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Claw-Free Graphs with Equal 2-Domination and Domination Numbers

Adriana Hansberg[a], **Bert Randerath**[b], **Lutz Volkmann**[c]

[a] *Instituto de Matemáticas, UNAM Juriquilla, 76230 Querétaro, Mexico*
[b] *Institut für Nachrichtentechnik, FB 07, Fachhochschule Köln, 50679 Köln, Germany*
[c] *Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany*

Abstract. For a graph G a subset D of the vertex set of G is a *k-dominating set* if every vertex not in D has at least k neighbors in D. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. Fink and Jacobson showed in 1985 that the inequality $\gamma_k(G) \geq \gamma(G) + k - 2$ is valid for every connected graph G. In this paper, we concentrate on the case $k = 2$, where γ_k can be equal to γ, and we characterize all claw-free graphs and all line graphs G with $\gamma(G) = \gamma_2(G)$.

1. Terminology and Introduction

We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n(G)$. The *neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. The *closed neighborhood* of v is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. By $\delta(G)$ and $\Delta(G)$, we denote the *minimum degree* and the *maximum degree* of the graph G, respectively. For a subset $S \subseteq V$, we define by $G[S]$ the subgraph induced by S. If x and y are vertices of a connected graph G, then we denote with $d_G(x, y)$ the *distance* between x and y in G, i.e. the length of a shortest path between x and y.

With K_n we denote the complete graph on n vertices and with C_n the cycle of length n. We refer to the complete bipartite graph with partition sets of cardinality p and q as the graph $K_{p,q}$. A *block* is a maximal connected subgraph without cut-vertices. A graph G is a *block-cactus graph* if every block of G is either a complete graph or a cycle. G is a *cactus graph* if every block of G is a cycle or a K_2. If we substitute each edge in a non-trivial tree by two parallel edges and then subdivide each edge, then we speak of a *C_4-cactus*. Let G and H be two graphs. For a vertex $v \in V(G)$, we say that the graph G' arises by *inflating* the vertex v to the graph H if the vertex v is substituted by a set S_v of $n(H)$ new vertices and a set of edges such that $G'[S_v] \cong H$ and every vertex in S_v is connected to every neighbor of v in G by an edge.

The *cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ with vertex set $V(G_1) \times V(G_2)$ and vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and

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Email addresses: ahansberg@im.unam.mx (Adriana Hansberg), hubert.randerath@fh-koeln.de (Bert Randerath), volkm@rwth-aachen.de (Lutz Volkmann)

$u_1v_1 \in E(G_1)$. Let u be a vertex of G_1 and v a vertex of G_2. Then the sets of vertices $\{(u, y) \mid y \in V(G_2)\}$ and $\{(x, v) \mid x \in V(G_1)\}$ are called a *row* and, respectively, a *column* of $G_1 \times G_2$. A set of vertices in $V(G_1 \times G_2)$ is called a *transversal* of $G_1 \times G_2$ if it contains exactly one vertex on every row and every column of $G_1 \times G_2$.

Let k be a positive integer. A subset $D \subseteq V$ is a *k-dominating set* of the graph G if $|N_G(v) \cap D| \geq k$ for every $v \in V - D$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. A k-dominating set of minimum cardinality of a graph G is called a $\gamma_k(G)$-set. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [16, 17]. More information on k-domination can be found in [2–6, 8–12, 15].

In [11] and [12], Fink and Jacobson introduced the concept of k-domination. The following theorem establishes a relation between the k-domination number γ_k and the domination number γ.

