h(X, Z)\xi, \tag{3.9}$$

which is known as Codazzi equation. Similarly we get the Ricci equation of a statistical hypersuface as follows

$$\begin{aligned}\tilde{R}(X, Y)\xi = {} & -(\nabla_X A^*)Y + (\nabla_Y A^*)X - \tau^*(Y)A^*X + \tau^*(X)A^*Y \\ & -h(X, A^*Y)\xi + h(A^*X, Y)\xi + d\tau^*(X, Y)\xi.\end{aligned} \tag{3.10}$$

The equations of Gauss, Codazzi and Ricci with respect to the dual connection $\tilde{\nabla}^*$ on $\tilde{M}$ are

$$\begin{aligned}\tilde{R}^*(X, Y)Z = {} & R^*(X, Y)Z - h^*(Y, Z)AX + h^*(X, Z)AY + \left(\nabla^*_X h^*\right)(Y, Z)\xi \\ & -\left(\nabla^*_Y h^*\right)(X, Z)\xi + \tau(X)h^*(Y, Z)\xi - \tau(Y)h^*(X, Z)\xi,\end{aligned} \tag{3.11}$$

$$\left(\tilde{R}^*(X, Y)Z\right)^{\perp} = \left(\nabla^*_X h^*\right)(Y, Z)\xi - \left(\nabla^*_Y h^*\right)(X, Z)\xi + \tau(X)h^*(Y, Z)\xi - \tau(Y)h^*(X, Z)\xi, \tag{3.12}$$

$$\tilde{R}^*(X, Y)\xi = -\left(\nabla^*_X A\right)Y + \left(\nabla^*_Y A\right)X - \tau(Y)AX + \tau(X)AY - h^*(X, AY)\xi + h^*(AX, Y)\xi + d\tau(X, Y)\xi. \tag{3.13}$$

In the case when the ambient space is of constant curvature c, the equations of Gauss, Codazzi and Ricci reduce to

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\} + \{h(Y, Z)A^*X - h(X, Z)A^*Y\}, \tag{3.14}$$

$$(\nabla_X h)(Y, Z) + \tau^*(X)h(Y, Z) = (\nabla_Y h)(X, Z) + \tau^*(Y)h(X, Z), \tag{3.15}$$

$$(\nabla_X A^*)Y - \tau^*(X)A^*Y = (\nabla_Y A^*)X - \tau^*(Y)A^*X, \tag{3.16}$$

$$h(X, A^*Y) - h(A^*X, Y) = d\tau^*(X, Y), \tag{3.17}$$

and the dual ones reduce to

$$R^*(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\} + \{h^*(Y,Z)AX - h^*(X,Z)AY\}, \tag{3.18}$$

$$\left(\nabla^*_X h^*\right)(Y,Z) + \tau(X)h^*(Y,Z) = \left(\nabla^*_Y h^*\right)(X,Z) + \tau(Y)h^*(X,Z), \tag{3.19}$$

$$\left(\nabla^*_X A\right)Y - \tau(X)AY = \left(\nabla^*_Y A\right)X - \tau(Y)AX, \tag{3.20}$$

$$h^*(X,AY) - h^*(AX,Y) = d\tau(X,Y). \tag{3.21}$$

4. General inequalities for statistical submanifolds

Let $\tilde{M}$ be an $(n+k)$-dimensional statistical manifold of constant curvature $c \in \mathbb{R}$, denoted by $\tilde{M}(c)$, and M an n-dimensional statistical submanifold of $\tilde{M}(c)$.

We use the notations

$$R(X,Y,Z,W) = g(R(X,Y)W,Z)$$

and

$$R^*(X,Y,Z,W) = g(R^*(X,Y)W,Z),$$

where R and R^* are the curvature tensor fields of ∇ and ∇^*. We mention that $R(X,Y,Z,W)$ is not skew-symmetric relative to Z and W.

Let $\{e_1,...,e_n\}$ and $\{e_{n+1},...,e_{n+k}\}$ be orthonormal tangent and normal frames, respectively, on M.

