Alternative Proofs of Some Classical Tauberian Theorems for The Weighted Mean Method of Integrals

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Abstract. Let $0 \neq p(x)$ be a nondecreasing real valued differentiable function on $[0, \infty)$ such that $p(0) = 0$ and $p(x) \to \infty$ as $x \to \infty$. Given a real valued function $f(x)$ which is continuous on $[0, \infty)$ and

$$s(x) = \int_0^x f(t)dt.$$ 

We define the weighted mean of $s(x)$ as

$$\sigma_p(x) = \frac{1}{p(x)} \int_0^x p'(t)s(t)dt,$$

where $p'(t)$ is derivative of $p(t)$. It is known that if the limit $\lim_{x \to \infty} s(x)$ exists, then $\lim_{x \to \infty} \sigma_p(x) = s$ also exists. However, the converse is not always true. Adding some suitable conditions to existence of $\lim_{x \to \infty} \sigma_p(x)$ which are called Tauberian conditions may imply convergence of the integral $\int_0^\infty f(t)dt$.

In this work, we give some classical type Tauberian theorems to retrieve convergence of $s(x)$ out of weighted mean integrability of $s(x)$ with some Tauberian conditions.

1. Introduction

Let $0 \neq p(x)$ be a nondecreasing real valued differentiable function on $[0, \infty)$ such that $p(0) = 0$ and $p(x) \to \infty$ as $x \to \infty$. Given a real valued continuous function $f$ on $[0, \infty)$ and $s(x) = \int_0^x f(t)dt$. The weighted mean of $s(x)$ is defined by

$$\sigma_p(x) = \frac{1}{p(x)} \int_0^x s(t)dp(t) = \frac{1}{p(x)} \int_0^x p'(t)s(t)dt.$$ 

The integral

$$\int_0^\infty f(t)dt$$

2010 Mathematics Subject Classification. Primary 40E05; Secondary 40A10

Keywords. Tauberian theorem, Tauberian condition, weighted mean, integral method, slowly oscillating function, slowly decreasing function, one-sided condition

Received: 01 August 2014; Accepted: 14 October 2014

Communicated by Dragana Ćvetković-Ilič

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is said to be integrable by weighted mean method determined by the function \( p(x) \), in short; \((N, p)\) integrable to a finite number \( s \) if

\[
\lim_{x \to \infty} \sigma_p(x) = s. \tag{1}
\]

If \( p(x) = x \) in the definition, then the \((N, p)\) integrability method reduces to Cesàro integrability method. If the integral

\[
\int_0^\infty f(t)dt = s \tag{2}
\]

exists, then limit (1) also exists. However, the converse is not always true. For example, \( \lim_{x \to \infty} \int_0^x \cos t dt \) does not exist. Also, by a special case choosing \( p(x) = x^2 \), from

\[
\sigma_p(x) = \frac{1}{p(x)} \int_0^x s(t)dp(t) = \frac{1}{p(x)} \int_0^x \left( \int_0^t f(u)du \right)dp(t) \\
= \frac{1}{p(x)} \int_0^x f(t) \left( \int_t^\infty dp(t) \right)dt \\
= \frac{1}{p(x)} \int_0^x (p(x) - p(t))f(t)dt \\
= \int_0^\infty (1 - \frac{p(t)}{p(x)})f(t)dt
\]

it follows that

\[
\lim_{x \to \infty} \sigma_p(x) = \lim_{x \to \infty} \int_0^\infty (1 - \frac{t^2}{x^2}) \cos t dt = 0.
\]

Notice that (1) may imply (2) by adding some suitable conditions on \( s(x) \). Such a condition is called a Tauberian condition and resulting theorem is said to be a Tauberian theorem.

The weighted De la Vallée Pousson means of \( s(x) \) are defined by

\[
\tau_p^\to(x) = \frac{1}{p(\lambda x) - p(x)} \int_x^{\lambda x} p'(t)s(t)dt
\]

for \( \lambda > 1 \), and

\[
\tau_p<^\to(x) = \frac{1}{p(x) - p(\lambda x)} \int_{\lambda x}^x p'(t)s(t)dt
\]

for \( 0 < \lambda < 1 \).

The concept of slowly decreasing for a sequence of real numbers was introduced by Schmidt [9]. Similarly, we can define for a real function.

