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Intuitionistic Fuzzy Stability of Jensen-Type Quadratic Functional Equations

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Abstract. In this paper, we prove some stability results for Jensen-type quadratic functional equations

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f(x)+f(y),$$
$$f(ax+ay)+f(ax-ay)=2a^2f(x)+2a^2f(y)$$

in intuitionistic fuzzy normed spaces for a nonzero real number a with $a \neq \pm\frac{1}{2}$.

1. Introduction

The study of stability problem for functional equations is related to a question of Ulam [37] concerning the stability of group homomorphism, which was affirmatively answered by Hyers [8] for Banach spaces. Subsequently, Hyers' result was generalized by Aoki [1] for additive mappings and Rassias [31] for linear mappings by considering an unbounded Cauchy difference. The paper by Rassias has provided a lot of influence in the development of what we now call the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Rassias [30] considered the Cauchy difference controlled by a product of different powers of norms. The above results have been generalized by Forti [4] and Găvruta [5] who permitted the Cauchy difference to become arbitrarily unbounded. For further new progress on such problems, the reader is referred to [2, 3, 6, 9, 11, 12, 17, 32, 33]. For a fuzzy version one is referred to [13–16, 28]. Quite recently, the stability problem for Jensen functional equations, Pexiderized quadratic functional equations, cubic functional equations, mixed type additive and cubic functional equations have been considered in [18, 20, 21, 26, 38]. The idea of intuitionistic fuzzy normed space was studied in [19, 22–25, 27, 34] in order to deal with some summability problems.

Recently, interesting results concerning Jensen-type functional equations

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f(x)+f(y), \tag{1}$$
$$f(ax+ay)+f(ax-ay)=2a^2f(x)+2a^2f(y) \tag{2}$$

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have been obtained in [10], where a is a nonzero real number and $a \neq \pm\frac{1}{2}$. The main purpose of this paper is to prove the stability of the Jensen-type functional equations (1) and (2) in the setting of intuitionistic fuzzy normed space. The results obtained in this paper extend a number of recent well-know results in the subject.

2. Preliminaries

In this section by using the idea of intuitionistic fuzzy metric spaces introduced by Park [29] and Saadati-Park [34], we define a new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous t-representable [7].

Lemma 2.1. (cf. [35]). *Consider the set L^* and the order relation $\leq_{L^*}$ defined by*

$$L^* = \{(x_1, x_2)|(x_1, x_2) \in [0,1]^2, x_1 + x_2 \leq 1\},$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1, x_2 \geq y_2, \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 2.2. (cf. [35]). *An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universal set U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta(u), \eta(u))|u \in U\}$, where, for all $u \in U, \zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ are called the membership degree and the nonmembership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, furthermore, they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.*

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a triangular norm $T = *$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1,x) = 1 * x = x$ for all $x \in [0,1]$. A triangular conorm $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \to [0,1]$ satisfying $S(0,x) = 0 \diamond x = x$ for all $x \in [0,1]$.

Using the lattice $(L^*, \leq_{L^*})$, these definitions can be extended in a straightforward manner.

Definition 2.3. (cf. [35]). *A triangular norm (t-norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \to L^*$ satisfying the following conditions:*
(a) $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$ (boundary condition);
(b) $(\forall (x,y) \in (L^*)^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$ (commutativity);
(c) $(\forall (x,y,z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y),z))$ (associativity);
(d) $(\forall (x,x',y,y') \in (L^*)^4)(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Longrightarrow \mathcal{T}(x,y) \leq_{L^*} \mathcal{T}(x',y')$ (monotonicity).

Definition 2.4. (cf. [35]). *A continuous t-norm $\mathcal{T}$ on L^* is said to be continuous t-representable if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0,1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,*

$$\mathcal{T}(x,y) = (x_1 * y_1, x_2 \diamond y_2).$$

Example 2.5. *For all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$, consider*

$$\mathcal{T}(a,b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$
$$\mathcal{M}(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\}).$$

Then $\mathcal{T}(a,b)$ and $\mathcal{M}(a,b)$ are continuous t-representable.

Now, we define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \dots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \dots, x^{(n)}), x^{(n+1)})$$

for all $n \geq 2$ and $x^{(i)} \in L^*$.

Definition 2.6. *A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \to L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N : [0,1] \to [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $([0,1], \leq)$ defined by $N_s = 1 - x$ for all $x \in [0,1]$.*

Definition 2.7. (cf. [35]). *The triple $(X, \mathcal{P}, \mathcal{T})$ is said to be an IFNS if X is a vector space, $\mathcal{T}$ is a continuous t-representable, and $\mathcal{P}$ is a mapping $X \times (0,\infty) \to L^*$, satisfying the following conditions for all $x, y \in X$ and $t, s > 0$:*
(i) $\mathcal{P}(x,t) >_{L^*} 0_{L^*}$;
(ii) $\mathcal{P}(x,t) = 1_{L^*}$ if and only if $x = 0$;
(iii) $\mathcal{P}(\alpha x, t) = \mathcal{P}\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;
(iv) $\mathcal{P}(x+y, t+s) \geq_{L^*} \mathcal{T}(\mathcal{P}(x,t), \mathcal{P}(y,t))$;
(v) $\mathcal{P}(x,\cdot) : (0,\infty) \to L^*$ is continuous;
(vi) $\lim_{t\to\infty} \mathcal{P}(x,t) = 1_{L^*}$.
In this case, $\mathcal{P}$ is called an intuitionistic fuzzy norm on X. Given μ and ν, membership and nonmembership degrees of an intuitionistic fuzzy set from $X \times (0,\infty)$ to $[0,1]$, such that

$$\mu(x,t) + \nu(x,t) \leq 1$$

for all $x \in X$ and $t > 0$, we write

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t)).$$

Example 2.8. (cf. [36]). *Let $(X, \|\cdot\|)$ be a normed space,*

$$\mathcal{T}(a,b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$, and μ, ν be membership and nonmembership degree of an intuitionistic fuzzy set defined by

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t)) = \left(\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|}\right) \quad \forall t \in \mathbb{R}^+.$$

Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an IFNS.

In Example 2.8, $\mu(x,t) + \nu(x,t) = 1$ for all $x \in X$. We present an example in which $\mu(x,t) + \nu(x,t) < 1$ for $x \neq 0$. This example is a modification of the example of Saadati and Park [34].