Theorem 1.1. (Fink, Jacobson [11] 1985) *If G is a graph with $\Delta(G) \geq k \geq 2$, then*

$$\gamma_k(G) \geq \gamma(G) + k - 2.$$

The inequality given above is sharp. However, the characterization of the graphs attaining equality is still an open problem. In [13], the author studied the extremal graphs for general k and gave several properties for them. Among other results, it was shown that if k is an integer with $k \geq 2$ and G a connected graph with $\Delta(G) \geq k$ and $\gamma_k(G) = \gamma(G) + k - 2$, then $\Delta(G[D]) \leq k - 2$ for any minimum k-dominating set D. In the case when $k = 2$, this implies that every minimum 2-dominating set is independent. We will state this fact in the next proposition and for the sake of completeness, we will give the proof, too.

Proposition 1.2. *Let G be a connected graph with $\Delta(G) \geq 2$. If $\gamma_2(G) = \gamma(G)$ and D is a minimum 2-dominating set, then D is independent.*

Proof. Let D be a minimum 2-dominating set. Then $|D| = \gamma_2(G) = \gamma(G)$. If D is not independent, then it contains two adjacent vertices $a, b \in D$. But then, $D - \{a\}$ is a dominating set of cardinality $\gamma(G) - 1$, a contradiction. □

In [14], the authors characterized the block-cactus graphs with equal domination and 2-domination numbers. They also presented some properties on graphs G with $\gamma_2(G) = \gamma(G)$.

In this paper, we center our attention on claw-free graphs. The graph $K_{1,3}$ is called a *claw*. A *claw-free graph* is a graph which does not contain a claw as an induced subgraph. A vast collection of results on claw-free graphs can be found in the survey [7]. If G is a graph, then the *line graph* of G, denoted by $L(G)$, is obtained by associating one vertex to each edge of G, and two vertices of $L(G)$ are joined by an edge if and only if the corresponding edges in G are incident with each other. If for a graph G there is a graph G' whose line graph is isomorphic to G, then G is called a *line graph*. In 1943, Krausz presented the following characterization of line graphs.

Theorem 1.3. (Krausz [18] 1943) *A graph G is a line graph if and only if it can be partitioned into edge disjoint complete graphs such that every vertex of G belongs to at most two of them.*

In 1968, Beineke [1] obtained a characterization of line graphs in terms of nine forbidden induced subgraphs. Since the claw is one of those subgraphs, every line graph is claw-free. In the figure below, we present three of the forbidden induced subgraphs, to which we will refer later.

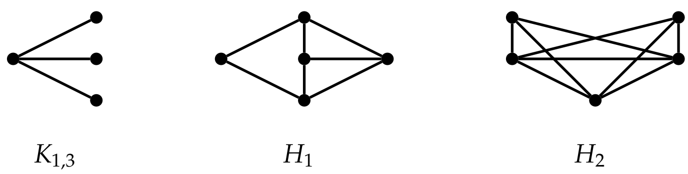

Figure 1: Three forbidden induced subgraphs in line graphs.

2. Claw-Free Graphs with $\gamma = \gamma_2$

In a graph G with $\gamma(G) = \gamma_2(G)$, every minimum 2-dominating set is independent by Proposition 1.2. This fact yields us the following lemma.

Lemma 2.1. *Let G be a connected nontrivial graph with $\gamma_2(G) = \gamma(G)$ and let D be a minimum 2-dominating set of G. Then, for each vertex $x \in V - D$ and $a, b \in D \cap N(x)$, there is a vertex $y \in V - D$ such that x, y, a and b induce a C_4.*

Proof. By Proposition 1.2, a and b are not adjacent. Let $X \subseteq V - D$ be the set of vertices that are not dominated by $D - \{a, b\}$. Since D is a 2-dominating set of G, all vertices of X are adjacent to both a and b. If all vertices of $X - \{x\}$ are adjacent to x, then the set $(D - \{a, b\}) \cup \{x\}$ is a dominating set of G of size $\gamma(G) - 1$, a contradiction. Hence, there is a vertex $y \in X$ such that x, y, a and b induce a C_4. □

Lemma 2.2. *Let G be a connected nontrivial claw-free graph. If $\gamma(G) = \gamma_2(G)$, then every minimum 2-dominating set D of G fulfills:*

(i) Every vertex in $V - D$ has exactly two neighbors in D.
(ii) Every two vertices $a, b \in D$ are at distance 2 in G.