A function \( s(x) \) is said to be slowly decreasing if

\[
\lim_{\lambda \to 1} \lim_{x \to \infty} \min_{x \leq t \leq \lambda x} (s(t) - s(x)) \geq 0 \tag{3}
\]

for \( \lambda > 1 \). The condition (3) can be equivalently reformulated as follows:

\[
\lim_{\lambda \to 1} \lim_{x \to \infty} \min_{\lambda x \leq t \leq x} (s(x) - s(t)) \geq 0 \tag{4}
\]

for \( 0 < \lambda < 1 \).

If the functions \( s(x) \) and \(-s(x)\) are slowly decreasing, then \( s(x) \) is slowly oscillating. An equivalent definition of slow oscillation is given as follows:
A real valued function \( s(x) \) is slowly oscillating \([1]\) if
\[
\lim_{\lambda \to 1^+} \limsup_{x \to \infty} \max_{x \leq t \leq \lambda x} |s(t) - s(x)| = 0,
\]
for \( \lambda > 1 \).

In \([1–4, 7]\), a number of authors presented some Tauberian theorems for Cesàro integrability method. Also, Çanak and Totur \([8]\) obtained a Tauberian condition, known as the Landau’s condition
\[
\frac{p(x)}{p'(x)} f(x) = O(1)
\]
(see \([6]\)), for weighted mean integrability order \( \alpha \), for some \( \alpha > -1 \).

In this paper, we establish that one-sided boundedness of the function \( \frac{p(x)}{p'(x)} f(x) \) is a Tauberian condition for weighted mean integrability. Furthermore, we prove that slow decrease of \( s(x) \) is a Tauber condition for weighted mean integrability.

2. Main Results

The results are some classical type Tauberian theorems for the weighted mean method of integrals.

Theorem 2.1. Let
\[
\liminf_{x \to \infty} \frac{p(\lambda x)}{p(x)} > 1, \text{ for } \lambda > 1,
\]
and
\[
\limsup_{x \to \infty} \frac{p(x)}{p(\lambda x)} > 1, \text{ for } 0 < \lambda < 1.
\]
If \( \int_0^\infty f(t)dt \) is \((\overline{N}, p)\) integrable to \( s \) and
\[
\frac{p(x)}{p'(x)} f(x) \geq -C,
\]
for some \( C \geq 0 \) and enough large \( x \), then the integral \( \int_0^\infty f(t)dt \) converges to \( s \).

Theorem 2.1 is a classical type Tauberian theorem known as the Hardy Littlewood’s Tauberian theorem \([5]\). A special case of Theorem 2.1 can be obtained by choosing \( p(x) = x \) as follows:

Corollary 2.2. If \( \int_0^\infty f(t)dt \) be Cesàro integrable to \( s \). If \( xf(x) \geq -C \) for some \( C \geq 0 \) and enough large \( x \), then the integral \( \int_0^\infty f(t)dt \) converges to \( s \).

Corollary 2.2 is given by Çanak and Totur \([3]\).

The following theorem is a version of the generalized Littlewood theorem \([9]\) for real functions.

Theorem 2.3. Let the conditions (6) and (7) be satisfied. If \( \int_0^\infty f(t)dt \) is \((\overline{N}, p)\) integrable to \( s \) and \( s(x) \) is slowly decreasing, then the integral \( \int_0^\infty f(t)dt \) converges to \( s \).

An obvious corollary of Theorem 2.3 is represented as follows:

Corollary 2.4. Let the conditions (6) and (7) be satisfied. If \( \int_0^\infty f(t)dt \) is \((\overline{N}, p)\) integrable to \( s \) and \( s(x) \) is slowly oscillating, then the integral \( \int_0^\infty f(t)dt \) converges to \( s \).

A special case of Theorem 2.3 can be obtained by choosing \( p(x) = x \).

Corollary 2.5. If \( \int_0^\infty f(t)dt \) is Cesàro integrable to \( s \) and \( s(x) \) is slowly oscillating, then the integral \( \int_0^\infty f(t)dt \) converges to \( s \).

Corollary 2.5 is given by Çanak and Totur \([3]\).
3. Proofs

We need the following lemma to be used in the proofs of main theorems.