Example 2.9. (cf. [36]). *Let $(X, \|\cdot\|)$ be a normed space,*

$$\mathcal{T}(a,b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$, and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$\mathcal{P}_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t)) = \left(\frac{t}{t+m\|x\|}, \frac{\|x\|}{t+\|x\|}\right)$$

for all $t \in \mathbb{R}^+$ in which $m > 1$. Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an IFNS. Here,

$$\mu(x,t) + \nu(x,t) = 1, \text{for } x = 0;$$
$$\mu(x,t) + \nu(x,t) < 1, \text{for } x \neq 0.$$

Lemma 2.10. (cf. [35]). *Let $\mathcal{P}_{\mu,\nu}$ be an intuitionistic fuzzy norm on X. Then $\mathcal{P}_{\mu,\nu}(x,t)$ is nondecreasing with respect to t for all $x \in X$.*

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [34].

Let $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be an IFNS. Then, a sequence $\{x_n\}$ is said to be intuitionistic fuzzy convergent to a point $x \in X$ (denoted by $x_n \xrightarrow{IF} x$) if $\mathcal{P}_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$. The sequence $\{x_n\}$ is said to be intuitionistic fuzzy Cauchy sequence if for every $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{P}_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon)$ for all $n, m \geq n_0$, where N_s is the standard negator. $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is intuitionistic fuzzy convergent in $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$. A complete IFNS is called an intuitionistic fuzzy Banach space.

3. Intuitionistic fuzzy stability

Throughout this section, assume that X, $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ and $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ are linear space, IFNS, intuitionistic fuzzy Banach space, respectively. We prove the intuitionistic fuzzy stability of Jesen-type quadratic functional equations (1) and (2) in the setting of intuitionistic fuzzy normed space,

Theorem 3.1. *Let $f : X \to Y$ be an even mapping with $f(0) = 0$. Suppose that φ is a mapping from X to an intuitionistic fuzzy normed space $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ such that*

$$\mathcal{P}_{\mu,\nu}\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t+s\right) \geq_{L^*} \mathcal{M}\{\mathcal{P}'_{\mu,\nu}(\varphi(x), t), \mathcal{P}'_{\mu,\nu}(\varphi(y), s)\} \tag{3}$$

for all $x, y \in X\backslash\{0\}$ and all positive real numbers t, s. If $\varphi(3x) = \alpha\varphi(x)$ for some positive real number α with $\alpha < 9$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{(9-\alpha)t}{18}\right) \tag{4}$$

for all $x \in X$ and $t > 0$, where

$$\mathcal{P}''_{\mu,\nu}(x,t) := \mathcal{M}^3\{\mathcal{P}'_{\mu,\nu}(\varphi(x), \frac{3}{2}t), \mathcal{P}'_{\mu,\nu}(\varphi(2x), \frac{3}{2}t), \mathcal{P}'_{\mu,\nu}(\varphi(3x), \frac{3}{2}t), \mathcal{P}'_{\mu,\nu}(\varphi(0), \frac{3}{2}t)\}.$$

Proof. Setting $y = 3x$ and $s = t$ in (3), we obtain

$$\mathcal{P}_{\mu,\nu}(2f(2x) + 2f(-x) - f(x) - f(3x), 2t) \geq_{L^*} \mathcal{M}\{\mathcal{P}'_{\mu,\nu}(\varphi(x), t), \mathcal{P}'_{\mu,\nu}(\varphi(3x), t)\} \tag{5}$$

for all $x \in X$ and all $t > 0$. Replacing x by $2x$, y by 0 and s by t in (3), we get

$$\mathcal{P}_{\mu,\nu}(4f(x) - f(2x), 2t) \geq_{L^*} \mathcal{M}\{\mathcal{P}'_{\mu,\nu}(\varphi(2x), t), \mathcal{P}'_{\mu,\nu}(\varphi(0), t)\}. \tag{6}$$

Thus

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(9f(x) - f(3x), 6t) \geq_{L^*} \mathcal{M}^3\{&\mathcal{P}'_{\mu,\nu}(\varphi(x), t), \mathcal{P}'_{\mu,\nu}(\varphi(2x), t),\\ &\mathcal{P}'_{\mu,\nu}(\varphi(3x), t)\mathcal{P}'_{\mu,\nu}(\varphi(0), t)\},\end{aligned} \tag{7}$$

and so

$$\mathcal{P}_{\mu,\nu}\left(f(x) - \frac{f(3x)}{9}, t\right) \geq_{L^*} \mathcal{P}''_{\mu,\nu}(x,t) \tag{8}$$

for all $x \in X$ and all $t > 0$, where

$$\mathcal{P}''_{\mu,\nu}(x,t) := \mathcal{M}^3\left\{\mathcal{P}'_{\mu,\nu}\left(\varphi(x), \frac{3}{2}t\right), \mathcal{P}'_{\mu,\nu}\left(\varphi(2x), \frac{3}{2}t\right), \mathcal{P}'_{\mu,\nu}\left(\varphi(3x), \frac{3}{2}t\right), \mathcal{P}'_{\mu,\nu}\left(\varphi(0), \frac{3}{2}t\right)\right\}.$$

Then by our assumption, we have

$$\mathcal{P}''_{\mu,\nu}(3x,t) = \mathcal{P}''_{\mu,\nu}\left(x, \frac{t}{\alpha}\right) \tag{9}$$

Replacing x by $3^n x$ in (8) and applying (9), we get

$$\begin{aligned}\mathcal{P}_{\mu,\nu}\left(\frac{f(3^n x)}{9^n} - \frac{f(3^{n+1}x)}{9^{n+1}}, \frac{\alpha^n t}{9^n}\right) &= \mathcal{P}_{\mu,\nu}\left(f(3^n x) - \frac{f(3^{n+1}x)}{9}, \alpha^n t\right)\\ &\geq_{L^*} \mathcal{P}''_{\mu,\nu}(3^n x, \alpha^n t)\\ &=_{L^*} \mathcal{P}''_{\mu,\nu}(x,t).\end{aligned} \tag{10}$$

