Filomat 29:3 (2015), 443–456
DOI 10.2298/FIL1503443G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Some Classes of Generalized Quasi Einstein Manifolds

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Abstract. In the present paper, we investigate generalized quasi Einstein manifolds satisfying some special curvature conditions $R \cdot S = 0$, $R \cdot S = L_S Q(g, S)$, $C \cdot S = 0$, $\tilde{C} \cdot S = 0$, $\tilde{W} \cdot S = 0$ and $W_2 \cdot S = 0$ where $R, S, C, \tilde{C}, \tilde{W}$ and W_2 respectively denote the Riemannian curvature tensor, Ricci tensor, conformal curvature tensor, concircular curvature tensor, quasi conformal curvature tensor and W_2-curvature tensor. Later, we find some sufficient conditions for a generalized quasi Einstein manifold to be a quasi Einstein manifold and we show the existence of a nearly quasi Einstein manifolds, by constructing a non trivial example.

1. Introduction

A Riemannian manifold (M^n, g), $(n \geqslant 2)$ is said to be an Einstein manifold if the Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$S(X, Y) = \frac{r}{n} g(X, Y) \tag{1}$$

where r is the scalar curvature of (M^n, g).

In 2000, M.C. Chaki and R.K. Maity introduced the notion of quasi-Einstein manifolds as generalization of the Einstein manifolds. According to them, a Riemannian manifold (M^n, g) $(n > 2)$ is said to be a quasi Einstein manifold [1] if its Ricci tensor of type $(0, 2)$ is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{2}$$

where a and b are real valued, non-zero scalar functions on (M^n, g), $X, Y \in \mathfrak{X}(M^n)$ and A is a non-zero 1-form, equivalent to the unit vector field U, that is,

$$g(X, U) = A(X)\ , \ g(U, U) = 1 \tag{3}$$

A is called an associated 1-form and U is called a generator of (M^n, g). If $b = 0$, then the manifold reduces to an Einstein manifold.

The notion of generalized quasi Einstein manifold has been first introduced by M.C. Chaki in 2001 [2]. A Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized quasi Einstein manifold if its Ricci tensor of type $(0, 2)$ is non-zero and satisfies the following condition [2]

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \tag{4}$$

2010 *Mathematics Subject Classification*. 53C15, 53C25, 53B15, 53B20

Keywords. Generalized quasi Einstein manifold, Ricci semi symmetric manifold, Ricci-pseudosymmetric manifold, conformal curvature tensor, concircular curvature tensor, quasi conformal curvature tensor,W_2-curvature tensor

Received: 18 August 2014; Accepted: 29 September 2014

Communicated by Ljubica Velimirović and Mića Stanković

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where $a,\ b,\ c$ are real valued, non-zero scalar functions on (M^n, g) of which $b \neq 0$, $c \neq 0$, A and B are two non-zero 1-forms such that

$$g(X,U) = A(X)\ \ ,\ g(X,V) = B(X)\ \ ,\ g(U,V) = 0\ \ g(U,U) = g(V,V) = 1 \tag{5}$$

That is, U and V are orthonormal vector fields corresponding to the 1-forms A and B, respectively. Similarly, a, b and c are called associated scalars, A and B are called associated 1-forms and U and V are generators of manifold. Such an n-dimensional manifold has been denoted by $G(QE)_n$. If $c = 0$, then the manifold reduces to a quasi Einstein manifolds and if $b = c = 0$, then the manifold reduces to an Einstein manifold. Also, the operator Q defined by $g(QX, Y) = S(X, Y)$ is called the Ricci operator.

Contracting (4) over X and Y, we get the scalar curvature function of the following form

$$r = an + b \tag{6}$$

In view of the equations (4) and (5), in a generalized quasi Einstein manifold, we have

$$S(Y,U) = (a+b)A(Y) + cB(Y)\ \ \textit{and}\ \ S(Y,V) = aB(Y) + cA(Y) \tag{7}$$

A non flat n-dimensional $(n > 2)$ Riemannian manifold is called nearly quasi Einstein manifold if its Ricci tensor $S(X,Y)$ of type $(0,2)$ is of the form

$$S(X,Y) = ag(X,Y) + bE(X,Y) \tag{8}$$

where a, b are non zero scalar functions and E is a non zero tensor of type $(0,2)$, [3].

Remark 1.1. *Since the multiplication of two covariant vectors is a covariant tensor of type* $(0,2)$, *every quasi Einstein manifold is a nearly quasi Einstein manifold. But converse is not true.*

Let R denote the Riemannian curvature tensor of M. The k-nullity distribution $N(k)$ [4] of a Riemannian manifold M is defined by

$$N(k) : p \longrightarrow N_p(k) = \left\{Z \in T_p(M) : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\right\} \tag{9}$$

for all $X, Y \in TM$, where k is some smooth function. In a quasi Einstein manifold M, if the generator U belongs to some k-nullity distribution $N(k)$, then M said to be an $N(k)$-quasi Einstein manifold [5]. Özgür and Triphati [6] proved that in an n-dimensional $N(k)$-quasi Einstein manifold, $k = \frac{a+b}{n-1}$.