The mean curvature vector fields are given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i) = \frac{1}{n}\sum_{\alpha=1}^{k}\left(\sum_{i=1}^{n} h^{\alpha}_{ii}\right)e_{n+\alpha},\ h^{\alpha}_{ij} = \tilde{g}\left(h\left(e_i,e_j\right),e_{n+\alpha}\right), \tag{4.1}$$

and

$$H^* = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i,e_i) = \frac{1}{n}\sum_{\alpha=1}^{k}\left(\sum_{i=1}^{n} h^{*\alpha}_{ii}\right)e_{n+\alpha},\ h^{*\alpha}_{ij} = \tilde{g}\left(h^*\left(e_i,e_j\right),e_{n+\alpha}\right). \tag{4.2}$$

Then we have the following.

Proposition 4.1. *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional statistical manifold $\tilde{M}(c)$ of constant curvature $c \in \mathbb{R}$. Assume that the imbedding curvature tensors h and h^* satisfy*

$$h(X,Y) = g(X,Y)H \text{ and } h^*(X,Y) = g(X,Y)H^*,$$

for any $X,Y \in \Gamma(TM)$. Then M is also a statistical manifold of constant curvature $c + g(H,H^*)$ whenever $g(H,H^*)$ is constant.

Proof. From the Gauss equation given by (2.11), the proof follows directly. □

Definition 4.2. *[10] Let $\tilde{M}$ be an $(n+k)$-dimensional statistical manifold. Then the Ricci tensor $\tilde{S}$ (of type $(0,2)$) is defined by*

$$\tilde{S}(Y,Z) = trace\left\{X \longrightarrow \tilde{R}(X,Y)Z\right\}, \tag{4.3}$$

where $\tilde{R}$ is the curvature tensor field of the affine connection $\tilde{\nabla}$ on $\tilde{M}$.

Thus we have the following result.

Theorem 4.3. *Let $\tilde{M}(c)$ be an $(n+k)$-dimensional statistical manifold of constant curvature $c \in \mathbb{R}$ and M an n-dimensional statistical submanifold of $\tilde{M}(c)$. Also let $\{e_1, ..., e_n\}$ and $\{n_1, ..., n_k\}$ be orthonormal tangent and normal frames, respectively, on M. Then the Ricci tensor S and the dual Ricci tensor S^* of M satisfy*

$$S(X,Y) = c(n-1)g(X,Y) + \sum_{i=1}^{k}\left[g\left(A_{n_i}X,Y\right)trA^*_{n_i} - g\left(A_{n_i}Y, A^*_{n_i}X\right)\right] \tag{4.4}$$

and

$$S^*(X,Y) = c(n-1)g(X,Y) + \sum_{i=1}^{k}\left[g\left(A^*_{n_i}X,Y\right)trA_{n_i} - g\left(A_{n_i}X, A^*_{n_i}Y\right)\right], \tag{4.5}$$

where A_{n_i} and $A^*_{n_i}$ are linear transformations defined by (2.7) *and* (2.8).

Proof. Let us assume that M is an n-dimensional submanifold of $\tilde{M}(c)$. Denote by R the Riemannian curvature tensor of M with respect to ∇. Then we write

$$S(X,Y) = \sum_{j=1}^{n} g\left(R\left(e_j, X\right)Y, e_j\right)$$

and by using the Gauss equation given by (2.11), we have

$$\begin{aligned} S(X,Y) &= \sum_{j=1}^{n} c\left\{g(X,Y)g\left(e_j,e_j\right) - g\left(X,e_j\right)g\left(Y,e_j\right)\right\} \\ &\quad + \tilde{g}\left(h^*\left(e_j,e_j\right), h(X,Y)\right) - \tilde{g}\left(h\left(e_j,Y\right), h^*\left(X,e_j\right)\right) \\ &= c(n-1)g(X,Y) + \sum_{j=1}^{n}\left[\tilde{g}\left(h^*\left(e_j,e_j\right), h(X,Y)\right) - \tilde{g}\left(h\left(e_j,Y\right), h^*\left(X,e_j\right)\right)\right]. \end{aligned} \tag{4.6}$$