For all $x \in X, t > 0$ and all non-negative integers n and m with $n > m$, we have

$$\begin{aligned}\mathcal{P}_{\mu,\nu}\left(\frac{f(3^n x)}{9^n} - \frac{f(3^m x)}{9^m}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{9^k}\right) &= \mathcal{P}_{\mu,\nu}\left(\sum_{k=m}^{n-1}\left[\frac{f(3^{k+1}x)}{9^{k+1}} - \frac{f(3^k x)}{9^k}\right], \sum_{k=m}^{n-1}\frac{\alpha^k t}{9^k}\right)\\ &\geq_{L^*} \mathcal{M}^{n-m-1}\left(\mathcal{P}_{\mu,\nu}\left(\frac{f(3^{m+1}x)}{9^{m+1}} - \frac{f(3^m x)}{9^m}, \frac{\alpha^m t}{9^m}\right), \ldots,\right.\\ &\left.\mathcal{P}_{\mu,\nu}\left(\frac{f(3^n x)}{9^n} - \frac{f(3^{n-1}x)}{9^{n-1}}, \frac{\alpha^{n-1}t}{9^{n-1}}\right)\right)\\ &\geq_{L^*} \mathcal{P}''_{\mu,\nu}(x,t).\end{aligned} \tag{11}$$

Hence

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(3^n x)}{9^n} - \frac{f(3^m x)}{9^m}, t\right) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{t}{\sum_{k=m}^{n-1}\frac{\alpha^k}{9^k}}\right) \tag{12}$$

for all $x \in X, t > 0$ and $m, n \in \mathbb{N}$ with $n > m$.

Since $0 < \alpha < 9$ and $\sum\limits_{k=0}^{\infty}\frac{\alpha^k}{9^k} < \infty$, then $\left\{\frac{f(3^n x)}{9^n}\right\}$ is a Cauchy sequence in $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ for each $x \in X$. Since $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ is an intuitionistic fuzzy Banach space, the sequence converges to some point $Q(x) \in Y$. So we can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{f(3^n x)}{9^n} \tag{13}$$

for all $x \in X$. Fix $x \in X$ and set $m = 0$ in (12) to obtain

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(3^n x)}{9^n} - f(x), t\right) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{9^k}}\right) \tag{14}$$

for all $x \in X, t > 0$. Thus, we obtain that

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) &= \mathcal{P}_{\mu,\nu}\left(Q(x) - \frac{f(3^n x)}{9^n} + \frac{f(3^n x)}{9^n} - f(x), t\right)\\ &\geq_{L^*} \mathcal{M}\left(\mathcal{P}_{\mu,\nu}\left(Q(x) - \frac{f(3^n x)}{9^n}, \frac{t}{2}\right), \mathcal{P}_{\mu,\nu}\left(\frac{f(3^n x)}{9^n} - f(x), \frac{t}{2}\right)\right)\\ &\geq_{L^*} \mathcal{M}\left(\mathcal{P}_{\mu,\nu}\left(Q(x) - \frac{f(3^n x)}{9^n}, \frac{t}{2}\right), \mathcal{P}''_{\mu,\nu}\left(x, \frac{t}{2\sum_{k=0}^{n-1}\frac{\alpha^k}{9^k}}\right)\right).\end{aligned} \tag{15}$$

Taking the limit as $n \to \infty$ in (15) and using (13), we get

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{(9-\alpha)t}{18}\right)$$

for all $x \in X$ and $t > 0$, which shows that Q satisfies (4).

Now we show that Q is quadratic. Let $x, y \in Y$. Then we have

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y),t\right)\\
&\geq\ _{L^*}\mathcal{M}^4\left\{\mathcal{P}_{\mu,\nu}\left(2Q\left(\frac{x+y}{2}\right)-\frac{2f(3^n(x+y)/2)}{9^n},\frac{t}{5}\right),\right.\\
&\mathcal{P}_{\mu,\nu}\left(2Q\left(\frac{x-y}{2}\right)-\frac{2f(3^n(x-y)/2)}{9^n},\frac{t}{5}\right),\\
&\mathcal{P}_{\mu,\nu}\left(\frac{f(3^n x)}{9^n}-Q(x),\frac{t}{5}\right),\mathcal{P}_{\mu,\nu}\left(\frac{f(3^n y)}{9^n}-Q(y),\frac{t}{5}\right),\\
&\left.\mathcal{P}_{\mu,\nu}\left(\frac{2f(3^n(x+y)/2)}{9^n}+\frac{2f(3^n(x-y)/2)}{9^n}-\frac{f(3^n x)}{9^n}-\frac{f(3^n y)}{9^n},\frac{t}{5}\right)\right\}.
\end{aligned} \tag{16}$$

The first four terms on the right hand side of the above inequality tend to 1_{L^*} as $n \to \infty$ by (13) and the fifth term, by (3), is greater than or equal to

$$\begin{aligned}
&_{L^*}\mathcal{M}\left\{\mathcal{P}'_{\mu,\nu}\left(\varphi(3^n x),\frac{9^n t}{10}\right),\mathcal{P}'_{\mu,\nu}\left(\varphi(3^n y),\frac{9^n t}{10}\right)\right\}\\
&=_{L^*}\mathcal{M}\left\{\mathcal{P}'_{\mu,\nu}\left(\varphi(x),\left(\frac{9}{\alpha}\right)^n\frac{t}{10}\right),\mathcal{P}'_{\mu,\nu}\left(\varphi(y),\left(\frac{9}{\alpha}\right)^n\frac{t}{10}\right)\right\},
\end{aligned} \tag{17}$$

which tends to 1_{L^*} as $n \to \infty$. Hence

$$\mathcal{P}_{\mu,\nu}\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y),t\right)=1_{L^*} \tag{18}$$

for all $x, y \in X$ and all $t > 0$. This means that Q satisfies the Jensen quadratic functional equation and so it is quadratic.