2. Ricci-pseudosymmetric $G(QE)_n$

An n-dimensional Riemannian manifold (M^n, g) is called Ricci-pseudosymmetric [7], if the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent where

$$(R(X,Y) \cdot S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W) \tag{10}$$

$$Q(g,S)(Z,W;X,Y) = -S((X \wedge_g Y)Z,W) - S(Z,(X \wedge_g Y)W) \tag{11}$$

and

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y \tag{12}$$

for all X, Y, Z, W on M. Then (M^n, g) is Ricci-pseudosymmetric if and only if

$$(R(X,Y) \cdot S)(Z,W) = L_S Q(g,S)(Z,W;X,Y) \tag{13}$$

holds on U_S where $U_S = \{x \in M : S \neq \frac{r}{n} g \; at \; x\}$ and L_S is a certain function on U_S. Then, by using (10)-(13), we can write (M^n, g) is Ricci-pseudosymmetric if and only if the equation

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = L_S[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(Z,X) - g(X,W)S(Y,Z)] \tag{14}$$

holds.

In this section, we consider Ricci-pseudosymmetric generalized quasi Einstein manifold. Then, by using (4) and (14), we obtain

$$\begin{aligned} & b[A(R(X,Y)Z)A(W) + A(Z)A(R(X,Y)W)] + c[A(R(X,Y)Z)B(W) \\ & \quad + A(W)B(R(X,Y)Z) + A(Z)B(R(X,Y)W) + A(R(X,Y)W)B(Z)] \\ = {} & L_S\Big[b\Big\{g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) + g(Y,W)A(Z)A(X) - g(X,W)A(Y)A(Z)\Big\} \\ & +c\Big\{g(Y,Z)[A(X)B(W) + A(W)B(X)] - g(X,Z)[A(Y)B(W) + A(W)B(Y)] \\ & + g(Y,W)[A(Z)B(X) + A(X)B(Z)] - g(X,W)[A(Y)B(Z) + A(Z)B(Y)]\Big\}\Big] \end{aligned} \tag{15}$$

Putting $Z = U$ and $W = V$ in (15), we get

$$bA(R(X,Y)V) = L_S\{b[A(X)B(Y) - A(Y)B(X)]\} \tag{16}$$

Since $A(R(X,Y,V)) = g(R(X,Y)V,U) = R(X,Y,V,U)$ and $b \neq 0$, we get

$$R(X,Y,U,V) = L_S[A(Y)B(X) - A(X)B(Y)] \tag{17}$$

Thus we obtain the following result:

Theorem 2.1. *In a Ricci-pseudosymmetric generalized quasi Einstein manifold, the curvature tensor R of the manifold satisfies the relation* (17).

Now, contracting (15) over X and W, we obtain

$$\begin{aligned} & b[A(R(U,Y)Z) - A(Z)S(Y,U)] + c[A(R(V,Y)Z) + B(R(U,Y)Z) - A(Z)S(Y,V) - B(Z)S(Y,U)] \\ & \qquad = L_S\Big\{b[g(Y,Z) - nA(Y)A(Z)] - cn[A(Y)B(Z) + A(Z)B(Y)]\Big\} \end{aligned} \tag{18}$$

Putting $Z = U$ in (18), we get

$$bS(Y,U) - cR(U,Y,U,V) + cS(Y,V) = L_S[b(n-1)A(Y) + cnB(Y)] \tag{19}$$

In view of (7) and (17), (19) yields

$$[ab + b^2 + c^2 - b(n-1)L_S]A(Y) + [bc + ac + c(1-n)L_S]B(Y) = 0 \tag{20}$$

Putting $Y = U$ in (20), we get

$$L_S = \frac{ab + b^2 + c^2}{b(n-1)} \tag{21}$$

Putting $Y = V$ in (20), we get

$$c[a + b - L_S(n-1)] = 0 \tag{22}$$

Thus we have

$$c = 0 \;\; or \;\; L_S = \frac{a+b}{n-1} \tag{23}$$

If $c = 0$, then by (21), we obtain $L_S = \frac{a+b}{n-1}$. If $c \neq 0$, then by (22), again we find $L_S = \frac{a+b}{n-1}$. Comparing this and the equation (21), we obtain $c = 0$.