Proof. (i) Because G is claw-free and, by Proposition 1.2, D is an independent 2-dominating set, every vertex in $V - D$ has exactly two neighbors in D and thus (i) follows.

(ii) Suppose that a and b are two vertices in D such that $d_G(a, b) > 2$. Without loss of generality, let b fulfill $d_G(a, b) = \min\{d_G(a, x) > 2 \mid x \in D\}$ and let P be a shortest path from a to b in G. Let u be the neighbor of a in P and v be the second neighbor of u in P. By Proposition 1.2, u does not belong to D. Suppose to the contrary that $v \in D$. By Lemma 2.1, there is a vertex $y \in V - D$ such that u, y, a and v induce a C_4. Let w denote the neighbor of v in P different from u. Since G is claw-free, w is adjacent to u or to y, contradicting the minimality of P. Hence, we may assume that $v \in V - D$, and both u and v have two neighbors in D. Let c be the second neighbor of u from D. Since G is claw-free and $ac \notin E$, v has to be adjacent to a or to c. Because of the minimality of the length of P, v cannot be adjacent to a and thus it is adjacent to another vertex from D. From the choice of the vertex b, we obtain that b is the second neighbor of v in D. Let S be the set of vertices in $V - D$ which have two neighbors from $\{a, b, c\}$, and let H be the graph induced by the set $S \cup \{a, b, c\}$. Since $d_G(a, b) > 2$, there are no vertices which have a and b as neighbors. Further, from Lemma 2.1, we obtain that there are vertices u' and v' in S such that u' is adjacent to a and c but not to u, and v' is adjacent to c and b but not to v. Besides, u and v' cannot be adjacent for otherwise the vertices u, a, v, v' would induce a claw in G. Hence, as $G[\{c, v', u', u\}]$ cannot be a claw, u' and v' are adjacent.

Now we will show that the set $D' = (D - \{a, b, c\}) \cup \{u, v'\}$ is a dominating set of G. Let $z \in V - D'$. From the construction of H and since D is 2-dominating, it is evident that if $z \in V - V(H)$, then it has at least one neighbor in $D - \{a, b, c\}$. If $z \in \{a, c, v\}$, it has u as neighbor in D' and if $z \in \{b, u'\}$, it is dominated by v' in D'. It remains the case that $z \in V(H) - \{a, b, c, u, u', v, v'\}$. Then z has exactly either a and c or c and b as neighbors in $\{a, b, c\}$. Suppose that z is neighbor of a and c. In that case it follows that z is either adjacent to u or to u', otherwise we would have a claw. If z is adjacent to u, we are done. If z is adjacent to u' and not to u, then z has to be adjacent to v', otherwise u, z, v' and c would induce a claw in G. Thus, z is dominated by v' in D'. The case that c and b are neighbors of z follows analogously. Hence, D' is a dominating set of G with less vertices than D and this is a contradiction to $\gamma(G) = \gamma_2(G) = |D|$. Thus, we obtain statement (ii). □

Given a connected claw-free graph G graph with $\gamma_2(G) = \gamma(G)$ and a minimum 2-dominating set D of G, then by Lemma 2.2 every two vertices of D have distance two in G. Hence, from Lemma 2.1 follows that each pair of vertices of D has two non-adjacent common neighbors in $V(G) - D$. This allows us to state the following lemma.