To prove the uniqueness of the mapping Q subject to (4), assume that there exists another quadratic mapping $Q' : X \to Y$ which satisfies (4). Then for each $x \in X$, clearly $Q(3^n x) = 9^n Q(x)$ and $Q'(3^n x) = 9^n Q'(x)$ for all $n \in \mathbb{N}$. It follows from (4) that

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(Q(x)-Q'(x),t)=\mathcal{P}_{\mu,\nu}\left(\frac{Q(3^n x)}{9^n}-\frac{Q'(3^n x)}{9^n},t\right)\\
&\geq\ _{L^*}\mathcal{M}\left\{\mathcal{P}_{\mu,\nu}\left(\frac{Q(3^n x)}{9^n}-\frac{f(3^n x)}{9^n},\frac{t}{2}\right),\mathcal{P}_{\mu,\nu}\left(\frac{f(3^n x)}{9^n}-\frac{Q'(3^n x)}{9^n},\frac{t}{2}\right)\right\}\\
&\geq\ _{L^*}\mathcal{P}''_{\mu,\nu}\left(x,\frac{(\frac{9}{\alpha})^n(9-\alpha)t}{36}\right)
\end{aligned} \tag{19}$$

for all $x \in X, t > 0$ and all $n \in \mathbb{N}$. Since $0 < \alpha < 9$ and $\lim_{n\to\infty}(\frac{9}{\alpha})^n = \infty$, the right hand side of the above inequality tends to 1_{L^*} as $n \to \infty$. Therefore, $\mathcal{P}_{\mu,\nu}(Q(x) - Q'(x), t) = 1_{L^*}$ for all $t > 0$, whence $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof of the theorem. □

Theorem 3.2. *Let $f : X \to Y$ be an even mapping with $f(0) = 0$. Suppose that φ is a mapping from X to an intuitionistic fuzzy normed space $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ satisfying (3). If $\varphi(3x) = \alpha\varphi(x)$ for some real number α with $\alpha > 9$, then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\mathcal{P}_{\mu,\nu}(Q(x)-f(x),t)\geq\ _{L^*}\mathcal{P}''_{\mu,\nu}\left(x,\frac{(\alpha-9)t}{2\alpha}\right) \tag{20}$$

for all $x \in X$ and $t > 0$, where

$$\mathcal{P}''_{\mu,\nu}(x,t):=\mathcal{M}^3\left\{\mathcal{P}'_{\mu,\nu}\left(\varphi\left(\frac{x}{3}\right),\frac{t}{6}\right),\mathcal{P}'_{\mu,\nu}\left(\varphi\left(\frac{2x}{3}\right),\frac{t}{6}\right),\mathcal{P}'_{\mu,\nu}\left(\varphi(x),\frac{t}{6}\right),\mathcal{P}'_{\mu,\nu}\left(\varphi(0),\frac{t}{6}\right)\right\}.$$

Proof. It follows from (8) that

$$\mathcal{P}_{\mu,\nu}\left(f(x)-9f\left(\frac{x}{3}\right),t\right)\geq_{L^*}\mathcal{P}''_{\mu,\nu}(x,t) \tag{21}$$

for all $x\in X$ and all $t>0$, where

$$\mathcal{P}''_{\mu,\nu}(x,t):=\mathcal{M}^3\left\{\mathcal{P}'_{\mu,\nu}\left(\varphi\left(\frac{x}{3}\right),\frac{t}{6}\right),\mathcal{P}'_{\mu,\nu}\left(\varphi\left(\frac{2x}{3}\right),\frac{t}{6}\right),\mathcal{P}'_{\mu,\nu}\left(\varphi(x),\frac{t}{6}\right),\mathcal{P}'_{\mu,\nu}\left(\varphi(0),\frac{t}{6}\right)\right\}.$$

Then by our assumption, we have

$$\mathcal{P}''_{\mu,\nu}\left(\frac{x}{3},t\right)=\mathcal{P}''_{\mu,\nu}(x,\alpha t) \tag{22}$$

Replacing x by $\frac{x}{3^n}$ in (21) and applying (22), we obtain

$$\begin{aligned}\mathcal{P}_{\mu,\nu}\left(9^nf\left(\frac{x}{3^n}\right)-9^{n+1}f\left(\frac{x}{3^{n+1}}\right),\frac{9^nt}{\alpha^n}\right) &=\mathcal{P}_{\mu,\nu}\left(f\left(\frac{x}{3^n}\right)-9f\left(\frac{x}{3^{n+1}}\right),\frac{t}{\alpha^n}\right)\\ &\geq_{L^*}\mathcal{P}''_{\mu,\nu}\left(\frac{x}{3^n},\frac{t}{\alpha^n}\right)\\ &=_{L^*}\mathcal{P}''_{\mu,\nu}(x,t).\end{aligned} \tag{23}$$

For all $x\in X, t>0$ and all non-negative integers n and m with $n>m$, we have

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}\left(9^nf\left(\frac{x}{3^n}\right)-9^mf\left(\frac{x}{3^m}\right),\sum_{k=m}^{n-1}\frac{9^kt}{\alpha^k}\right)=\mathcal{P}_{\mu,\nu}\left(\sum_{k=m}^{n-1}\left[9^{k+1}f\left(\frac{x}{3^{k+1}}\right)-9^kf\left(\frac{x}{3^k}\right)\right],\sum_{k=m}^{n-1}\frac{9^kt}{\alpha^k}\right)\\ &\geq_{L^*}\mathcal{M}^{n-m-1}\left(\mathcal{P}_{\mu,\nu}\left(9^{m+1}f\left(\frac{x}{3^{m+1}}\right)-9^mf\left(\frac{x}{3^m}\right),\frac{9^mt}{\alpha^m}\right),\ldots,\right.\\ &\left.\mathcal{P}_{\mu,\nu}\left(9^nf\left(\frac{x}{3^n}\right)-9^{n-1}f\left(\frac{x}{3^{n-1}}\right),\frac{9^{n-1}t}{\alpha^{n-1}}\right)\right)\\ &\geq_{L^*}\mathcal{P}''_{\mu,\nu}(x,t).\end{aligned} \tag{24}$$

Hence

$$\mathcal{P}_{\mu,\nu}\left(9^nf\left(\frac{x}{3^n}\right)-9^mf\left(\frac{x}{3^m}\right),t\right)\geq_{L^*}\mathcal{P}''_{\mu,\nu}\left(x,\frac{t}{\sum_{k=m}^{n-1}\frac{9^k}{\alpha^k}}\right) \tag{25}$$

for all $x\in X, t>0$ and $m,n\in\mathbb{N}$ with $n>m$.