Thus in each case we have $c = 0$ and $L_S = \frac{a+b}{n-1}$ which means that the manifold reduces to a quasi Einstein manifold. Also, from (17), we have

$$R(X,Y)U = \frac{a+b}{n-1}[A(Y)X - A(X)Y] \tag{24}$$

which means that the generator U belongs to some k-nullity distribution. Hence we can state the following theorem:

Theorem 2.2. *Every Ricci-pseudosymmetric non-Einstein generalized quasi Einstein manifold is an $N(k)$-quasi Einstein manifold with $k = \frac{a+b}{n-1}$.*

Remark 2.3. *Every Ricci semi symmetric manifold is Ricci-pseudosymmetric. But the converse is not true.*

Using above theorem and remark, we can state that:

Theorem 2.4. *Every Ricci-pseudosymmetric generalized quasi Einstein manifold is Ricci semi symmetric if and only if $a + b = 0$.*

3. $G(QE)_n$ with the condition $R \cdot S = 0$

In this section, we consider generalized quasi Einstein manifold satisfying the condition $R \cdot S = 0$. Then

$$(R(X,Y) \cdot S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W) = 0 \tag{25}$$

In view of (4), (25) yields

$$\begin{aligned} &b[A(R(X,Y)Z)A(W) + A(Z)A(R(X,Y)W)] + c[A(R(X,Y)Z)B(W) + A(W)B(R(X,Y)Z) \\ &+ A(Z)B(R(X,Y)W) + A(R(X,Y)W)B(Z)] = 0 \end{aligned} \tag{26}$$

Putting $Z = U$, $W = V$ in (26), we get $bR(X,Y,U,V) = 0$. Since $b \neq 0$, we have

$$R(X,Y,U,V) = 0 \tag{27}$$

Putting $W = U$ in (26) and using (27), we get

$$bR(X,Y,Z,U) + cR(X,Y,Z,V) = 0 \tag{28}$$

Contracting (28) over Y and Z, we get

$$bS(X,U) + cS(X,V) = 0 \tag{29}$$

In view of (7), (29) yields

$$abA(X) + b^2A(X) + bcB(X) + acB(X) + c^2A(X) = 0 \tag{30}$$

Putting $X = U$ in (30), we get

$$ab + b^2 + c^2 = 0 \tag{31}$$

In view of (31), (30) yields

$$c(a+b)B(X) = 0 \tag{32}$$

Putting $X = V$ in (32), we obtain $c(a+b) = 0$. Then either $c = 0$ or $a+b = 0$. If $c = 0$, then by (31), $b(a+b) = 0$ so either $b = 0$ or $a+b = 0$. If $b = 0$, then we also have $c = 0$ and so the manifold reduces to an Einstein manifolds, which is a contradiction. Thus b is always different than zero and so $a+b = 0$. On the other hand, if $c \neq 0$, then $a+b = 0$ and by (31), again we obtain $c = 0$. That is, in each case, $c = 0$ and $a+b = 0$. Thus the Ricci tensor becomes

$$S(X,Y) = a[g(X,Y) - A(X)A(Y)] \tag{33}$$

Hence we can state that:

Theorem 3.1. *Every generalized quasi Einstein manifold satisfying the condition $R \cdot S = 0$ is a quasi Einstein manifold and the sum of the associated scalar functions is zero.*

From Remark 1.1, we can state that:

Corollary 3.2. *Every generalized quasi Einstein manifold satisfying the condition $R \cdot S = 0$ is a nearly quasi Einstein manifold.*

4. An Example of Nearly Quasi Einstein Manifolds

In this section, we show the existence of a nearly quasi Einstein manifolds with non-zero and non-constant scalar curvature, by constructing a non trivial example. Since the multiplication of two 1-forms is a covariant tensor of type $(0,2)$ so every generalized quasi Einstein manifold can be considered as a nearly quasi Einstein manifold.

Let us consider a Riemannian metric g on the 4-dimensional real number space $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{4/3}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2 \tag{34}$$

where $i, j = 1,2,3,4$ and x^1, x^2, x^3, x^4 are the standard coordinates of $\mathbb{R}^4$. Then the only non vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3(x^4)} \ , \quad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3(x^4)^{1/3}} \tag{35}$$

$$R_{1221} = R_{1331} = R_{2332} = \frac{4(x^4)^{2/3}}{9} \tag{36}$$

$$R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{2/3}} \tag{37}$$

$$R_{11} = R_{22} = R_{33} = \frac{2}{3(x^4)^{2/3}} \ , \quad R_{44} = -\frac{2}{3(x^4)^2} \tag{38}$$

and the components which can be obtained from these by symmetry properties. Also it can be shown that the scalar curvature is

$$r = \frac{4}{3(x^4)^2} \tag{39}$$

is non zero and non constant.

Let us now define associated scalar functions as

$$a = \frac{2}{3(x^4)^2} \ , \quad b = -\frac{4}{3(x^4)^2} \tag{40}$$

and associated tensor E of type $(0,2)$

$$E_{ij}(x) = \begin{cases} 1 & \text{if } i = j = 4 \\ 0 & \text{if } i = j = 1,2,3 \text{ or } i \neq j \end{cases} \tag{41}$$

Then we can easily show that for all $i, j = 1,2,3,4$

$$R_{ij} = ag_{ij} + bE_{ij} \tag{42}$$

Thus the manifold $\mathbb{R}^4$ endowed with the above metric is a nearly quasi Einstein manifold.