Lemma 2.3. *Let G be a connected claw-free graph with $\gamma(G) = \gamma_2(G)$ and let D be a minimum 2-dominating set of G. Let S be a subset of $V(G) - D$ containing exactly two non-adjacent common neighbors of every pair of vertices of D and $H = G[D \cup S]$. Then, for every $v \in V(H)$, the graph $H[N_H(v)]$ consists of two disjoint cliques.*

Proof. Note that H is again claw-free and $|V(H)| = |D| + 2\binom{|D|}{2} = |D|^2 = p^2$. If $p = 2$, then $H = C_4$ and we are done. So suppose that $p \geq 3$. Assume first that v is a vertex in D. From the construction of H and since D is independent, v is adjacent to exactly $|D| - 1 = p - 1$ pairs of non-adjacent vertices from S, such that each pair has the same two neighbors in D. Let x and y be such a pair. Let z be a neighbor of v different from x and y. As G is claw-free, z is adjacent to x or to y. Hence, $N_H[v] \subseteq N_H[x] \cup N_H[y]$. Suppose that the set $N_H[x] \cap N_H[y] \cap N_H(v)$ contains a vertex w. Let b be the second neighbor of x and y in D and c the second neighbor of w in D. Evidently $w \notin \{x, y\}$ and $c \notin \{v, b\}$. Since x, y and c are pairwise non-adjacent, together with w, they build a claw and we obtain a contradiction. It follows that the sets $N_H[x] \cap N_H(v)$ and $N_H[y] \cap N_H(v)$ are disjoint. Because of G being claw-free, each of these sets is a clique. Since $N_H(v) = (N_H[x] \cup N_H[y]) \cap N_H(v) = (N_H[x] \cap N_H(v)) \cup (N_H[y] \cap N_H(v))$, it follows that $H[N_H(v)]$ is the disjoint union of two cliques.

Assume now that $v \in S$. Let a and b be the two neighbors of v in D. Since there is only a second vertex which is adjacent to both a and b in H and as it is not a neighbor of v in H, it follows that the set $N_H[a] \cap N_H[b] \cap N_H(v)$ is empty. As G is claw-free, the sets $N_H[a] \cap N_H(v)$ and $N_H[b] \cap N_H(v)$ build two disjoint cliques and, for the same reason, every other neighbor of v in H is adjacent either to a or to b. Hence, $N_H(v) = (N_H[a] \cap N_H(v)) \cup (N_H[b] \cap N_H(v))$ and $H[N_H(v)]$ is the disjoint union of two cliques. □

Let $\mathcal{H}_1$ be the family of claw-free graphs G with $\Delta(G) = n(G) - 2$ containing two non-adjacent vertices of maximum degree and let $\mathcal{H}_2$ be the family of graphs G that arise from $K_p \times K_p$, $p \geq 3$, by inflating every vertex but the ones on a transversal (we call it the *diagonal*) to a clique of arbitrary order (see Figure 2).

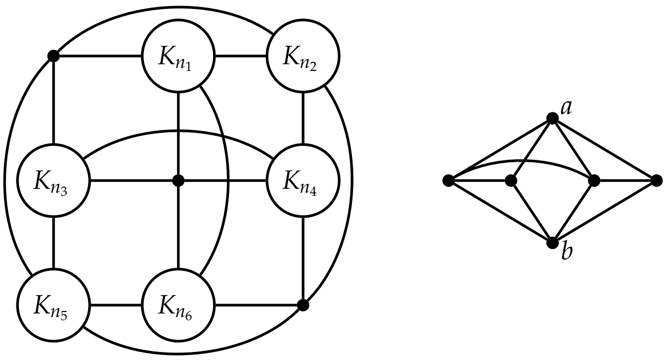

Figure 2: Examples of graphs from the families $\mathcal{H}_2$ and $\mathcal{H}_1$ (here, $n_i \in \mathbb{N}$ for $1 \leq i \leq 6$)

Theorem 2.4. *Let G be a connected claw-free graph. Then $\gamma(G) = \gamma_2(G)$ if and only if $G \in \mathcal{H}_1 \cup \mathcal{H}_2$.*

Proof. Let G be a connected graph. We prove the statement in two parts.