On the other hand we get

$$\tilde{g}\left(h^*\left(e_j,e_j\right), h(X,Y)\right) = \sum_{i=1}^{k} g\left(A_{n_i}X,Y\right)g\left(A^*_{n_i}e_j, e_j\right) \tag{4.7}$$

and

$$\tilde{g}\left(h\left(e_j,Y\right), h^*\left(X,e_j\right)\right) = \sum_{i=1}^{k} g\left(A^*_{n_i}X, e_j\right)g\left(A_{n_i}Y, e_j\right). \tag{4.8}$$

By substituting (4.7) and (4.8) into (4.6), we obtain

$$\begin{aligned} S(X,Y) &= c(n-1)g(X,Y) + \sum_{j=1}^{n}\sum_{i=1}^{k}\left[g\left(A_{n_i}X,Y\right)g\left(A^*_{n_i}e_j,e_j\right) - g\left(A_{n_i}X,e_j\right)g\left(A^*_{n_i}Y,e_j\right)\right] \\ &= c(n-1)g(X,Y) + \sum_{i=1}^{k}\left[g\left(A_{n_i}X,Y\right)trA^*_{n_i} - g\left(A_{n_i}X, A^*_{n_i}Y\right)\right], \end{aligned}$$

which gives the equality (4.4). For dual Ricci tensor S^*, similar calculations can be done.

Thus the proof is complete. □

Definition 4.4. *[10] Let ∇ be a torsion-free affine connection on a Riemannian manifold M that admits a parallel volume element ω. If ω is a volume element on M such that $\nabla\omega = 0$, then (∇, ω) is called an equiaffine structure on M.*

Proposition 4.5. *[10] An affine connection ∇ with zero torsion has symmetric Ricci tensor if and only if it is locally equiaffine.*

Thus we have the following result for statistical manifolds having equiaffine connection.

Lemma 4.6. *Let $\tilde{M}(c)$ be an $(n+k)$-dimensional statistical manifold of constant curvature $c \in \mathbb{R}$ and M an n-dimensional submanifold of $\tilde{M}(c)$. Assume that the affine connection ∇ of M is equiaffine. If A_{n_i} and $A^*_{n_i}$ are the linear transformations satisfying* (2.7) *and* (2.8), *then*

$$\sum_{i=1}^{k} \left[A_{n_i}, A^*_{n_i}\right] = 0, \tag{4.9}$$

where $\left[A_{n_i}, A^*_{n_i}\right] = A_{n_i}A^*_{n_i} - A^*_{n_i}A_{n_i}$.

Proof. Denote by S the Ricci tensor of the manifold M. Since M is an equiaffine submanifold of $\tilde{M}(c)$, the Ricci tensor S is symmetric and we have

$$\begin{aligned} 0 = S(X,Y) - S(Y,X) &= -\sum_{i=1}^{k}\left[g\left(A_{n_i}Y, A^*_{n_i}X\right) - g\left(A_{n_i}X, A^*_{n_i}Y\right)\right] \\ &= -\sum_{i=1}^{k} g\left(Y, \left(A_{n_i}A^*_{n_i} - A^*_{n_i}A_{n_i}\right)(X)\right) = -g\left(Y, \sum_{i=1}^{k}\left[A_{n_i}, A^*_{n_i}\right]X\right), \end{aligned}$$

which implies

$$\sum_{i=1}^{k} \left[A_{n_i}, A^*_{n_i}\right] = 0.$$

□

Corollary 4.7. *Let $\tilde{M}(c)$ be an $(n+k)$-dimensional statistical manifold of constant curvature $c \in \mathbb{R}$ and M an n-dimensional equiaffine submanifold M of $\tilde{M}(c)$. Let S and S^*denote the dual Ricci tensors of M. Then we have*

$$(S - S^*)(X,Y) = \sum_{i=1}^{k} g\left(\left(A_{n_i} - A^*_{n_i}\right)X, Y\right) tr\left(A^*_{n_i} - A_{n_i}\right)$$

for the linear transformations A_{n_i} and $A^*_{n_i}$ defined by (2.7) *and* (2.8).