Lemma 3.1. (i) For $\lambda > 1$,

$$s(x) - \sigma_p(x) = \frac{p(\lambda x)}{p(\lambda x) - p(x)} \left( \sigma_p(\lambda x) - \sigma_p(x) \right) - \frac{1}{p(\lambda x) - p(x)} \int_x^{\lambda x} p'(t)(s(t) - s(x))dt$$

(ii) For $0 < \lambda < 1$,

$$s(x) - \sigma_p(x) = \frac{p(\lambda x)}{p(x) - p(\lambda x)} \left( \sigma_p(x) - \sigma_p(\lambda x) \right) + \frac{1}{p(x) - p(\lambda x)} \int_x^{\lambda x} p'(t)(s(x) - s(t))dt$$

Proof. (i) From the definition of weighted de la Vallée Poussin means of $s(x)$, we have

$$s(x) = \tau_p^\gamma(x) - \frac{1}{p(\lambda x) - p(x)} \int_x^{\lambda x} p'(t)(s(t) - s(x))dt.$$  

(8)

Subtracting $\sigma_p(x)$ from the identity (8), we get

$$s(x) - \sigma_p(x) = \tau_p^\gamma(x) - \sigma_p(x) - \frac{1}{p(\lambda x) - p(x)} \int_x^{\lambda x} p'(t)(s(t) - s(x))dt.$$  

(9)

Also $\tau_p^\gamma(x)$ can be written as

$$\tau_p^\gamma(x) = \frac{1}{p(\lambda x) - p(x)} \left( \int_0^{\lambda x} p'(t)s(t)dt - \int_0^x p'(t)s(t)dt \right)$$

$$= \frac{1}{p(\lambda x) - p(x)} \left( \sigma_p(\lambda x)p(\lambda x) - \sigma_p(x)p(x) \right)$$

$$= \frac{p(\lambda x)}{p(\lambda x) - p(x)} \sigma_p(\lambda x) - \frac{p(x)}{p(\lambda x) - p(x)} \sigma_p(x).$$

Therefore, we have

$$\tau_p^\gamma(x) = \frac{p(\lambda x)}{p(\lambda x) - p(x)} \sigma_p(\lambda x) - \left( \frac{p(\lambda x)}{p(\lambda x) - p(x)} - 1 \right) \sigma_p(x).$$

Subtracting $\sigma_p(x)$ from the last identity, we get

$$\tau_p^\gamma(x) - \sigma_p(x) = \frac{p(\lambda x)}{p(\lambda x) - p(x)} \sigma_p(\lambda x) - \frac{p(\lambda x)}{p(\lambda x) - p(x)} \sigma_p(x)$$

Writing last identity in (9), we obtain

$$s(x) - \sigma_p(x) = \frac{p(\lambda x)}{p(\lambda x) - p(x)} \left( \sigma_p(\lambda x) - \sigma_p(x) \right) - \frac{1}{p(\lambda x) - p(x)} \int_x^{\lambda x} p'(t)(s(t) - s(x))dt.$$  

This completes the proof.

(ii) The proof of Lemma 3.1(ii) is similar to that of Lemma 3.1(i).
Proof of Theorem 2.1

Suppose that \( \frac{p(lx)}{p(x)} f(x) \geq -C \) for some \( C \geq 0 \). Then, we obtain \( -s'(x) \leq C \frac{p'(x)}{p(x)} \) for all \( x \). From Lemma 3.1 (i), we have

\[
\begin{align*}
s(x) - \sigma_p(x) &= p(\lambda x) \frac{(\sigma_p(\lambda x) - \sigma_p(x)) - 1}{p(\lambda x) - p(x)} \int_s^{\lambda x} p'(t)(s(t) - s(x)) dt \\
&= \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(\lambda x) - \sigma_p(x)) - \frac{1}{p(\lambda x) - p(x)} \int_s^{\lambda x} \left( \int_t^s s'(z) dz \right) p'(t) dt \\
&\leq \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(\lambda x) - \sigma_p(x)) + \frac{1}{p(\lambda x) - p(x)} \int_s^{\lambda x} \left( \int_t^s \frac{Cp'(z)}{p(z)} dz \right) p'(t) dt \\
&= \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(\lambda x) - \sigma_p(x)) + C \log \frac{p(\lambda x)}{p(x)},
\end{align*}
\]

for \( \lambda > 1 \).