First, we show that $\gamma(G) = \gamma_2(G) = 2$ if and only if $G \in \mathcal{H}_1$. Clearly, $\Delta(G) \leq n(G) - 2$ if and only if $\gamma(G) \geq 2$. Hence, if G is a connected graph such that $\gamma(G) = \gamma_2(G) = 2$, then $\Delta(G) \leq n(G) - 2$ and every minimum 2-dominating set is independent. Hence, there are two non-adjacent vertices a and b such that every other vertex is adjacent to both of them, that is, $d_G(a) = d_G(b) = n(G) - 2 = \Delta(G)$. Thus, $G \in \mathcal{H}_1$. Conversely, if G is a graph with $\Delta(G) = n(G) - 2$ containing two non-adjacent vertices a and b with $d_G(a) = d_G(b) = \Delta(G)$, then every vertex $x \in V(G) - \{a, b\}$ is adjacent to both a and b. This implies that $2 \leq \gamma(G) \leq \gamma_2(G) \leq 2$ and so $\gamma(G) = \gamma_2(G) = 2$.

We will show now that $\gamma(G) = \gamma_2(G) = p \geq 3$ holds if and only if $G \in \mathcal{H}_2$. Let $H \in \mathcal{H}_2$ be a graph isomorphic to the cartesian product $K_p \times K_p$ of two complete graphs of order p, let $T \subset V(H)$ be a transversal of H and let G be a graph that arises from H by inflating every vertex $x \in V(H) - T$ to a clique C_x of arbitrary order. It is evident that every dominating set of G has to contain vertices on every *row* or every *column* of G and thus $p \leq \gamma(G)$. Since T is a 2-dominating set of G, we obtain $p \leq \gamma(G) \leq \gamma_2(G) \leq p$ and hence, $\gamma(G) = \gamma_2(G) = p$.

We prove the converse. Let $\gamma(G) = \gamma_2(G) = p \geq 3$, let $D = \{a_1, a_2, \ldots, a_p\}$ be a minimum 2-dominating set and let S be a subset of $V(G) - D$ containing exactly two non-adjacent common neighbors of every pair of vertices of D and $H = G[D \cup S]$, as in Lemma 2.3. Let C_1 and C_2 be the two complete graphs induced by $N_H[a_1]$ in H such that $V(C_1) \cap V(C_2) = \{a_1\}$, given also by Lemma 2.3. Thus C_1 and C_2 contain exactly one vertex of each pair of non-adjacent vertices from S which have a_1 and a second common neighbor in D. Then, for every vertex $a_i \in D - \{a_1\}$, there are vertices $u_i \in V(C_1)$ and $v_i \in V(C_2)$ such that u_i and v_i are common neighbors of a_1 and a_i. We define $u_1 := a_1$ and $v_1 := a_1$. By the construction of H, it follows that $V(C_1) = \{u_1, u_2, \ldots, u_p\}$ and $V(C_2) = \{v_1, v_2, \ldots, v_p\}$. Further, for every vertex $u_i \in V(C_1)$, let C_{u_i} be the clique in H such that $N_H[u_i] = V(C_1) \cup V(C_{u_i})$ and $V(C_1) \cap V(C_{u_i}) = \{u_i\}$. Analogously for every $v_j \in V(C_2)$, let C_{v_j} be the clique in H such that $N_H[v_j] = V(C_2) \cup V(C_{v_j})$ and $V(C_2) \cap V(C_{v_j}) = \{v_j\}$. Note that $C_1 = C_{v_1}$ and $C_2 = C_{u_1}$.

Claim 1. For every pair of different indices $i, j \in \{1, 2, \ldots, p\}$, $V(C_{u_i}) \cap V(C_{u_j}) = \emptyset$ and $V(C_{v_i}) \cap V(C_{v_j}) = \emptyset$.