Since $\alpha>9$ and $\sum_{k=0}^{\infty}\frac{9^k}{\alpha^k}<\infty$, then $\{9^nf(\frac{x}{3^n})\}$ is a Cauchy sequence in $(Y,\mathcal{P}_{\mu,\nu},\mathcal{M})$ for each $x\in X$. Since $(Y,\mathcal{P}_{\mu,\nu},\mathcal{M})$ is an intuitionistic fuzzy Banach space, the sequence converges to some point $Q(x)\in Y$. So we can define the mapping $Q:X\to Y$ by

$$Q(x):=\lim_{n\to\infty}9^nf\left(\frac{x}{3^n}\right) \tag{26}$$

for all $x\in X$. Fix $x\in X$ and set $m=0$ in (25) to obtain

$$\mathcal{P}_{\mu,\nu}\left(9^nf\left(\frac{x}{3^n}\right)-f(x),t\right)\geq_{L^*}\mathcal{P}''_{\mu,\nu}\left(x,\frac{t}{\sum_{k=0}^{n-1}\frac{9^k}{\alpha^k}}\right) \tag{27}$$

for all $x\in X, t>0$ and so we have that

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(Q(x)-f(x),t) &=\mathcal{P}_{\mu,\nu}\left(Q(x)-9^nf\left(\frac{x}{3^n}\right)+9^nf\left(\frac{x}{3^n}\right)-f(x),t\right)\\ &\geq_{L^*}\mathcal{M}\left(\mathcal{P}_{\mu,\nu}\left(Q(x)-9^nf\left(\frac{x}{3^n}\right),\frac{t}{2}\right),\mathcal{P}_{\mu,\nu}\left(9^nf\left(\frac{x}{3^n}\right)-f(x),\frac{t}{2}\right)\right)\\ &\geq_{L^*}\mathcal{M}\left(\mathcal{P}_{\mu,\nu}\left(Q(x)-9^nf\left(\frac{x}{3^n}\right),\frac{t}{2}\right),\mathcal{P}''_{\mu,\nu}\left(x,\frac{t}{2\sum_{k=0}^{n-1}\frac{9^k}{\alpha^k}}\right)\right).\end{aligned} \tag{28}$$

Taking the limit as $n \to \infty$ in (28) and using (26), we get

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{(\alpha - 9)t}{2\alpha}\right)$$

for all $x \in X$ and $t > 0$, which shows that Q satisfies (20). The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof of the theorem. □

Theorem 3.3. *Let $f : X \to Y$ be a mapping with $f(0) = 0$. Suppose that φ is a mapping from X to an intuitionistic fuzzy normed space $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ satisfying (3). If $\varphi(2x) = \alpha\varphi(x)$ for some positive real number α with $\alpha < 4$, then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{(4 - \alpha)t}{8}\right) \tag{29}$$

for all $x \in X$ and $t > 0$, where $\mathcal{P}''_{\mu,\nu}(x, t) := \mathcal{M}\{\mathcal{P}'_{\mu,\nu}(\varphi(2x), 2t), \mathcal{P}'_{\mu,\nu}(\varphi(0), 2t)\}$.

Proof. Setting $y = 0$ and replacing x by $2x$ and s by t in (3), we get

$$\mathcal{P}_{\mu,\nu}(4f(x) - f(2x), 2t) \geq_{L^*} \mathcal{M}\{\mathcal{P}'_{\mu,\nu}(\varphi(2x), t), \mathcal{P}'_{\mu,\nu}(\varphi(0), t)\} \tag{30}$$

for all $x \in X$ and all $t > 0$. Thus

$$\mathcal{P}_{\mu,\nu}\left(f(x) - \frac{f(2x)}{4}, t\right) \geq_{L^*} \mathcal{P}''_{\mu,\nu}(x, t) \tag{31}$$

for all $x \in X$ and all $t > 0$, where

$$\mathcal{P}''_{\mu,\nu}(x, t) := \mathcal{M}\{\mathcal{P}'_{\mu,\nu}(\varphi(2x), 2t), \mathcal{P}'_{\mu,\nu}(\varphi(0), 2t)\}.$$

Then by our assumption, we have

$$\mathcal{P}''_{\mu,\nu}(2x, t) = \mathcal{P}''_{\mu,\nu}\left(x, \frac{t}{\alpha}\right) \tag{32}$$

Replacing x by $2^n x$ in (31) and applying (32), we get

$$\begin{aligned}\mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1}x)}{4^{n+1}}, \frac{\alpha^n t}{4^n}\right) &= \mathcal{P}_{\mu,\nu}\left(f(2^n x) - \frac{f(2^{n+1}x)}{4}, \alpha^n t\right)\\ &\geq_{L^*} \mathcal{P}''_{\mu,\nu}(2^n x, \alpha^n t)\\ &=_{L^*} \mathcal{P}''_{\mu,\nu}(x, t).\end{aligned} \tag{33}$$

For all $x \in X, t > 0$ and all non-negative integers n and m with $n > m$, we have

$$\begin{aligned}\mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{4^k}\right) &= \mathcal{P}_{\mu,\nu}\left(\sum_{k=m}^{n-1}\left[\frac{f(2^{k+1}x)}{4^{k+1}} - \frac{f(2^k x)}{4^k}\right], \sum_{k=m}^{n-1} \frac{\alpha^k t}{4^k}\right)\\ &\geq_{L^*} \mathcal{M}^{n-m-1}\left(\mathcal{P}_{\mu,\nu}\left(\frac{f(2^{m+1}x)}{4^{m+1}} - \frac{f(2^m x)}{4^m}, \frac{\alpha^m t}{4^m}\right), \ldots,\right.\\ &\left.\mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{4^n} - \frac{f(2^{n-1}x)}{4^{n-1}}, \frac{\alpha^{n-1} t}{4^{n-1}}\right)\right)\\ &\geq_{L^*} \mathcal{P}''_{\mu,\nu}(x, t).\end{aligned} \tag{34}$$

Hence

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, t\right) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{t}{\sum_{k=m}^{n-1} \frac{\alpha^k}{4^k}}\right) \tag{35}$$

for all $x \in X, t > 0$ and $m, n \in \mathbb{N}$ with $n > m$.