Proof. It is easily seen by using (4.4), (4.5) and (4.9). □

Proposition 4.8. *Let $\tilde{M}(c)$ be an $(n+k)$-dimensional statistical manifold of constant curvature $c \in \mathbb{R}$ and M an n-dimensional statistical submanifold of $\tilde{M}(c)$. Then*

$$2\tau \geq n(n-1)c + n^2\tilde{g}(H, H^*) - \|h\|\,\|h^*\|, \tag{4.10}$$

where τ is the scalar curvature of (M, ∇, g), i.e., $\tau = \sum_{1 \leq i < j \leq n} g(R(e_i, e_j)e_j, e_i)$.

Proof. From (2.11), we have the Gauss equation as follows

$$\begin{aligned} R(X,Y,Z,W) &= c\left[g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\right] \\ &+ \tilde{g}\left(h^*(X,Z), h(Y,W)\right) - \tilde{g}\left(h(X,W), h^*(Y,Z)\right), \end{aligned}$$

where X, Y, Z and $W \in \Gamma(TM)$. Putting $X = Z = e_i$ and $Y = W = e_j$, $i, j = 1, ..., n$, we write

$$R\left(e_i, e_j, e_i, e_j\right) = c\left[g\left(e_i, e_i\right) g\left(e_j, e_j\right) - g\left(e_i, e_j\right)^2\right] + \tilde{g}\left(h^*\left(e_i, e_i\right), h\left(e_j, e_j\right)\right) - \tilde{g}\left(h\left(e_i, e_j\right), h^*\left(e_j, e_i\right)\right). \tag{4.11}$$

We denote by $\|h\|^2 = \sum_{\alpha=1}^{k} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2$ and similarly $\|h^*\|$. By summing over $1 \leq i, j \leq n$, it follows from (4.11) that

$$2\tau = \left(n^2 - n\right) c + n^2 g\left(H, H^*\right) - \sum_{i,j=1}^{n} \sum_{\alpha=1}^{k} h_{ij}^{\alpha} h_{ij}^{*\alpha} \geq n\left(n-1\right) c + n^2 \tilde{g}\left(H, H^*\right) - \|h\| \, \|h^*\|, \tag{4.12}$$

for H and H^* defined by (4.1) and (4.2), which gives (4.10). □

Remark 4.9. *On any statistical submanifold M of $\tilde{M}(c)$ one has $\tau = \tau^*$.*

Let ∇^0 be the Levi-Civita connection of an n-dimensional submanifold M in an $(n + k)$-dimensional statistical manifold $\tilde{M}(c)$ of constant curvature c. Denote by H^0 the mean curvature vector field. Then a sharp relationship between the Ricci curvature and the squared mean curvature obtained by B.-Y. Chen [5] is the following:

$$Ric^0(X) \leq \frac{n^2}{4} \left\|H^0\right\|^2 + (n-1)c, \tag{4.13}$$

which is known as the Chen-Ricci inequality.

From (2.2), we get $2H^0 = H + H^*$ and thus

$$\left\|H^0\right\|^2 = \frac{1}{4}\left(\|H\|^2 + \|H^*\|^2 + 2g\left(H, H^*\right)\right), \tag{4.14}$$

where H and H^* are defined by (4.1) and (4.2). Therefore, from (4.13) and (4.14), we derive

$$Ric^0(X) \leq \frac{n^2}{16} \|H\|^2 + \frac{n^2}{16} \|H^*\|^2 + \frac{n^2}{8} \tilde{g}\left(H, H^*\right) + (n-1)c. \tag{4.15}$$

5. Inequalities for statistical hypersurfaces

By analogy with Proposition 4.8, we have an inequality for statistical hypersurfaces as follows:

Proposition 5.1. *Let M be a statistical hypersurface of an $(n + 1)$-dimensional statistical manifold $\tilde{M}(c)$ of constant curvature $c \in \mathbb{R}$. We have*

$$2\tau \geq n(n-1)c + n^2 \|H\| \, \|H^*\| - \|h\| \, \|h^*\|, \tag{5.1}$$

where τ is the scalar curvature of M.