5. $G(QE)_n$ with the condition $P \cdot S = 0$

The projective curvature tensor P [8] of type $(1,3)$ of an n-dimensional Riemannian manifold (M^n, g); $(n > 3)$ is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] \tag{43}$$

Note that; the projective curvature tensor P satisfies the following symmetry properties:

- $P(X,Y,Z,W) = -P(Y,X,Z,W)$
- $P(X,Y,Z,W) \neq -P(X,Y,W,Z)$

for all $X, Y, Z, W \in TM$, where $P(X,Y,Z,W) = g(P(X,Y)Z, W)$ is the projective curvature tensor of type $(0,4)$.

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leqslant i \leqslant n$. Now, from (43), we have

$$\sum_{i=1}^{n} P(e_i, Y, e_i, U) = -S(Y,U) + \frac{1}{n-1}[rA(Y) - S(Y,U)] \tag{44}$$

In this section, we consider a generalized quasi Einstein manifold satisfying the condition $P \cdot S = 0$. Then for all $X, Y, Z \in \mathfrak{X}(M^n)$;

$$(P(X,Y) \cdot S)(Z,W) = -S(P(X,Y)Z, W) - S(Z, P(X,Y)W) = 0 \tag{45}$$

Combining (4) and (45), we get

$$\begin{aligned} &\frac{-a}{n-1}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + S(Y,W)g(X,Z) - S(X,W)g(Y,Z)] \\ &+ b[A(P(X,Y)Z)A(W) + A(Z)A(P(X,Y)W)] + c[A(P(X,Y)Z)B(W) \\ &+ A(W)B(P(X,Y)Z) + A(Z)B(P(X,Y)W) + A(P(X,Y)W)B(Z)] = 0 \end{aligned} \tag{46}$$

Putting $Z = U$ and $W = V$ in (46), we get

$$\begin{aligned} &\frac{-a}{n-1}[S(Y,U)B(X) - S(X,U)B(Y) + S(Y,V)A(X) - S(X,V)A(Y)] + bP(X,Y,V,U) \\ &+ c[P(X,Y,U,U) + P(X,Y,V,V)] = 0 \end{aligned} \tag{47}$$

In view of (7) and (43), (47) yields

$$bR(X,Y,U,V) = 0 \tag{48}$$

Since $b \neq 0$, we obtain $R(X, Y, U, V) = 0$.
Contracting (46) over X and W and using the equation (44), we get

$$\begin{aligned}&\frac{-a}{n-1}[nS(Y,Z) - rg(Y,Z)] + b[P(U,Y,Z,U) + A(Z)\{-S(Y,U) + \frac{1}{n-1}[rA(Y) - S(Y,U)]\}]\\&+ c\Big[P(V,Y,Z,U) + P(U,Y,Z,V) + A(Z)\{-S(Y,V) + \frac{1}{n-1}[rB(Y) - S(Y,V)]\}\\&+ B(Z)\{-S(Y,U) + \frac{1}{n-1}[rA(Y) - S(Y,U)]\}\Big] = 0\end{aligned} \tag{49}$$

Putting $Z = U$ in (49), we get

$$\begin{aligned}&\frac{-a}{n-1}[nS(Y,U) - rA(Y)] + b[P(U,Y,U,U) + A(Z) - S(Y,U) + \frac{1}{n-1}[rA(Y) - S(Y,U)]]\\&+ c\Big[P(V,Y,U,U) + P(U,Y,U,V) + A(Z) - S(Y,V) + \frac{1}{n-1}[rB(Y) - S(Y,V)]\Big] = 0\end{aligned} \tag{50}$$

In view of (7) and (43), (50) yields

$$[-b(a+b) - c^2]A(Y) - c(a+b)B(Y) = 0 \tag{51}$$

Putting $Y = U$ in (51), we get:

$$b(a+b) + c^2 = 0 \tag{52}$$

Putting $Y = U$ in (51), we get:

$$c(a+b) = 0 \tag{53}$$

so $c = 0$ or $a + b = 0$. If $c = 0$, then by (52), we obtain $b = 0$ or $a + b = 0$. If $b = 0$, then we have $b = c = 0$ which means that the manifold reduces to an Einstein manifold. But this is a contradiction. Thus b is always different that zero. Hence $a + b = 0$.

On the other hand, if $a + b = 0$, then by (52), again we obtain $c = 0$. Hence in each case, $a + b = 0$ and $c = 0$ which means that the manifold becomes a quasi Einstein manifold.

Thus we can state the following theorems:

Theorem 5.1. *Every generalized quasi Einstein manifold satisfying the condition $P \cdot S = 0$ is a quasi Einstein manifold and sum of the associated scalar functions a and b is zero.*

And also we obtain:

Theorem 5.2. *In a generalized quasi Einstein manifold satisfying the condition $P \cdot S = 0$, for all $X, Y, Z, W \in TM$*

$$P(X, Y, U, V) = 0$$

6. $G(QE)_n$ with the condition $C \cdot S = 0$

The conformal curvature tensor C [8] of type $(1, 3)$ of an n-dimensional Riemannian manifold (M^n, g); $(n > 3)$ is defined by

$$\begin{aligned}C(X,Y)Z = R(X,Y)Z &- \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]\\&+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]\end{aligned} \tag{54}$$