Since $0 < \alpha < 4$ and $\sum_{k=0}^{\infty} \frac{\alpha^k}{4^k} < \infty$, then $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence in $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ for each $x \in X$. Since $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ is an intuitionistic fuzzy Banach space, the sequence converges to some point $Q(x) \in Y$. So we can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \tag{36}$$

for all $x \in X$. Fix $x \in X$ and set $m = 0$ in (35) to obtain

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{4^n} - f(x), t\right) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{4^k}}\right) \tag{37}$$

for all $x \in X, t > 0$. Thus, we have that

$$\begin{aligned}
\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) &= \mathcal{P}_{\mu,\nu}\left(Q(x) - \frac{f(2^n x)}{4^n} + \frac{f(2^n x)}{4^n} - f(x), t\right) \\
&\geq_{L^*} \mathcal{M}\left(\mathcal{P}_{\mu,\nu}\left(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}\right), \mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{4^n} - f(x), \frac{t}{2}\right)\right) \\
&\geq_{L^*} \mathcal{M}\left(\mathcal{P}_{\mu,\nu}\left(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}\right), \mathcal{P}''_{\mu,\nu}\left(x, \frac{t}{2\sum_{k=0}^{n-1} \frac{\alpha^k}{4^k}}\right)\right).
\end{aligned} \tag{38}$$

Taking the limit as $n \to \infty$ in (38) and using (36), we get

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{(4-\alpha)t}{8}\right)$$

for all $x \in X$ and $t > 0$, which shows that Q satisfies (30). The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof of the theorem. □

Theorem 3.4. *Let $f : X \to Y$ be a mapping with $f(0) = 0$. Suppose that φ is a mapping from X to an intuitionistic fuzzy normed space $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ satisfying (3). If $\varphi(2x) = \alpha\varphi(x)$ for some real number α with $\alpha > 4$, then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{(\alpha-4)t}{2\alpha}\right) \tag{39}$$

for all $x \in X$ and $t > 0$, where $\mathcal{P}''_{\mu,\nu}(x, t) := \mathcal{M}\{\mathcal{P}'_{\mu,\nu}(\varphi(x), \frac{t}{2}), \mathcal{P}'_{\mu,\nu}(\varphi(0), \frac{t}{2})\}$.

Proof. It follows from (31) that

$$\mathcal{P}_{\mu,\nu}\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \geq_{L^*} \mathcal{P}''_{\mu,\nu}(x, t) \tag{40}$$

for all $x \in X$ and all $t > 0$, where

$$\mathcal{P}''_{\mu,\nu}(x, t) := \mathcal{M}\left\{\mathcal{P}'_{\mu,\nu}\left(\varphi(x), \frac{t}{2}\right), \mathcal{P}'_{\mu,\nu}\left(\varphi(0), \frac{t}{2}\right)\right\}.$$

Then by our assumption, we have

$$\mathcal{P}''_{\mu,\nu}\left(\frac{x}{2}, t\right) = \mathcal{P}''_{\mu,\nu}(x, \alpha t) \tag{41}$$

Replacing x by $\frac{x}{2^n}$ in (40) and applying (41), we obtain

$$\begin{aligned}
\mathcal{P}_{\mu,\nu}\left(4^n f\left(\frac{x}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}\right), \frac{4^n t}{\alpha^n}\right) &= \mathcal{P}_{\mu,\nu}\left(f\left(\frac{x}{2^n}\right) - 4f\left(\frac{x}{2^{n+1}}\right), \frac{t}{\alpha^n}\right) \\
&\geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(\frac{x}{2^n}, \frac{t}{\alpha^n}\right) \\
&=_{L^*} \mathcal{P}''_{\mu,\nu}(x, t).
\end{aligned} \tag{42}$$

For all $x \in X, t > 0$ and all non-negative integers n and m with $n > m$, we have

$$\begin{aligned} &\mathcal{P}_{\mu,\nu}\left(4^n f\left(\frac{x}{2^n}\right) - 4^m f\left(\frac{x}{2^m}\right), \sum_{k=m}^{n-1} \frac{4^k t}{\alpha^k}\right) = \mathcal{P}_{\mu,\nu}\left(\sum_{k=m}^{n-1}\left[4^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 4^k f\left(\frac{x}{2^k}\right)\right], \sum_{k=m}^{n-1} \frac{4^k t}{\alpha^k}\right) \\ &\geq_{L^*} \mathcal{M}^{n-m-1}\left(\mathcal{P}_{\mu,\nu}\left(4^{m+1} f\left(\frac{x}{2^{m+1}}\right) - 4^m f\left(\frac{x}{2^m}\right), \frac{4^m t}{\alpha^m}\right), \ldots,\right. \\ &\left.\mathcal{P}_{\mu,\nu}\left(4^n f\left(\frac{x}{2^n}\right) - 4^{n-1} f\left(\frac{x}{2^{n-1}}\right), \frac{4^{n-1} t}{\alpha^{n-1}}\right)\right) \\ &\geq_{L^*} \mathcal{P}''_{\mu,\nu}(x,t). \end{aligned} \tag{43}$$

Hence

$$\mathcal{P}_{\mu,\nu}\left(4^n f\left(\frac{x}{2^n}\right) - 4^m f\left(\frac{x}{2^m}\right), t\right) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{t}{\sum_{k=m}^{n-1} \frac{4^k}{\alpha^k}}\right) \tag{44}$$

for all $x \in X, t > 0$ and $m, n \in \mathbb{N}$ with $n > m$.

Since $\alpha > 4$ and $\sum_{k=0}^{\infty} \frac{4^k}{\alpha^k} < \infty$, then $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ for each $x \in X$. Since $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ is an intuitionistic fuzzy Banach space, the sequence converges to some point $Q(x) \in Y$. Therefore we can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) \tag{45}$$

for all $x \in X$. Fix $x \in X$ and set $m = 0$ in (44) to obtain

$$\mathcal{P}_{\mu,\nu}\left(4^n f\left(\frac{x}{2^n}\right) - f(x), t\right) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{t}{\sum_{k=0}^{n-1} \frac{4^k}{\alpha^k}}\right) \tag{46}$$

for all $x \in X, t > 0$. Thus, we obtain that

$$\begin{aligned} \mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) &= \mathcal{P}_{\mu,\nu}\left(Q(x) - 4^n f\left(\frac{x}{2^n}\right) + 4^n f\left(\frac{x}{2^n}\right) - f(x), t\right) \\ &\geq_{L^*} \mathcal{M}(\mathcal{P}_{\mu,\nu}(Q(x) - 4^n f(\frac{x}{2^n}), \frac{t}{2}), \mathcal{P}_{\mu,\nu}(4^n f(\frac{x}{2^n}) - f(x), \frac{t}{2})) \\ &\geq_{L^*} \mathcal{M}\left(\mathcal{P}_{\mu,\nu}\left(Q(x) - 4^n f\left(\frac{x}{2^n}\right), \frac{t}{2}\right), \mathcal{P}''_{\mu,\nu}\left(x, \frac{t}{2\sum_{k=0}^{n-1} \frac{4^k}{\alpha^k}}\right)\right). \end{aligned} \tag{47}$$