Proof. Let $\{e_1, ..., e_n\}$ be an orthonormal frame of M and e_{n+1} unit normal vector to M. From (3.14), we get

$$R\left(e_i, e_j, e_i, e_j\right) = c\left[g\left(e_i, e_i\right) g\left(e_j, e_j\right) - g\left(e_i, e_j\right)^2\right] + \tilde{g}\left(h^*\left(e_i, e_i\right), h\left(e_j, e_j\right)\right) - \tilde{g}\left(h\left(e_i, e_j\right), h^*\left(e_j, e_i\right)\right). \tag{5.2}$$

We define the mean curvature vector fields H and H^* by

$$H = \frac{1}{n}\left(\sum_{i=1}^{n} h_{ii}\right) e_{n+1}, \; h_{ij} = \tilde{g}\left(h\left(e_i, e_j\right), e_{n+1}\right)$$

and

$$H^* = \frac{1}{n}\left(\sum_{i=1}^{n} h_{ii}^*\right) e_{n+1}, \; h_{ij}^* = \tilde{g}\left(h^*\left(e_i, e_j\right), e_{n+1}\right).$$

After summing (5.2) over all $i, j = 1, ..., n$, we obtain

$$2\tau = n(n-1)c + n^2 \|H\| \|H^*\| - \sum_{i,j=1}^{n} h_{ij} h^*_{ij}. \tag{5.3}$$

Applying Cauchy-Buniakowski-Schwarz to (5.3), we deduce

$$2\tau \geq n(n-1)c + n^2 \|H\| \|H^*\| - \|h\| \|h^*\|.$$

□

Theorem 5.2. *Let M be a statistical hypersurface of an $(n+1)$-dimensional statistical manifold $\tilde{M}(c)$. For each $X \in T_p(M)$ we have*

$$Ric(X) = (n-1)c + n\tilde{g}(h^*(X,X), H) - \sum_{i=1}^{n} h_{i1} h^*_{i1},$$

and

$$Ric^*(X) = (n-1)c + n\tilde{g}(h(X,X), H^*) - \sum_{i=1}^{n} h_{i1} h^*_{i1}.$$

Proof. Let us choose the orthonormal frame $\{e_1, ..., e_n\}$ such that $X = Z = e_1$ and $Y = W = e_i$, $i = 2, ..., n$. From (3.14), we get

$$R(X, e_i, X, e_i) = c\left(g(X,X)g(e_i,e_i) - g(X,e_i)^2\right) + \tilde{g}(h^*(X,X), h(e_i,e_i)) - \tilde{g}(h(X,e_i), h^*(X,e_i)),$$

and after summing over $2 \leq j \leq n$, we derive

$$Ric(X) = (n-1)c + n\tilde{g}(h^*(X,X), H) - \sum_{i=1}^{n} h_{i1} h^*_{i1}.$$

The proof is similar for Ric^*. □

Example. Recall the example 5.4 from [6]. Let $(\mathbb{H}, \tilde{g})$ be the upper half space of constant curvature -1,

$$\mathbb{H} := \left\{ y = \left(y^1, ..., y^{n+1}\right) \in \mathbb{R}^{n+1} | y^{n+1} > 0 \right\}, \tilde{g} := \left(y^{n+1}\right)^{-2} \sum_{k=1}^{n+1} dy^k dy^k.$$

An affine connection $\tilde{\nabla}$ on $\mathbb{H}$ is given by

$$\tilde{\nabla}_{\frac{\partial}{\partial y^{n+1}}} \frac{\partial}{\partial y^{n+1}} = \left(y^{n+1}\right)^{-1} \frac{\partial}{\partial y^{n+1}},$$

$$\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 2\delta_{ij} \left(y^{n+1}\right)^{-1} \frac{\partial}{\partial y^{n+1}},$$

$$\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^{n+1}} = \tilde{\nabla}_{\frac{\partial}{\partial y^{n+1}}} \frac{\partial}{\partial y^j} = 0,$$

where $i, j = 1, ..., n$. The curvature tensor field $\tilde{R}$ of $\tilde{\nabla}$ is identically zero, i.e., $c = 0$. Thus $\left(\mathbb{H}, \tilde{\nabla}, \tilde{g}\right)$ is a Hessian manifold of constant Hessian curvature 4.