In this section, we consider a generalized quasi Einstein manifold satisfying the condition $C \cdot S = 0$. Then for all $X, Y, Z \in \mathfrak{X}(M^n)$;

$$(C(X,Y) \cdot S)(Z,W) = -S(C(X,Y)Z,W) - S(Z,C(X,Y)W) = 0 \tag{55}$$

Combining (4) and (55), we get

$$\begin{aligned} & b[A(C(X,Y)Z)A(W) + A(Z)A(C(X,Y)W)] + c[A(C(X,Y)Z)B(W) \\ & + A(W)B(C(X,Y)Z) + A(Z)B(C(X,Y)W) + A(C(X,Y)W)B(Z)] = 0 \end{aligned} \tag{56}$$

Putting $Z = U$ and $W = V$ in (56), we get

$$bA(C(X,Y)V) = 0 \tag{57}$$

Since $A(C(X,Y)V) = g(C(X,Y)V,U) = C(X,Y,V,U)$ and $b \neq 0$,from (57) we obtain

$$C(X,Y,U,V) = 0 \tag{58}$$

In view of (54), (58) yields

$$\begin{aligned} R(X,Y,U,V) = & \frac{1}{n-2}[S(Y,U)g(X,V) - S(X,U)g(Y,V) + g(Y,U)S(X,V) - g(X,U)S(Y,V)] \\ & - \frac{r}{(n-1)(n-2)}[g(Y,U)g(X,V) - g(X,U)g(Y,V)] \end{aligned} \tag{59}$$

In view of (7), (59) yields

$$R(X,Y,U,V) = \frac{a+b}{n-1}[A(Y)B(X) - A(X)B(Y)] \tag{60}$$

Hence we can state the following:

Theorem 6.1. *In a generalized quasi Einstein manifold satisfying the condition $C \cdot S = 0$, the curvature tensor R of the manifold satisfies the relation* (60).

From (60), we have

$$R(X,Y)U = \frac{a+b}{n-1}[A(Y)X - A(X)Y] \tag{61}$$

Contracting (61) over X, we get

$$S(Y,U) = (a+b)g(Y,U) \tag{62}$$

i.e.; $QY = (a+b)Y$, for all $Y \in TM$. Thus we get the following result:

Theorem 6.2. *In a generalized quasi Einstein manifold satisfying the condition $C \cdot S = 0$, $(a+b)$ is an eigenvalue of the Ricci operator Q.*

7. $G(QE)_n$ with the condition $\tilde{C} \cdot S = 0$

The concircular curvature tensor $\tilde{C}$ [8] of type $(1,3)$ of an n-dimensional Riemannian manifold (M^n, g), $(n > 3)$ is defined by

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{63}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Note that; the concircular curvature tensor $\tilde{C}$ satisfies the following symmetry properties:

- $\tilde{C}(X,Y,Z,W) = -\tilde{C}(Y,X,Z,W) = -\tilde{C}(X,Y,W,Z)$

for all $X,Y,Z,W \in TM$, where $\tilde{C}(X,Y,Z,W) = g(\tilde{C}(X,Y)Z,W)$ is the concircular curvature tensor of type $(0,4)$.

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leqslant i \leqslant n$. Now, from (63), we have

$$\sum_{i=1}^{n} \tilde{C}(e_i,Y,e_i,U) = -S(Y,U) + \frac{r}{n}A(Y) \tag{64}$$

In this section, we consider generalized quasi Einstein manifold satisfying the condition $\tilde{C}\cdot S = 0$. Then we have

$$(\tilde{C}\cdot S)(Z,W) = -S(\tilde{C}(X,Y)Z,W) - S(Z,\tilde{C}(X,Y)W) = 0 \tag{65}$$

In view of (4), (65) yields

$$\begin{aligned} &b[A(\tilde{C}(X,Y)Z)A(W) + A(Z)A(\tilde{C}(X,Y)W)] + c[A(\tilde{C}(X,Y)Z)B(W) \\ &+ A(W)B(\tilde{C}(X,Y)Z) + A(Z)B(\tilde{C}(X,Y)W) + A(\tilde{C}(X,Y)W)B(Z)] = 0 \end{aligned} \tag{66}$$

Putting $Z = U$, $W = V$ in (66), we get

$$bA(\tilde{C}(X,Y)V) = 0 \tag{67}$$

Since $A(\tilde{C}(X,Y)V) = g(\tilde{C}(X,Y)V,U) = \tilde{C}(X,Y,V,U)$ and $b \neq 0$,

$$\tilde{C}(X,Y,U,V) = 0 \tag{68}$$

Thus, in view of (63), (68) yields

$$R(X,Y,U,V) = \frac{r}{n(n-1)}[A(Y)B(X) - A(X)B(Y)] \tag{69}$$

Now, contracting (66) over X and W and using (64), we get

$$\begin{aligned} &b\Big[A(\tilde{C}(U,Y)Z) - A(Z)[S(Y,U) - \frac{r}{n}g(Y,U)]\Big] \\ &+c\Big[A(\tilde{C}(V,Y)Z) + B(\tilde{C}(U,Y)Z) - A(Z)[S(Y,V) - \frac{r}{n}g(Y,V)] - B(Z)[S(Y,U) - \frac{r}{n}g(Y,U)]\Big] = 0 \end{aligned} \tag{70}$$

