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Additive ρ-Functional Inequalities in β-Homogeneous Normed Spaces

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Abstract. In this paper, we solve the following additive ρ-functional inequalities

$$\|f(x+y)-f(x)-f(y)\| \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)+-f(y)\right)\right\|, \tag{1}$$

where ρ is a fixed complex number with $|\rho|<1$, and

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \le \|\rho(f(x+y)-f(x)-f(y))\|, \tag{2}$$

where ρ is a fixed complex number with $|\rho|<\frac{1}{2}$, and prove the Hyers-Ulam stability of the additive ρ-functional inequalities (1) and (2) in β-homogeneous complex Banach spaces and prove the Hyers-Ulam stability of additive ρ-functional equations associated with the additive ρ-functional inequalities (1) and (2) in β-homogeneous complex Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [16] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right)=\frac{1}{2}f(x)+\frac{1}{2}f(y)$ is called the *Jensen equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning these problems (see [1, 3, 5, 17]).

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In [9], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{3}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [4, 7, 14] for functional inequalities. Gilányi [10] and Fechner [6] proved the Hyers-Ulam stability of the functional inequality (3). Park, Cho and Han [12] proved the Hyers-Ulam stability of additive functional inequalities.

Definition 1.1. *Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:*
(FN_1) $\|x\| = 0$ if and only if $x = 0$;
(FN_2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;
(FN_3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;
(FN_4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;
(FN_5) $\|\lambda x_n\| \to 0$ provided $x_n \to 0$.
Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{C}$ (see [15]).

In Section 2, we solve the additive ρ-functional inequality (1) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (1) in β-homogeneous complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive ρ-functional equation associated with the additive ρ-functional inequality (1) in β-homogeneous complex Banach spaces.

In Section 3, we solve the additive ρ-functional inequality (2) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (2) in β-homogeneous complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive ρ-functional equation associated with the additive ρ-functional inequality (2) in β-homogeneous complex Banach spaces.

Throughout this paper, let β_1, β_2 be positive real numbers with $\beta_1 \leq 1$ and $\beta_2 \leq 1$. Assume that X is a β_1-homogeneous real or complex normed space with norm $\|\cdot\|$ and that Y is a β_2-homogeneous complex Banach space with norm $\|\cdot\|$.

2. Additive ρ-Functional Inequality (1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 1$.

In this section, we solve and investigate the additive ρ-functional inequality (1) in β-homogeneous complex Banach spaces.

Lemma 2.1. *A mapping $f : X \to Y$ satisfies*

$$\|f(x + y) - f(x) - f(y)\| \leq \left\| \rho \left(2f\left(\frac{x + y}{2}\right) - f(x) - f(y) \right) \right\| \tag{4}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (4).

Letting $x = y = 0$ in (4), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = x$ in (4), we get

$$\|f(2x) - 2f(x)\| \leq 0$$

and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{5}$$

for all $x \in X$.

It follows from (4) and (5) that

$$\|f(x+y) - f(x) - f(y)\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| = |\rho|^{\beta_2}\|f(x+y) - f(x) - f(y)\|$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

Corollary 2.2. *A mapping $f : X \to Y$ satisfies*

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \tag{6}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

The functional equation (6) is called an *additive ρ-functional equation*.

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (4) in β-homogeneous complex Banach spaces.

Theorem 2.3. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| + \theta(\|x\|^r + \|y\|^r) \tag{7}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^{\beta_1 r} - 2^{\beta_2}}\|x\|^r \tag{8}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (7), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = x$ in (7), we get

$$\|f(2x) - 2f(x)\| \leq 2\theta\|x\|^r \tag{9}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \frac{2}{2^{\beta_1 r}}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \leq \frac{2}{2^{\beta_1 r}}\sum_{j=l}^{m-1}\frac{2^{\beta_2 j}}{2^{\beta_1 r j}}\theta\|x\|^r \tag{10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (10) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (10), we get (8).

It follows from (7) that

$$\begin{aligned}\|A(x+y)-A(x)-A(y)\| &= \lim_{n\to\infty} 2^{\beta_2 n}\left\|f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 2^{\beta_2 n}|\rho|^{\beta_2}\left(\left\|2f\left(\frac{x+y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right\|\right)+\lim_{n\to\infty}\frac{2^{\beta_2 n}\theta}{2^{\beta_1 nr}}(\|x\|^r+\|y\|^r) \\ &= |\rho|^{\beta_2}\left\|2A\left(\frac{x+y}{2}\right)-A(x)-A(y)\right\|\end{aligned}$$

for all $x, y \in X$. So

$$\|A(x+y)-A(x)-A(y)\| \leq \left\|\rho\left(2A\left(\frac{x+y}{2}\right)-A(x)-A(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (8). Then we have

$$\begin{aligned}\|A(x)-T(x)\| &= 2^{\beta_2 n}\left\|A\left(\frac{x}{2^n}\right)-T\left(\frac{x}{2^n}\right)\right\| \\ &\leq 2^{\beta_2 n}\left(\left\|A\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|+\left\|T\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|\right) \\ &\leq \frac{4\cdot 2^{\beta_2 n}}{(2^{\beta_1 r}-2^{\beta_2})2^{\beta_1 nr}}\theta\|x\|^r,\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (8). □

Theorem 2.4. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (7). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \leq \frac{2\theta}{2^{\beta_2}-2^{\beta_1 r}}\|x\|^r \tag{11}$$

for all $x \in X$.