For a constant $y_0 > 0$, we consider the following immersion

$$f_0 : \mathbb{R}^n \longrightarrow \mathbb{H}, f_0\left(y^1, ..., y^n\right) = \left(y^1, ..., y^n, y_0\right).$$

Let (∇, g) be the statistical structure on $\mathbb{R}^n$ induced by f_0 from $\left(\tilde{\nabla}, \tilde{g}\right)$. We then get that (∇, g) is a Hessian structure. In other words, f_0 is a statistical immersion of the trivial Hessian manifold $(\mathbb{R}^n, \nabla, g)$ into the upper half Hessian space $\left(\mathbb{H}, \tilde{\nabla}, \tilde{g}\right)$. It is easy to calculate that

$$\xi = y_0 \frac{\partial}{\partial y^{n+1}}, \quad h = 2g, \quad h^* = 0, \quad \|H^*\| = 0, \tag{5.4}$$

which means that the equality case of (5.1) is satisfied for $(\mathbb{R}^n, \nabla, g)$ and $\left(\mathbb{H}, \tilde{\nabla}, \tilde{g}\right)$.

On the other hand this example can be generalized by using the Lemma 5.3 of [6]. Let $\left(\mathbb{H}, \tilde{\nabla}, \tilde{g}\right)$ be a Hessian manifold of constant Hessian curvature $\tilde{c} \neq 0$, (M, ∇, g) a trivial Hessian manifold and $f : M \longrightarrow \mathbb{H}$ a statistical immmersion of codimension one. Then the following hold:

$$A^* = 0, \quad h^* = 0, \quad \|H^*\| = 0, \tag{5.5}$$

thus the immersion f has codimension one and satisfies the equality case of (5.1).

6. Chen-Ricci inequalities for statistical submanifolds in statistical manifold of constant curvature

Let $\tilde{M}(c)$ be an $(n+k)$-dimensional statistical manifold of constant curvature $c \in \mathbb{R}$ and M an n-dimensional statistical submanifold of $\tilde{M}(c)$. Then the Gauss equation is

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + \tilde{g}(h(X, Z), h^*(Y, W)) - \tilde{g}(h^*(X, W), h(Y, Z)).$$

By setting $X = Z = e_i$ and $Y = W = e_j$, $i, j = 1, ..., n$, and summing over $1 \leq i, j \leq n$, then we have

$$n(n-1)c = 2\tau - n^2 \tilde{g}(H, H^*) + \sum_{i,j=1}^{n} \tilde{g}\left(h^*\left(e_i, e_j\right), h\left(e_i, e_j\right)\right),$$

where H and H^* are the mean curvature vector fields defined by (4.1) and (4.2). From this, we get

$$\begin{aligned} n(n-1)c = 2\tau &- \frac{n^2}{2}\left[\tilde{g}(H + H^*, H + H^*) - \tilde{g}(H, H) - g(H^*, H^*)\right] \\ &+ \frac{1}{2}\sum_{i,j=1}^{n}\left[\tilde{g}\left(h^*\left(e_i, e_j\right) + h\left(e_i, e_j\right), h^*\left(e_i, e_j\right) + h\left(e_i, e_j\right)\right)\right. \\ &\left. - \tilde{g}\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right) - \tilde{g}\left(h^*\left(e_i, e_j\right), h^*\left(e_i, e_j\right)\right)\right]. \end{aligned}$$