Proof of Claim 1. Since $a_i \in V(C_{u_i})$ and $a_j \in V(C_{u_j})$ and a_i and a_j are non-adjacent, it follows that $C_{u_i} \neq C_{u_j}$ and thus by Lemma 2.3 we obtain $V(C_{u_i}) \cap V(C_{u_j}) = \emptyset$. $V(C_{v_i}) \cap V(C_{v_j}) = \emptyset$ follows analogously. ||

Claim 2. $V(H) = \bigcup_{i=1}^{p} V(C_{u_i}) = \bigcup_{j=1}^{p} V(C_{v_j})$ and each union is a disjoint one.

Proof of Claim 2. Let $x \in V(H)$. We will show that $x \in V(C_{u_i})$ and $x \in V(C_{v_j})$ for some $i, j \in \{1, 2, \ldots, p\}$. If $x = a_i \in D$, then $a_i \in V(C_{u_i}) \cap V(C_{v_i})$ and we are done. Thus suppose that $x \notin D$ and let $\{a_i, a_j\} = D \cap N_H(x)$. Then $x \in V(C_{u_i})$ or $x \in V(C_{u_j})$ but not both because of Claim 1. Analogously $x \in V(C_{v_i})$ or $x \in V(C_{v_j})$ but not both. Since $\{a_i\} = V(C_{u_i}) \cap V(C_{v_i})$ and $\{a_j\} = V(C_{u_j}) \cap V(C_{v_j})$, it follows that $x \in V(C_{u_i}) \cap V(C_{v_j})$ or $x \in V(C_{u_j}) \cap V(C_{v_i})$ but not both. Hence $V(H) \subseteq \bigcup_{i=1}^{p} V(C_{u_i})$ and $V(H) \subseteq \bigcup_{j=1}^{p} V(C_{v_j})$ and each union is a disjoint one. Since the inclusions the other way around are obvious, the claim is proved. ||

Claims 1 and 2 imply that every vertex $x \in V(H) - (V(C_1) \cup V(C_2))$ is adjacent to exactly one vertex $u_x \in V(C_1)$ and one vertex $v_x \in V(C_2)$. Moreover, we obtain that $N_H[x] = V(C_{u_x}) \cup V(C_{v_x})$ and $V(C_{u_x}) \cap V(C_{v_x}) = \{x\}$. Now we can define the mapping

$$\begin{aligned} \phi : V(H) \longrightarrow V(C_1 \times C_2) :\ & u_i \mapsto (u_i, v_1), \text{ for } u_i \in V(C_1) \\ & v_i \mapsto (u_1, v_i), \text{ for } v_i \in V(C_2) \\ & x \mapsto (u_x, v_x), \text{otherwise.} \end{aligned}$$

Claim 3. The mapping ϕ is bijective.

Proof of Claim 3. Let x and y be two vertices from $V(H) - (V(C_1) \cup V(C_2))$ such that $\phi(x) = (u_i, v_j) = \phi(y)$. Then x and y are contained in $V(C_{u_i}) \cup V(C_{v_j})$. By Lemma 2.3, we obtain that $\{x\} = V(C_{u_i}) \cap V(C_{v_j}) = \{y\}$ and thus $x = y$. Hence, ϕ is injective. Since

$$\begin{aligned} |V(H) - (V(C_1) \cup V(C_2))| &= |D|^2 - 2|D| + 1 = (|D| - 1)^2 \\ &= |(V(C_1) - \{u_1\}) \times (V(C_2) - \{v_1\})|, \end{aligned}$$

it follows that ϕ is bijective. ||

Claim 4. $H \cong C_1 \times C_2 \cong K_p \times K_p$.