Putting $Z = U$ in (70), we get

$$b[-S(Y,U) + \frac{r}{n}g(Y,U)] + c[\tilde{C}(U,Y,U,V) - S(Y,V) + \frac{r}{n}g(Y,V)] = 0 \tag{71}$$

In view of (7) and (68), (71) yields

$$[-ab - b^2 + \frac{rb}{n} - c^2]A(Y) + [-ac - bc + \frac{cr}{n}]B(Y) = 0 \tag{72}$$

Putting $Y = U$ in (72), we get

$$-ab - b^2 + \frac{rb}{n} - c^2 = 0 \tag{73}$$

Putting $Y = V$ in (72), we get

$$c(-a - b + \frac{r}{n}) = 0 \tag{74}$$

so $c = 0$ or $a + b = \frac{r}{n}$.

If $c = 0$, then by (73), we get $b = 0$ or $a + b = \frac{r}{n}$. If $b = 0$, then as both of b and c are zero, the manifold reduces to an Einstein manifold. If $a + b = \frac{r}{n}$, then by (6), again we obtain $b = 0$ and using this in (73), we get $c = 0$. Thus, in each case, $b = c = 0$, which means that the manifold reduces to an Einstein manifold. But this contradicts with our assumption. Hence we can state that:

Theorem 7.1. *There exists no non-Einstein generalized quasi Einstein manifold satisfying the condition* $\tilde{C} \cdot S = 0$.

8. $G(QE)_n$ with the condition $\tilde{W} \cdot S = 0$

In 1968, Yano and Sawaki introduced the notion of quasi conformal curvature tensor $\tilde{W}$ [9] of type $(1, 3)$ which icludes both the conformal curvature tensor C and the concircular curvature tensor $\tilde{C}$. The quasi conformal curvature tensor $\tilde{W}$ of type $(1, 3)$ is defined by

$$\tilde{W}(X, Y)Z = -(n-2)\beta C(X, Y)Z + [\alpha + (n-2)\beta]\tilde{C}(X, Y)Z \tag{75}$$

where α and β are arbitrary non-zero constants. In particular, if $\alpha = 1$, $\beta = 0$, then $\tilde{W}$ reduces to the concircular curvature tensor and if $\alpha = 1$, $\beta = \frac{-1}{n-2}$, then $\tilde{W}$ reduces to the conformal curvature tensor.

Note that; the quasi conformal curvature tensor $\tilde{W}$ satisfies the following symmetry properties:

- $\tilde{W}(X, Y, Z, W) = -\tilde{W}(Y, X, Z, W) = -\tilde{W}(X, Y, W, Z)$

for all $X, Y, Z, W \in TM$, where $\tilde{W}(X, Y, Z, W) = g(\tilde{W}(X, Y)Z, W)$ is the quasi conformal curvature tensor of type $(0, 4)$.

In view of (54) and (63), (75) can be written as

$$\begin{aligned}\tilde{W}(X, Y)Z =& \alpha R(X, Y)Z + \beta[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &- \frac{r}{n}\Big(\frac{\alpha}{n-1} + 2\beta\Big)[g(Y, Z)X - g(X, Z)Y]\end{aligned} \tag{76}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leqslant i \leqslant n$. Now, from (76), we have

$$\sum_{i=1}^{n} \tilde{W}(Y, e_i, e_i, U) = \alpha S(Y, U) + \beta[rA(Y) + (n-2)S(Y, U)] - \frac{r}{n}\Big(\frac{\alpha}{n-1} + 2\beta\Big)(n-1)A(Y) \tag{77}$$

In this section, we consider generalized quasi Einstein manifold satisfying the condition $\tilde{W} \cdot S = 0$. Then we have

$$(\tilde{W}(X, Y) \cdot S)(Z, W) = -S(\tilde{W}(X, Y)Z, W) - S(Z, \tilde{W}(X, Y)W) = 0 \tag{78}$$

In view of (4), (78) yields

$$\begin{aligned}&b[A(\tilde{W}(X, Y)Z)A(W) + A(Z)A(\tilde{W}(X, Y)W)] + c[A(\tilde{W}(X, Y)Z)B(W) \\ &+ A(W)B(\tilde{W}(X, Y)Z) + A(Z)B(\tilde{W}(X, Y)W) + A(\tilde{W}(X, Y)W)B(Z)] = 0\end{aligned} \tag{79}$$