Taking the limit as $n \to \infty$ in (47) and using (45), we get

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}''_{\mu,\nu}\left(x, \frac{(\alpha-4)t}{2\alpha}\right)$$

for all $x \in X$ and $t > 0$, which shows that Q satisfies (39). The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof of the theorem. □

Theorem 3.5. *Let $|2a| > 1$ and $f : X \to Y$ be a mapping with $f(0) = 0$. Suppose that φ is a mapping from X to an intuitionistic fuzzy normed space $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ such that*

$$\begin{aligned} &\mathcal{P}_{\mu,\nu}(f(ax+ay) + f(ax-ay) - 2a^2 f(x) - 2a^2 f(y), t+s) \\ &\qquad \geq_{L^*} \mathcal{M}\{\mathcal{P}'_{\mu,\nu}(\varphi(x), t), \mathcal{P}'_{\mu,\nu}(\varphi(y), s)\} \end{aligned} \tag{48}$$

for all $x, y \in X\backslash\{0\}$ and all positive real numbers t, s. If $\varphi(2ax) = \alpha\varphi(x)$ for some positive real number α with $0 < \alpha < 4a^2$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(x, \frac{(4a^2-\alpha)t}{4}\right) \tag{49}$$

for all $x \in X$ and $t > 0$.

Proof. Setting $y = x$ and $s = t$ in (48), we get

$$\mathcal{P}_{\mu,\nu}(f(2ax) - 4a^2 f(x), 2t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x), t) \tag{50}$$

for all $x \in X$ and all $t > 0$. Thus

$$\mathcal{P}_{\mu,\nu}\left(f(x) - \frac{f(2ax)}{4a^2}, t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x), 2a^2 t) \tag{51}$$

for all $x \in X$ and all $t > 0$.

Replacing x by $(2a)^n x$ in (51), we have

$$\begin{aligned}\mathcal{P}_{\mu,\nu}\left(\frac{f((2a)^n x)}{(2a)^{2n}} - \frac{f((2a)^{n+1}x)}{(2a)^{2n+2}}, \frac{\alpha^n t}{(2a)^{2n}}\right) &= \mathcal{P}_{\mu,\nu}\left(f((2a)^n x) - \frac{f((2a)^{n+1}x)}{4a^2}, \alpha^n t\right) \\ &\geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x), 2a^2 t).\end{aligned} \tag{52}$$

For all $x \in X, t > 0$ and all non-negative integers n and m with $n > m$, we have

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}\left(\frac{f((2a)^n x)}{(2a)^{2n}} - \frac{f((2a)^m x)}{(2a)^m}, \sum_{k=m}^{n-1} \frac{\alpha^k t}{(2a)^{2k}}\right) \\ &= \mathcal{P}_{\mu,\nu}\left(\sum_{k=m}^{n-1}\left[\frac{f((2a)^{k+1}x)}{(2a)^{2k+2}} - \frac{f((2a)^k x)}{(2a)^{2k}}\right], \sum_{k=m}^{n-1} \frac{\alpha^k t}{(2a)^{2k}}\right) \\ &\geq_{L^*} \mathcal{M}^{n-m-1}\Bigg(\mathcal{P}_{\mu,\nu}\left(\frac{f((2a)^{m+1}x)}{(2a)^{2m+2}} - \frac{f((2a)^m x)}{(2a)^{2m}}, \frac{\alpha^m t}{(2a)^{2m}}\right), \ldots, \\ &\mathcal{P}_{\mu,\nu}\left(\frac{f((2a)^n x)}{(2a)^{2n}} - \frac{f((2a)^{n-1}x)}{(2a)^{2n-2}}, \frac{\alpha^{n-1} t}{(2a)^{2n-2}}\right)\Bigg) \\ &\geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x), 2a^2 t).\end{aligned} \tag{53}$$

Hence

$$\mathcal{P}_{\mu,\nu}\left(\frac{f((2a)^n x)}{(2a)^{2n}} - \frac{f((2a)^m x)}{(2a)^m}, t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x), \frac{2a^2 t}{\sum_{k=m}^{n-1} \frac{\alpha^k}{(2a)^{2k}}}\right) \tag{54}$$

for all $x \in X, t > 0$ and $m, n \in \mathbb{N}$ with $n > m$.

Since $0 < \alpha < 4a^2$ and $\sum_{k=0}^{\infty} \frac{\alpha^k}{(2a)^{2k}} < \infty$, then $\{\frac{f((2a)^n x)}{(2a)^{2n}}\}$ is a Cauchy sequence in $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ for each $x \in X$. Since $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ is an intuitionistic fuzzy Banach space, the sequence converges to some point $Q(x) \in Y$. So we can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{f((2a)^n x)}{(2a)^{2n}} \tag{55}$$

for all $x \in X$. Fix $x \in X$ and put $m = 0$ in (54) to obtain

$$\mathcal{P}_{\mu,\nu}\left(\frac{f((2a)^n x)}{(2a)^{2n}} - f(x), t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x), \frac{2a^2 t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{(2a)^{2k}}}\right) \tag{56}$$

for all $x \in X, t > 0$. Thus

$$\begin{aligned}\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) &= \mathcal{P}_{\mu,\nu}\left(Q(x) - \frac{f((2a)^n x)}{(2a)^{2n}} + \frac{f((2a)^n x)}{(2a)^{2n}} - f(x), t\right) \\ &\geq_{L^*} \mathcal{M}\left(\mathcal{P}_{\mu,\nu}\left(Q(x) - \frac{f((2a)^n x)}{(2a)^{2n}}, \frac{t}{2}\right), \mathcal{P}_{\mu,\nu}\left(\frac{f((2a)^n x)}{(2a)^{2n}} - f(x), \frac{t}{2}\right)\right) \\ &\geq_{L^*} \mathcal{M}\left(\mathcal{P}_{\mu,\nu}\left(Q(x) - \frac{f((2a)^n x)}{(2a)^{2n}}, \frac{t}{2}\right), \mathcal{P}'_{\mu,\nu}\left(\varphi(x), \frac{a^2 t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{(2a)^{2k}}}\right)\right).\end{aligned} \tag{57}$$

Taking the limit as $n \to \infty$ in (57) and using (55), we get

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(x, \frac{(4a^2 - \alpha)t}{4}\right)$$

for all $x \in X$ and $t > 0$, which shows that Q satisfies (49). The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof of the theorem. □

Theorem 3.6. *Let $|2a| < 1$ and $f : X \to Y$ be a mapping with $f(0) = 0$. Suppose that φ is a mapping from X to an intuitionistic fuzzy normed space $(Z, \mathcal{P}'_{\mu,\nu}, \mathcal{M})$ satisfying (48). If $\varphi(2ax) = \alpha\varphi(x)$ for some real number α with $\alpha > 4a^2$, then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(x, \frac{(\alpha - 4a^2)t}{4}\right) \tag{58}$$

for all $x \in X$ and $t > 0$.