Proof of Claim 4. Let x and y be two vertices in $V(H)$ and let $\phi(x) = (u_i, v_j)$ and $\phi(y) = (u_l, v_m)$. We will show that x and y are adjacent if and only if $i = l$ or $j = m$. Suppose that x is a neighbor of y. From the definition of the mapping ϕ we have that x is adjacent to u_i and v_j and that y is adjacent to u_l and v_m. From Lemma 2.3 it follows that y is adjacent either to u_i or to v_j. This implies that $i = l$ or $j = m$. Conversely, if $i = l$ or $j = m$, it follows again by Lemma 2.3 that x and y are in a clique together with either $u_i = u_l$ or with $v_j = v_m$. ||

By Proposition 1.2, D is independent. Therefore, every row and every column of H contains at most one vertex of D. Since $|D| = p$, every row and every column of H contains exactly one vertex of D. Hence, D is a transversal of H. Let x be a vertex in $V(G) - V(H)$ and let a and b be the neighbors of x in D. Then H contains exactly two non-adjacent vertices u and v having both a and b as neighbors. As G is claw-free, x is adjacent to u or to v. Suppose that x is adjacent to both u and v. By Lemma 2.1, there is a vertex $y \in V - D$ such that x, y, a and b induce a C_4. Clearly, y is distinct from u and v. Now the set $S' = (S - \{u, v\}) \cup \{x, y\}$ has the same properties as S and thus the graph H' induced by $(V(H) - \{u, v\}) \cup \{x, y\}$ is isomorphic to $K_p \times K_p$. By symmetry, we can assume that $N_H(u) = N_{H'}(x)$ and $N_H(v) = N_{H'}(y)$. Since $p \geq 3$, there is a vertex $z_1 \in V(H) - \{a, b, v\}$ that belongs to the column of H that contains v and there is a vertex z_2 that belongs to the row of H that contains v. Clearly, z_1 and z_2 are distinct and z_1, z_2 and x are pairwise non-adjacent, and so together with v they build a claw in G and we obtain a contradiction. Hence, without loss of generality, we can assume that x is adjacent to u but not to v. Then the set $S' = (S - \{u\}) \cup \{x\}$ has the same properties as S and thus the graph induced by the set $(V(H)-\{u\}) \cup \{x\}$ is again isomorphic to $K_p \times K_p$.

For every vertex $u \in V(H) - D$, let a_u and b_u be the neighbors of u in D, let C_u^* be the set of vertices in G that are adjacent to a_u, b_u and u and let $C_u = C_u^* \cup \{u\}$. Clearly, $\bigcup_{u \in V(H)-D} C_u \cup D = V(G)$. It is now easy to see that, for every vertex $u \in V(H) - D$, the set C_u induces a clique in G and that $N_G[x] = N_G[u]$ for every vertex $x \in C_u$.

Hence, if we melt all vertices of every clique C_u for each vertex $u \in V(H) - D$ to a unique vertex $\hat{u}$, we obtain a graph $\hat{H}$ isomorphic to $K_p \times K_p$. Reverting the process, that is, inflating each vertex $\hat{u}$ to the original clique C_u, we obtain again G. Therefore, $G \in \mathcal{H}_2$. □

Theorem 2.5. *Let G be a connected line graph. Then $\gamma_2(G) = \gamma(G)$ if and only if G is either the cartesian product $K_p \times K_p$ of two complete graphs of the same cardinality p or G is isomorphic to the graph J depicted in Figure 3.*

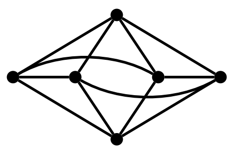

Figure 3: Graph J

Proof. Since every line graph is claw-free, the set of line graphs with $\gamma = \gamma_2$ is contained in $\mathcal{H}_1 \cup \mathcal{H}_2$. If G is a cartesian product of two complete graphs K_p for an integer $p \geq 2$, then the graphs induced by the vertices of every row and of every column of G are complete graphs K_p and form a partition of G into edge disjoint complete subgraphs such that every vertex of G is contained in at most two of them. Hence, by Theorem 1.3, G is a line graph. If $G \cong J$, it is not difficult to obtain a partition of the graph J into edge disjoint complete subgraphs such that every vertex of J is contained in at most two of them and thus J is a line graph.