After taking lim sup of both sides as \( x \to \infty \), we obtain

\[
\limsup_{x \to \infty} \left( s(x) - \sigma_p(x) \right) \leq \limsup_{x \to \infty} \left( \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(\lambda x) - \sigma_p(x)) + C \log \frac{p(\lambda x)}{p(x)} \right)
\]

\[
\leq \limsup_{x \to \infty} \frac{p(\lambda x)}{p(\lambda x) - p(x)} \limsup_{x \to \infty} (\sigma_p(\lambda x) - \sigma_p(x)) \leq \limsup_{x \to \infty} \left( C \log \frac{p(\lambda x)}{p(x)} \right).
\]

Since \( s(x) \) is weighted mean integrable to \( s \), we have \( \sigma_p(x) \to s \) as \( x \to \infty \). By the condition (6), we get

\[
0 \leq \limsup_{x \to \infty} \frac{p(\lambda x)}{p(\lambda x) - p(x)} \leq 1 + (\liminf_{x \to \infty} \frac{p(\lambda x)}{p(x)} - 1)^{-1} < \infty.
\]

Therefore the first term on the right-hand side of the inequality above vanishes and we obtain

\[
\limsup_{x \to \infty} \left( s(x) - \sigma_p(x) \right) \leq \limsup_{x \to \infty} \left( C \log \frac{p(\lambda x)}{p(x)} \right),
\]

for some \( C > 0 \). After taking the limit of both sides as \( \lambda \to 1^+ \), we get

\[
\limsup_{x \to \infty} \left( s(x) - \sigma_p(x) \right) \leq 0. \tag{10}
\]

From Lemma 3.1 (ii) and the hypothesis \( -s'(x) \leq C \frac{p'(x)}{p(x)} \) for all \( x \), we have

\[
\begin{align*}
s(x) - \sigma_p(x) &= \frac{p(\lambda x)}{p(x) - p(\lambda x)} (\sigma_p(\lambda x) - \sigma_p(x)) + \frac{1}{p(x) - p(\lambda x)} \int_s^x p'(t)(s(t) - s(x)) dt \\
&= \frac{p(\lambda x)}{p(x) - p(\lambda x)} (\sigma_p(\lambda x) - \sigma_p(x)) + \frac{1}{p(x) - p(\lambda x)} \int_s^x \left( \int_t^x s'(z) dz \right) p'(t) dt \\
&\geq \frac{p(\lambda x)}{p(x) - p(\lambda x)} (\sigma_p(\lambda x) - \sigma_p(x)) - \frac{1}{p(x) - p(\lambda x)} \int_s^x \left( \int_t^x \frac{Cp'(z)}{p(z)} dz \right) p'(t) dt \\
&= \frac{p(\lambda x)}{p(x) - p(\lambda x)} (\sigma_p(\lambda x) - \sigma_p(x)) - \frac{C}{p(x) - p(\lambda x)} \int_s^x \log \frac{p(t)}{p(x)} p'(t) dt \\
&\geq \frac{p(\lambda x)}{p(x) - p(\lambda x)} (\sigma_p(\lambda x) - \sigma_p(x)) - C \log \frac{p(\lambda x)}{p(x)}.
\end{align*}
\]
After taking lim inf of both sides as $x \to \infty$, we have
\[
\liminf_{x \to \infty} (s(x) - \sigma_p(x)) \geq \liminf_{x \to \infty} \left( \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(\lambda x) - \sigma_p(x)) - C \log \frac{p(\lambda x)}{p(x)} \right) \\
\geq \liminf_{x \to \infty} \frac{p(\lambda x)}{p(\lambda x) - p(x)} \liminf_{x \to \infty} (\sigma_p(x) - \sigma_p(\lambda x)) + \liminf_{x \to \infty} \left( -C \log \frac{p(\lambda x)}{p(x)} \right).
\]

By the condition (7), we have
\[
0 \leq \liminf_{x \to \infty} \frac{p(\lambda x)}{p(\lambda x) - p(x)} = (\limsup_{x \to \infty} \frac{p(x)}{p(\lambda x)})^{-1} < \infty.
\]

From $\sigma_p(x) \to s$ as $x \to \infty$, the first term on the right-hand side of the equality above vanishes and we obtain
\[
\liminf_{x \to \infty} (s(x) - \sigma_p(\lambda x)) \geq \liminf_{x \to \infty} \left( -C \log \frac{p(\lambda x)}{p(x)} \right).
\]
for some $C > 0$. After taking the limit of both sides as $\lambda \to 1^-$, we get
\[
\liminf_{x \to \infty} (s(x) - \sigma_p(x)) \geq 0.
\]
\[\text{(11)}\]