From $2H^0 = H + H^*$ it follows that

$$\begin{aligned} n(n-1)c = 2\tau - 2n^2 \tilde{g}\left(H^0, H^0\right) &+ \frac{n^2}{2}\tilde{g}(H, H) + \frac{n^2}{2}\tilde{g}(H^*, H^*) \\ &+ 2\sum_{i,j=1}^{n} \tilde{g}\left(h^0\left(e_i, e_j\right), h^0\left(e_i, e_j\right)\right) - \frac{1}{2}\left(\|h\|^2 + \|h^*\|^2\right). \end{aligned} \tag{6.1}$$

On the other hand we can write

$$\|h\|^2 = \sum_{\alpha=1}^{k}\left\{\left(h_{11}^{\alpha}\right)^2 + \left(h_{22}^{\alpha} + ... + h_{nn}^{\alpha}\right)^2 + 2\sum_{1\leq i<j\leq n}\left(h_{ij}^{\alpha}\right)^2\right\} - \sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n} h_{ii}^{\alpha} h_{jj}^{\alpha}$$

$$= \frac{1}{2}\sum_{\alpha=1}^{k}\left\{\left(h_{11}^{\alpha}+h_{22}^{\alpha}+\ldots+h_{nn}^{\alpha}\right)^{2}+\left(h_{11}^{\alpha}-h_{22}^{\alpha}-\ldots-h_{nn}^{\alpha}\right)^{2}\right\}$$

$$+2\sum_{\alpha=1}^{k}\sum_{1\leq i<j\leq n}\left(h_{ij}^{\alpha}\right)^{2}-\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}h_{ii}^{\alpha}h_{jj}^{\alpha}\geq\frac{1}{2}n^{2}\left\|H\right\|^{2}-\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}[h_{ii}^{\alpha}h_{jj}^{\alpha}-(h_{ij}^{\alpha})^{2}].$$

We similarly derive

$$\|h^{*}\|^{2}\geq\frac{1}{2}n^{2}\left\|H^{*}\right\|^{2}-\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}[h_{ii}^{*\alpha}h_{jj}^{*\alpha}-(h_{ij}^{*\alpha})^{2}]. \tag{6.2}$$

Thus we have the following inequality

$$\|h\|^{2}+\|h^{*}\|^{2}\geq\frac{1}{2}n^{2}\left\|H\right\|^{2}+\frac{1}{2}n^{2}\left\|H^{*}\right\|^{2}-\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}\left(h_{ii}^{\alpha}+h_{ii}^{*\alpha}\right)\left(h_{jj}^{\alpha}+h_{jj}^{*\alpha}\right) \tag{6.3}$$

$$+2\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}h_{ii}^{\alpha}h_{jj}^{*\alpha}+\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}[(h_{ij}^{\alpha})^{2}+(h_{ij}^{*\alpha})^{2}].$$

Substituting (6.3) into (6.1) , we obtain

$$n(n-1)c\leq 2\tau-2n^{2}\tilde{g}(H^{0},H^{0})+\frac{n^{2}}{2}\tilde{g}(H,H)+\frac{n^{2}}{2}\tilde{g}(H^{*},H^{*})+2\|h^{0}\|^{2}+2\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}h_{ii}^{0\alpha}h_{jj}^{0\alpha}$$

$$-\frac{n^{2}}{4}\tilde{g}(H,H)-\frac{n^{2}}{4}\tilde{g}(H^{*},H^{*})-\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}h_{ii}^{\alpha}h_{jj}^{*\alpha}-\frac{1}{2}\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}[((h_{ij}^{\alpha}))^{2}+(h_{ij}^{*\alpha})^{2}].$$

Since

$$\sum_{2\leq i\neq j\leq n}R\left(e_{i},e_{j},e_{i},e_{j}\right)=(n-1)(n-2)c+\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}\left(h_{ii}^{\alpha}h_{jj}^{*\alpha}-h_{ij}^{\alpha}h_{ij}^{*\alpha}\right),$$

the previous inequality becomes

$$n(n-1)c\leq 2\tau-2n^{2}\tilde{g}(H^{0},H^{0})+\frac{n^{2}}{4}\tilde{g}(H,H)+\frac{n^{2}}{4}\tilde{g}(H^{*},H^{*})+2\||h^{0}\|^{2}$$

$$+2\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}h_{ii}^{0\alpha}h_{jj}^{0\alpha}-\sum_{2\leq i\neq j\leq n}R(e_{i},e_{j},e_{i},e_{j})+(n-1)(n-2)c-\frac{1}{2}\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}(h_{ij}^{\alpha}+h_{ij}^{*\alpha})^{2}.$$