Conversely, suppose that $G \in \mathcal{H}_1 \cup \mathcal{H}_2$ is a line graph.

Case 1. Assume that $G \in \mathcal{H}_2$, that is, G is a cartesian product $K_p \times K_p$ of two complete graphs of order p for an integer $p \geq 2$ such that the vertices not in a certain transversal T of G are inflated into a clique of arbitrary order. Let a and b be two elements of T and U_1 and U_2 the two inflated vertices which are neighbors of both a and b. Suppose that U_1 has order at least 2 and let x and y be vertices in U_1 and z a vertex in U_2. It is now easy to see that the vertices a, b, x, y and z induce the graph H_1 of Figure 1. Hence, G cannot be a line graph, which contradicts to our hypothesis. Thus, G contains no inflated vertices, that is, it is a cartesian product of two complete graphs of order $p \geq 2$.

Case 2. Assume that $G \in \mathcal{H}_1$, that is, G is a graph of maximum degree $\Delta(G) = n(G) - 2$ containing two non-adjacent vertices a and b such that every vertex $x \in V(G)$ is adjacent to both a and b. If $n(G) = 4$, then obviously it is a C_4 and thus isomorphic to $K_2 \times K_2$. Since the only claw-free graph in $\mathcal{H}_1$ of order 5 is isomorphic to H_1, which is not a line graph, we can assume that $n(G) \geq 6$. As $\Delta(G) = n(G) - 2$, there are two non-adjacent vertices x and y different from a and b. Let $z \in V(G) - \{a, b, x, y\}$. Since G is claw-free and every vertex in $V(G) - \{a, b\}$ is adjacent to both a and b, without loss of generality, we can suppose that z is neighbor of x. If z is not adjacent to y, the vertices a, b, x, z and y would induce a graph isomorphic to H_1 and G would not be a line graph. Hence, z is neighbor of y. Since $\Delta(G) = n(G) - 2$, there is another vertex z' which is not adjacent to z, but, as before, adjacent to x and y and of course to a and b. If $n(G) = 6$, we are ready and $G \cong J$. If $n(G) \geq 7$, then there is another vertex w adjacent to x, y, z and z' (with the same arguments as before). But then, the vertices a, b, x, z and w induce a graph isomorphic to H_2 of Figure 1 and G is not a line graph. Therefore, G cannot have order greater than 6 and, thus, the only possibility for G is to be isomorphic to the graph J.

It follows that $\gamma_2(G) = \gamma(G)$ if and only if G is either the cartesian product $K_p \times K_p$ of two complete graphs of the same cardinality $p \geq 2$ or G is isomorphic to the graph J of Figure 3. □

3. Open Problems and Further Research

We close with the following list of open problems that we have yet to settle.

Problem 3.1. *Characterize further families of graphs G with $\gamma_2(G) = \gamma(G)$ (for instance outerplanar graphs, diamond-free graphs, etc.).*

Problem 3.2. *Find necessary and/or sufficient conditions for a graph having $\gamma_k(G) = \gamma(G) + k - 2$.*

As mentioned in the introduction, we know that, when a graph G fulfills $\gamma_k(G) = \gamma(G) + k - 2$, then the maximum degree of the graph induced by a minimum k-dominating set it at most $k - 2$. This property was the key in characterizing the claw free graphs G with $\gamma_2(G) = \gamma(G)$, as every vertex outside a minimum 2-dominating set has to have exactly two neighbors in it. Similarly for larger k, one could analyze families of graphs with some forbidden structures. For instance, when $k = 3$ and G is $K_{1,4}$-free and $K_{1,3} + e$-free (i.e. a claw provided with an additional edge e), then every vertex outside any minimum 3-dominating set D has exactly three neighbors in D. Thus, we pose the following problem.

Problem 3.3. *Characterize the $\{K_{1,4}, K_{1,3} + e\}$-free graphs G with $\gamma_3(G) = \gamma(G) + 1$.*

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