Proof. It follows from (9) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \leq \frac{2\theta}{2^{\beta_2}}\|x\|^r$$

for all $x \in X$. Hence

$$\left\|\frac{1}{2^l}f(2^l x)-\frac{1}{2^m}f(2^m x)\right\| \leq \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x)-\frac{1}{2^{j+1}}f(2^{j+1}x)\right\| \leq \sum_{j=l}^{m-1}\frac{2^{\beta_1 rj}}{2^{\beta_2 j}}\frac{2\theta}{2^{\beta_2}}\|x\|^r \tag{12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (12) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (12), we get (11).

The rest of the proof is similar to the proof of Theorem 2.3. □

By the triangle inequality, we have

$$\begin{aligned}&\left\|f(x+y)-f(x)-f(y)\right\|-\left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\|\\ &\leq\left\|f(x+y)-f(x)-f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\|.\end{aligned}$$

As corollaries of Theorems 2.3 and 2.4, we obtain the Hyers-Ulam stability results for the additive ρ-functional equation (6) in β-homogeneous complex Banach spaces.

Corollary 2.5. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\|f(x+y)-f(x)-f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\|\leq\theta(\|x\|^r+\|y\|^r) \tag{13}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (8).

Corollary 2.6. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (13). *Then there exists a unique additive mapping $A : X \to Y$ satisfying* (11).

Remark 2.7. *If ρ is a real number such that $-1 < \rho < 1$ and Y is a β_2-homogeneous real Banach space, then all the assertions in this section remain valid.*

3. Additive ρ-Functional Inequality (2)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < \frac{1}{2}$.

In this section, we solve and investigate the additive ρ-functional inequality (2) in β-homogeneous complex Banach spaces.

Lemma 3.1. *A mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|\leq\|\rho(f(x+y)-f(x)-f(y))\| \tag{14}$$

for all $x, y \in X$ if and onlt if $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (14).

Letting $y = 0$ in (14), we get $\left\|2f\left(\frac{x}{2}\right)-f(x)\right\| \leq 0$and so

$$2f\left(\frac{x}{2}\right)=f(x) \tag{15}$$

for all $x \in X$.

It follows from (14) and (15) that

$$\|f(x+y)-f(x)-f(y)\|=\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|\leq|\rho|^{\beta_2}\|f(x+y)-f(x)-f(y)\|$$

and so

$$f(x+y)=f(x)+2f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

Corollary 3.2. *A mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right)-f(x)-f(y)=\rho\left(f(x+y)-f(x)-f(y)\right) \tag{16}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

The functional equation (16) is called an *additive ρ-functional equation*.

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (14) in β-homogeneous complex Banach spaces.

Theorem 3.3. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \left\|\rho(f(x+y) - f(x) - f(y))\right\| + \theta(\|x\|^r + \|y\|^r) \tag{17}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^{\beta_1 r}\theta}{2^{\beta_1 r} - 2^{\beta_2}}\|x\|^r \tag{18}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (17), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \tag{19}$$

for all $x \in X$. So

$$\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \leq \sum_{j=l}^{m-1}\frac{2^{\beta_2 j}}{2^{\beta_1 r j}}\theta\|x\|^r \tag{20}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (20) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (20), we get (18).

It follows from (17) that

$$\begin{aligned}
\left\|2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right\| &= \lim_{n\to\infty} 2^{\beta_2 n}\left(\left\|2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\|\right) \\
&\leq \lim_{n\to\infty} 2^{\beta_2 n}\left\|\rho\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| + \lim_{n\to\infty}\frac{2^{\beta_2 n}\theta}{2^{\beta_1 n r}}(\|x\|^r + \|y\|^r) \\
&= \|\rho(A(x+y) - A(x) - A(y))\|
\end{aligned}$$

for all $x, y \in X$. So

$$\left\|2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right\| \leq \|\rho(A(x+y) - A(x) - A(y))\|$$

for all $x, y \in X$. By Lemma 3.1, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (18). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= 2^{\beta_2 n}\left\|A\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\| \\
&\leq 2^{\beta_2 n}\left(\left\|A\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right) \\
&\leq \frac{2 \cdot 2^{\beta_2 n} \cdot 2^{\beta_1 r}}{(2^{\beta_1 r} - 2^{\beta_2})2^{\beta_1 r n}}\theta\|x\|^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (18). $\square$

Theorem 3.4. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (17)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^{\beta_1 r}\theta}{2^{\beta_2} - 2^{\beta_1 r}}\|x\|^r \tag{21}$$

for all $x \in X$.

Proof. It follows from (19) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{2^{\beta_1 r}\theta}{2^{\beta_2}}\|x\|^r$$

for all $x \in X$. Hence

$$\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| \le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\right\| \le \frac{2^{\beta_1 r}\theta}{2^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 rj}}{2^{\beta_2 j}}\|x\|^r \tag{22}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (22) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (22), we get (21).

The rest of the proof is similar to the proof of Theorem 3.3. □

By the triangle inequality, we have

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| - \left\|\rho\left(f(x+y) - f(x) - f(y)\right)\right\| \\ &\le \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right)\right\|.\end{aligned}$$

As corollaries of Theorems 3.3 and 3.4, we obtain the Hyers-Ulam stability results for the additive ρ-functional equation (16) in β-homogeneous complex Banach spaces.

Corollary 3.5. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right)\right\| \le \theta(\|x\|^r + \|y\|^r) \tag{23}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (18)*.*

Corollary 3.6. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (23)*. Then there exists a unique additive mapping $A : X \to Y$ satisfying* (21)*.*

Remark 3.7. *If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a β_2-homogeneous real Banach space, then all the assertions in this section remain valid.*

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