Proof. It follows from (50) that

$$\mathcal{P}_{\mu,\nu}\left(f(x) - (2a)^2 f\left(\frac{x}{2a}\right), 2t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi\left(\frac{x}{2a}\right), t\right) \tag{59}$$

for all $x \in X$ and all $t > 0$. Thus

$$\mathcal{P}_{\mu,\nu}\left(f(x) - (2a)^2 f\left(\frac{x}{2a}\right), t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi\left(\frac{x}{2a}\right), \frac{t}{2}\right) =_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x), \frac{\alpha}{2}t\right). \tag{60}$$

Replacing x by $\frac{x}{(2a)^n}$ in (60), we have

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}\left((2a)^{2n} f\left(\frac{x}{(2a)^n}\right) - (2a)^{2n+2} f\left(\frac{x}{(2a)^{n+1}}\right), \frac{(2a)^{2n}t}{\alpha^n}\right)\\
&= \mathcal{P}_{\mu,\nu}\left(f\left(\frac{x}{(2a)^n}\right) - 4a^2 f\left(\frac{x}{(2a)^{n+1}}\right), \frac{t}{\alpha^n}\right)\\
&\geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x), \frac{\alpha}{2}t\right).
\end{aligned} \tag{61}$$

For all $x \in X, t > 0$ and all non-negative integers n and m with $n > m$, we have

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}\left((2a)^{2n} f\left(\frac{x}{(2a)^n}\right) - (2a)^{2m} f\left(\frac{x}{(2a)^m}\right), \sum_{k=m}^{n-1} \frac{(2a)^{2k}t}{\alpha^k}\right)\\
&= \mathcal{P}_{\mu,\nu}\left(\sum_{k=m}^{n-1}\left[(2a)^{2k+2} f\left(\frac{x}{(2a)^{k+1}}\right) - (2a)^{2k} f\left(\frac{x}{(2a)^k}\right)\right], \sum_{k=m}^{n-1} \frac{(2a)^{2k}t}{\alpha^k}\right)\\
&\geq_{L^*} \mathcal{M}^{n-m-1}\left(\mathcal{P}_{\mu,\nu}\left((2a)^{2m+2} f\left(\frac{x}{(2a)^{m+1}}\right) - (2a)^{2m} f\left(\frac{x}{(2a)^m}\right), \frac{(2a)^{2m}t}{\alpha^m}\right), \ldots,\right.\\
&\left.\mathcal{P}_{\mu,\nu}\left((2a)^{2n} f\left(\frac{x}{(2a)^n}\right) - (2a)^{2n-2} f\left(\frac{x}{(2a)^{n-1}}\right), \frac{(2a)^{2n-2}t}{\alpha^{n-1}}\right)\right)\\
&\geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x), \frac{\alpha}{2}t\right).
\end{aligned} \tag{62}$$

Hence

$$\mathcal{P}_{\mu,\nu}\left((2a)^{2n} f\left(\frac{x}{(2a)^n}\right) - (2a)^{2m} f\left(\frac{x}{(2a)^m}\right), t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x), \frac{\alpha t}{2\sum_{k=m}^{n-1} \frac{(2a)^{2k}t}{\alpha^k}}\right) \tag{63}$$

for all $x \in X, t > 0$ and $m, n \in \mathbb{N}$ with $n > m$.

Since $\alpha > 4a^2$ and $\sum_{k=0}^{\infty} \frac{(2a)^{2k}}{\alpha^k} < \infty$, then $\{(2a)^{2n} f(\frac{x}{(2a)^n})\}$ is a Cauchy sequence in $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ for each $x \in X$. Since $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{M})$ is an intuitionistic fuzzy Banach space, the sequence converges to some point $Q(x) \in Y$. So we can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} (2a)^{2n} f\left(\frac{x}{(2a)^n}\right) \tag{64}$$

for all $x \in X$. Fix $x \in X$ and set $m = 0$ in (63) to obtain

$$\mathcal{P}_{\mu,\nu}\left((2a)^{2n} f\left(\frac{x}{(2a)^n}\right) - f(x), t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x), \frac{\alpha t}{2\sum_{k=0}^{n-1} \frac{(2a)^{2k} t}{\alpha^k}}\right) \tag{65}$$

for all $x \in X, t > 0$ and so we have that

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) = \mathcal{P}_{\mu,\nu}\left(Q(x) - (2a)^{2n} f\left(\frac{x}{(2a)^n}\right) + (2a)^{2n} f\left(\frac{x}{(2a)^n}\right) - f(x), t\right) \\
&\geq_{L^*} \mathcal{M}\left(\mathcal{P}_{\mu,\nu}\left(Q(x) - (2a)^{2n} f\left(\frac{x}{(2a)^n}\right), \frac{t}{2}\right), \mathcal{P}_{\mu,\nu}\left((2a)^{2n} f\left(\frac{x}{(2a)^n}\right) - f(x), \frac{t}{2}\right)\right) \\
&\geq_{L^*} \mathcal{M}\left(\mathcal{P}_{\mu,\nu}\left(Q(x) - (2a)^{2n} f\left(\frac{x}{(2a)^n}\right), \frac{t}{2}\right), \mathcal{P}'_{\mu,\nu}\left(\varphi(x), \frac{\alpha t}{4\sum_{k=0}^{n-1} \frac{(2a)^{2k} t}{\alpha^k}}\right)\right).
\end{aligned} \tag{66}$$

Taking the limit as $n \to \infty$ in (66) and using (64), we get

$$\mathcal{P}_{\mu,\nu}(Q(x) - f(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(x, \frac{(4a^2 - \alpha)t}{4}\right)$$

for all $x \in X$ and $t > 0$, which shows that Q satisfies (58). The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof of the theorem. □

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