From (10) and (11), we obtain $\lim s(x) = \lim s_p(x)$. □

**Proof of Theorem 2.3**

Let $s(x)$ be slowly decreasing. By Lemma 3.1 (i), we have
\[
s(x) - \sigma_p(x) = \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(\lambda x) - \sigma_p(x)) - \frac{1}{p(\lambda x) - p(x)} \int_x^{\lambda x} p'(t)(s(t) - s(x)) dt \\
\leq \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(\lambda x) - \sigma_p(x)) - \frac{1}{p(\lambda x) - p(x)} \int_x^{\lambda x} p'(t) \min_{s(t) \leq s(x)} (s(t) - s(x)) dt \\
\leq \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(\lambda x) - \sigma_p(x)) - \min_{s(t) \leq s(x)} (s(t) - s(x))
\]

After taking lim sup of both sides as $x \to \infty$, we have
\[
\limsup_{x \to \infty} (s(x) - \sigma_p(x)) \leq \limsup_{x \to \infty} \left( \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(\lambda x) - \sigma_p(x)) - \min_{s(t) \leq s(x)} (s(t) - s(x)) \right) \\
\leq \limsup_{x \to \infty} \frac{p(\lambda x)}{p(\lambda x) - p(x)} \limsup_{x \to \infty} (\sigma_p(\lambda x) - \sigma_p(x)) + \limsup_{x \to \infty} \left( -\min_{s(t) \leq s(x)} (s(t) - s(x)) \right)
\]

Since $\sigma_p(x) \to s$ as $x \to \infty$, by the condition (6), the first term on the right-hand side of the equality above vanishes and we obtain
\[
\limsup_{x \to \infty} (s(x) - \sigma_p(x)) \leq -\liminf_{x \to \infty} \min_{s(t) \leq s(x)} (s(t) - s(x))
\]

After taking the limit of both sides as $\lambda \to 1^-$, we get
\[
\limsup_{x \to \infty} (s(x) - \sigma_p(x)) \leq 0.
\]
\[\text{(12)}\]

On the other hand, from Lemma 3.1 (ii), we have
\[
s(x) - \sigma_p(x) = \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(x) - \sigma_p(\lambda x)) + \frac{1}{p(\lambda x) - p(x)} \int_{\lambda x}^{x} p'(t)(s(t) - s(t)) dt \\
\geq \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(x) - \sigma_p(\lambda x)) + \frac{1}{p(\lambda x) - p(x)} \int_{\lambda x}^{x} p'(t) \min_{s(t) \leq s(x)} (s(t) - s(t)) dt \\
\geq \frac{p(\lambda x)}{p(\lambda x) - p(x)} (\sigma_p(x) - \sigma_p(\lambda x)) + \min_{s(t) \leq s(x)} (s(t) - s(t))
\]
After taking $\lim \inf$ of both sides as $x \to \infty$, we have

$$
\liminf_{x \to \infty} (s(x) - \sigma_p(x)) \geq \liminf_{x \to \infty} \left( \frac{p(\lambda x)}{p(x) - p(\lambda x)} (\sigma_p(x) - \sigma_p(\lambda x)) + \min_{\lambda x \leq t \leq x} (s(x) - s(t)) \right)
$$

$$
\geq \liminf_{x \to \infty} \left( \frac{p(\lambda x)}{p(x) - p(\lambda x)} (\sigma_p(x) - \sigma_p(\lambda x)) \right) + \liminf_{x \to \infty} \left( \min_{\lambda x \leq t \leq x} (s(x) - s(t)) \right)
$$

Since $\sigma_p(x) \to s$ as $x \to \infty$, by the condition (7), the first term on the right-hand side of the equality above vanishes and we obtain

$$
\liminf_{x \to \infty} (s(x) - \sigma_p(x)) \geq \liminf_{x \to \infty} \left( \min_{\lambda x \leq t \leq x} (s(x) - s(t)) \right)
$$

After taking the limit of both sides as $\lambda \to 1^-$, we get

$$
\liminf_{x \to \infty} (s(x) - \sigma_p(x)) \geq 0.
$$

(13)

Combining (12) and (13), we have $\lim_{x \to \infty} s(x) = \lim_{x \to \infty} \sigma_p(x)$. □

References