Putting $Z = U$, $W = V$ in (79), we get

$$bA(\tilde{W}(X,Y)V) = 0 \tag{80}$$

Since $A(\tilde{W}(X,Y)V) = g(\tilde{W}(X,Y)V,U) = \tilde{W}(X,Y,V,U)$ and $b \neq 0$,

$$\tilde{W}(X,Y,U,V) = 0 \tag{81}$$

In view of (81), (76) yields

$$\begin{aligned}\alpha R(X,Y,U,V) =& \beta[S(X,U)B(Y) - S(Y,U)B(X) + S(Y,V)A(X) - S(X,V)A(Y)] \\ &+ \frac{r}{n}\Big(\frac{\alpha}{n-1} + 2\beta\Big)[A(Y)B(X) - A(X)B(Y)]\end{aligned} \tag{82}$$

By virtue (7), (82) yields

$$R(X,Y,U,V) = \gamma[A(Y)B(X) - A(X)B(Y)] \tag{83}$$

where $\gamma = \frac{1}{\alpha}\{-(2a+b)\beta + \frac{r}{n}\Big(\frac{\alpha}{n-1} + 2\beta\Big)\}$. Now, contracting (79) over X and W and using (77), we get

$$\begin{aligned}&b\Big\{A(\tilde{W}(U,Y)Z) - A(Z)\Big[\alpha S(Y,U) + \beta\{rA(Y) + (n-2)S(Y,U)\} - \frac{r}{n}\Big(\frac{\alpha}{n-1} + 2\beta\Big)(n-1)A(Y)\Big]\Big\} \\ &+c\Big\{A(\tilde{W}(V,Y)Z) + B(\tilde{W}(U,Y)Z) - A(Z)\Big[\alpha S(Y,V) + \beta\{rB(Y) + (n-2)S(Y,V)\} - \frac{r}{n}\Big(\frac{\alpha}{n-1} + 2\beta\Big)(n-1)B(Y)\Big] \\ &-B(Z)\Big[\alpha S(Y,U) + \beta\{rA(Y) + (n-2)S(Y,U)\} - \frac{r}{n}\Big(\frac{\alpha}{n-1} + 2\beta\Big)(n-1)A(Y)\Big]\Big\} = 0\end{aligned} \tag{84}$$

Putting $Z = U$ in (84), we get

$$\begin{aligned}&-b\Big[\alpha S(Y,U) + \beta\{rA(Y) + (n-2)S(Y,U)\} - \frac{r}{n}\Big(\frac{\alpha}{n-1} + 2\beta\Big)(n-1)A(Y)\Big] \\ &+c\tilde{W}(U,Y,U,V) - c\Big[\alpha S(Y,V) + \beta\{rB(Y) + (n-2)S(Y,V)\} - \frac{r}{n}\Big(\frac{\alpha}{n-1} + 2\beta\Big)(n-1)B(Y)\Big] = 0\end{aligned} \tag{85}$$

In view of (7) and (81), (85) yields,

$$\begin{aligned}&\Big[ab\alpha + b^2\alpha + c^2\alpha + \beta(n-1)(b^2 + c^2 + 2ab) - \beta c^2 - \frac{br}{n}\Big(\alpha + 2(n-1)\beta\Big)\Big]A(Y) \\ &+\Big[c\alpha(a+b) + c\beta(n-2)(a+b) + c\beta r - \frac{cr}{n}\Big(\alpha + 2(n-1)\beta\Big)\Big]B(Y) = 0\end{aligned} \tag{86}$$

Putting $Y = U$ in (86), we get

$$ab\alpha + b^2\alpha + c^2\alpha + \beta(n-1)(b^2 + c^2 + 2ab) - \beta c^2 - \frac{br}{n}\Big(\alpha + 2(n-1)\beta\Big) = 0 \tag{87}$$

Putting $Y = V$ in (86) and by (6), we get

$$c\Big[\alpha(a+b) + \beta(n-1)(2a+b) - \frac{r}{n}\Big(\alpha + 2(n-1)\beta\Big)\Big] = 0 \tag{88}$$

From the equation (88), we have either $c = 0$ or $\alpha(a+b) + \beta(n-1)(2a+b) - \frac{r}{n}\Big(\alpha + 2(n-1)\beta\Big) = 0$.

If $c = 0$, then by (87), we get $b\Big[\alpha(a+b) + \beta(n-1)(2a+b) - \frac{r}{n}\Big(\alpha + 2(n-1)\beta\Big)\Big] = 0$ which implies $b = 0$ or $\alpha(a+b) + \beta(n-1)(2a+b) - \frac{r}{n}\Big(\alpha + 2(n-1)\beta\Big) = 0$.

If $b = 0$, then the manifold reduces to an Einstein manifold. Thus $b \neq 0$. But in this case, we have $\alpha(a+b) + \beta(n-1)(2a+b) - \frac{r}{n}\Big(\alpha + 2(n-1)\beta\Big) = 0$ which means that $b = 0$ or $\alpha + (n-2)\beta = 0$. From the definition of quasi-conformal curvature tensor, it is known that $\alpha + (n-2)\beta \neq 0$. Thus, again we obtain $b = 0$. Therefore, in each case, both of b and c are zero and so the manifold reduces to an Einstein manifold. But this contradicts with our assumption.