Then we get

$$Ric(X)\geq n^{2}\tilde{g}(H^{0},H^{0})-\frac{n^{2}}{8}\tilde{g}(H,H)+\frac{n^{2}}{8}\tilde{g}(H^{*},H^{*})+(n-1)c-\|h^{0}\|^{2}-\sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}[h_{ii}^{0\alpha}h_{jj}^{0\alpha}-(h_{ij}^{0\alpha})^{2}]. \tag{6.4}$$

By the Gauss equation with respect to the Levi-Civita connection, we have

$$\sum_{1\leq i\neq j\leq n}\tilde{R}^{0}(e_{i},e_{j},e_{i},e_{j})=2\tau^{0}-n^{2}\tilde{g}(H^{0},H^{0})+\|h^{0}\|^{2},$$

and, respectively,

$$\sum_{2\leq i\neq j\leq n} \tilde{R}^0(e_i,e_j,e_i,e_j) = \sum_{2\leq i\neq j\leq n} R^0(e_i,e_j,e_i,e_j) - \sum_{\alpha=1}^{k}\sum_{2\leq i\neq j\leq n}\left[h_{ii}^{0\alpha}h_{jj}^{0\alpha} - (h_{ij}^{0\alpha})^2\right].$$

Substituting in (6.4) it follows that

$$Ric(X) \geq 2\tau^0 - \sum_{1\leq i\neq j\leq n} \tilde{R}^0(e_i,e_j,e_i,e_j) - \frac{n^2}{8}\tilde{g}(H,H) - \frac{n^2}{8}\tilde{g}(H^*,H^*)+$$

$$+(n-1)c - \sum_{2\leq i\neq j\leq n} R^0(e_i,e_j,e_i,e_j) + \sum_{2\leq i\neq j\leq n} \tilde{R}^0(e_i,e_j,e_i,e_j).$$

Finally we obtain

$$Ric(X) \geq 2Ric^0(X) - \frac{n^2}{8}\tilde{g}(H,H) - \frac{n^2}{8}\tilde{g}(H^*,H^*) + (n-1)c - 2\sum_{i=2}^{n} \tilde{K}^0(X\wedge e_i).$$

We denote by $\max \tilde{K}^0(X\wedge\cdot)$ the maximum of the sectional curvature function of $\tilde{M}(c)$ with respect to $\tilde{\nabla}$ restricted to 2-plane sections of the tangent space T_pM which are tangent to X.

Summing up, we can state the following.

Theorem 6.1. *Let M be an n-dimensional statistical submanifold of an $(n+k)$-dimensional statistical manifold $\tilde{M}(c)$. For each unit $X \in T_p(M)$, we have*

$$Ric(X) \geq 2Ric^0(X) - \frac{n^2}{8}\tilde{g}(H,H) - \frac{n^2}{8}\tilde{g}(H^*,H^*) + (n-1)c - 2(n-1)\max \tilde{K}^0(X\wedge\cdot).$$

Particular Case: M is a minimal submanifold. Because $H^0 = 0$, we have $H + H^* = 0$. Then the previous inequality implies

Corollary 6.2. *Let M be a minimal n-dimensional statistical submanifold of an $(n+k)$-dimensional statistical manifold $\tilde{M}(c)$. For each unit $X \in T_p(M)$, we have*

$$Ric(X) \geq 2Ric^0(X) + \frac{n^2}{4}\tilde{g}(H,H^*) + (n-1)c - 2(n-1)\max \tilde{K}^0(X\wedge\cdot).$$

Remark 6.3. *Similar inequalities can be stated for the Ricci curvature Ric^*.*

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