On the other hand, if $c \neq 0$, then we have $\alpha(a+b) + \beta(n-1)(2a+b) - \frac{r}{n}\Big(\alpha + 2(n-1)\beta\Big) = 0$. From similar calculations with above, we obtain $b = 0$ and using this in (87), we have $c = 0$ which means that the manifold reduces to an Einstein manifold. But this contradicts with our assumption. Hence we can state that:

Theorem 8.1. *There exists no non-Einstein generalized quasi Einstein manifold satisfying the condition* $\tilde{W} \cdot S = 0$.

9. $G(QE)_n$ with the condition $W_2 \cdot S = 0$

In 1970, Pokhariyal and Mishra [10] have introduced a W-curvature tensor or W_2-curvature tensor and studied its properties and this tensor is defined as

$$W_2(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY] \tag{89}$$

Note that; the W_2 curvature tensor satisfies the following symmetry properties:

- $W_2(X,Y,Z,W) = -W_2(Y,X,Z,W)$
- $W_2(X,Y,Z,W) \neq -W_2(X,Y,W,Z)$

for all $X, Y, Z, W \in TM$.

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leqslant i \leqslant n$. Now, from (89), we have

$$\sum_{i=1}^{n} W_2(Y, e_i, e_i, U) = 0 \tag{90}$$

In this section, we consider a generalized quasi Einstein manifold satisfying the condition $W_2 \cdot S = 0$. Then for all $X, Y, Z \in \mathfrak{X}(M^n)$;

$$(W_2(X,Y) \cdot S)(Z,W) = -S(W_2(X,Y)Z,W) - S(Z,W_2(X,Y)W) = 0 \tag{91}$$

Combining (4) and (91), we get

$$\begin{aligned} &\frac{-a}{n-1}[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z)] \\ &+ b[A(W_2(X,Y)Z)A(W) + A(Z)A(W_2(X,Y)W)] + c[A(W_2(X,Y)Z)B(W) \\ &+ A(W)B(W_2(X,Y)Z) + A(Z)B(W_2(X,Y)W) + A(W_2(X,Y)W)B(Z)] = 0 \end{aligned} \tag{92}$$

Putting $Z = U$ and $W = V$ in (92), we get

$$\begin{aligned} &\frac{-a}{n-1}[A(Y)S(X,V) - A(X)S(Y,V) + B(Y)S(X,U) - B(X)S(Y,U)] + bW_2(X,Y,V,U) \\ &+ c[W_2(X,Y,U,U) + W_2(X,Y,V,V)] = 0 \end{aligned} \tag{93}$$

In view of (7) and (89), (93) yields

$$bR(X,Y,U,V) = \frac{b(2a+b)}{n-1}[A(Y)B(X) - A(X)B(Y)] \tag{94}$$

Since $b \neq 0$, we obtain

$$R(X,Y,U,V) = \frac{2a+b}{n-1}[A(Y)B(X) - A(X)B(Y)] \tag{95}$$

Hence we state the following theorem:

Theorem 9.1. *In a generalized quasi Einstein manifold satisfying the condition $W_2 \cdot S = 0$, the curvature tensor R satisfies the relation* (95).

Contracting (92) over X and W and using the equation (90), we get

$$\frac{-a}{n-1}[rg(Y,Z) - nS(Y,Z)] + bW_2(U,Y,Z,U) + c[W_2(V,Y,Z,U) + W_2(U,Y,Z,V)] = 0 \tag{96}$$

Putting $Z = U$ in (49), we get

$$\frac{-a}{n-1}[rA(Y) - nS(Y,U)] + bW_2(U,Y,U,U) + c[W_2(V,Y,U,U) + W_2(U,Y,U,V)] = 0 \tag{97}$$

In view of (7) and (89), (97) yields

$$\Big[\frac{-ar}{n-1} + \frac{an(a+b)}{n-1} - \frac{c^2}{n-1}\Big]A(Y) - \Big[\frac{acn}{n-1} + \frac{bc}{n-1} - \frac{c(2a+b)}{n-1} + \frac{ac}{n-1}\Big]B(Y) = 0 \tag{98}$$

Putting $Y = U$ in (98), we get:

$$ab = \frac{c^2}{n-1} \tag{99}$$

Putting $Y = V$ in (98), we get:

$$ac = 0 \tag{100}$$

Then, $a = 0$ or $c = 0$. If $a = 0$, then by (99), $c = 0$.

On the other hand, if $a \neq 0$, then $c = 0$. Then by (99), $ab = 0$. Since $a \neq 0$, we get $b = 0$. But in this case, $b = c = 0$ which means that the manifold reduces to an Einstein manifold. This is a contradiction. Thus again a must be zero.

Hence, in each case the Ricci tensor can be written as

$$S(X,Y) = bA(X)A(Y) \tag{101}$$

Also, contracting (101) over X and Y, we obtain $r = b$. Thus we can state the following theorem:

Theorem 9.2. *In a non-Einstein generalized quasi Einstein manifold satisfying the condition $W_2 \cdot S = 0$, the Ricci tensor is of the form*

$$S(X,Y) = rA(X)A(Y)$$

where r is the scalar curvature of the manifold.

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