A “q-deformed” Generalization of the Hosszú-Gluskin Theorem

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Abstract. In this paper a new form of the Hosszú-Gluskin theorem is presented in terms of polyadic powers and using the language of diagrams. It is shown that the Hosszú-Gluskin chain formula is not unique and can be generalized (“deformed”) using a parameter q which takes special integer values. A version of the “q-deformed” analog of the Hosszú-Gluskin theorem in the form of an invariance is formulated, and some examples are considered. The “q-deformed” homomorphism theorem is also given.

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1. Introduction

Since the early days of “polyadic history” [1–3], the interconnection between polyadic systems and binary ones has been one of the main areas of interest [4, 5]. Early constructions were confined to building some special polyadic (mostly ternary [6, 7]) operations on elements of binary groups [8–10]. A very special form of n-ary multiplication in terms of binary multiplication and a special mapping as a chain formula was found in [11] and [12, 13]. The theorem that any n-ary multiplication can be presented in this form is called the Hosszú-Gluskin theorem (for review see [14, 15]). A concise and clear proof of the Hosszú-Gluskin chain formula was presented in [16].

In this paper we give a new form of the Hosszú-Gluskin theorem in terms of polyadic powers. Then we show that the Hosszú-Gluskin chain formula is not unique and can be generalized (“deformed”) using a parameter q which takes special integer values. We present the “q-deformed” analog of the Hosszú-Gluskin
2. Preliminaries

We will use the concise notations from our previous review paper [17], while here we repeat some necessary definitions using the language of diagrams. For a non-empty set \( G \), we denote its elements by lower-case Latin letters \( g_i \in G \) and the \( n \)-tuple (or polyad) \( g_1, \ldots, g_n \) will be written by \((g_1, \ldots, g_n)\) or using one bold letter with index \( g^{(n)} \), and an \( n \)-tuple with equal elements by \( g^n \). In case the number of elements in the \( n \)-tuple is clear from the context or is not important, we denote it in one bold letter \( g \) without indices. We omit \( g \in G \), if it is obvious from the context.

The Cartesian product \( G \times \cdots \times G = G^n \) consists of all \( n \)-tuples \((g_1, \ldots, g_n)\), such that \( g_i \in G \), \( i = 1, \ldots, n \). The \( i \)-projection of the Cartesian product \( G^n \) on its \( i \)-th “axis” is the map \( \text{Pr}_{i}^{(n)} : G^n \to G \) such that \((g_1, \ldots, g_i, \ldots, g_n) \mapsto g_i \). The \( i \)-diagonal \( \text{Diag}_i : G \to G^n \) sends one element to the equal \( n \)-tuple \( g \mapsto (g^n) \). The one-point set \( \{\bullet\} \) is treated as a unit for the Cartesian product, since there are bijections between \( G \) and \( G \times \{\bullet\}^n \), where \( G \) can be on any place. In diagrams, if the place is unimportant, we denote such bijections by \( c \). On the Cartesian product \( G^n \) one can define a polyadic (\( n \)-ary or \( n \)-adic, if it is necessary to specify \( n \), its arity or rank) operation \( \mu_n : G^n \to G \). For operations we use small Greek letters and place arguments in square brackets \( \mu_n[g] \). The operations with \( n = 1, 2, 3 \) are called unary, binary and ternary. The case \( n = 0 \) is special and corresponds to fixing a distinguished element of \( G \), a “constant” \( c \in G \), and it is called a 0-ary operation \( \mu_n^{(0)} \), which maps the one-point set \( \{\bullet\} \) to \( G \), such that \( \mu_n^{(0)} : \{\bullet\} \to G \), and (formally) has the value \( \mu_n^{(0)}[\{\bullet\}] = c \in G \). The composition of \( n \)-ary and \( m \)-ary operations \( \mu_n \circ \mu_m \) gives a \((n + m - 1)\)-ary operation by the iteration \( \mu_{n+m-1}[g_0, h] = \mu_n[g_0, \mu_m[h]] \). If we compose \( \mu_n \) with the 0-ary operation \( \mu_n^{(0)} \), then we obtain the arity “collapsing” \( \mu_{n-1}^{(0)}[g] = c \), because \( g \) is a polyad of length \((n-1)\). A universal algebra is a set which is closed under several polyadic operations [18]. If a concrete universal algebra has one fundamental \( n \)-ary operation, called a polyadic multiplication (or \( n \)-ary multiplication) \( \mu_n \), we name it a “polyadic system” [1].

Definition 2.1. A polyadic system \( G = \langle \text{set|one fundamental operation} \rangle \) is a set \( G \) which is closed under polyadic multiplication.

More specifically, a \( n \)-ary system \( G_n = \langle G \mid \mu_n \rangle \) is a set \( G \) closed under one \( n \)-ary operation \( \mu_n \) (without any other additional structure).

For a given \( n \)-ary system \( \langle G \mid \mu_n \rangle \) one can construct another polyadic system \( \langle G \mid \mu_n' \rangle \) over the same set \( G \), but with another multiplication \( \mu_n' \) of different arity \( n' \). In general, there are three ways of changing the arity:

1. Iterating. Composition of the operation \( \mu_n \) with itself increases the arity from \( n \) to \( n' = n_{\text{iter}} > n \). We denote the number of iterating multiplications by \( \ell_n \) and call the resulting composition an iterated product\(^3\) \( \mu_n^{\ell_n} \) (using the bold Greek letters) as \( \mu_n^{\ell_n} \) if \( \ell_n \) is obvious or not important

\[
\mu_n^{\ell_n} = \mu_n^\ell \overset{\text{def}}{=} \mu_n \circ \cdots \circ \left( \mu_n \times \text{id}^{(n-1)} \right) \cdots \times \text{id}^{(n-1)} ,
\]

(2.1)

\(^3\)A set with one closed binary operation without any other relations was called a groupoid by Hausmann and Ore [19] (see, also [20]). Nowadays the term “groupoid” is widely used in the category theory and homotopy theory for a different construction, the so-called Brandt groupoid [21]. Bourbaki [22] introduced the term “magma”. To avoid misreading we will use the neutral notation “polyadic system”.

\(^4\)Sometimes \( \mu_n^{\ell_n} \) is named a long product [3].
where the final arity is
\[ n' = n_{\text{iter}} = \ell_\mu (n - 1) + 1. \] (2.2)

There are many variants of placing \( \mu_n \)'s among id's in the r.h.s. of (2.1), if no associativity is assumed. An example of the iterated product can be given for a ternary operation \( \mu_3 (n = 3) \), where we can construct a 7-ary operation \( (n' = 7) \) by \( \ell_\mu = 3 \) compositions
\[ \mu_7^* [g_1, \ldots, g_7] = \mu_3 [\mu_3 [g_1, g_2, g_3], g_4, g_5, g_6, g_7], \] (2.3)

and the corresponding commutative diagram is

\[ G^7 \xrightarrow{\mu \times \text{id}^{\times 4}} G^5 \xrightarrow{\mu \times \text{id}^{\times 2}} G^3 \xrightarrow{\mu_3} G \]

(2.4)

In the general case, the horizontal part of the (iterating) diagram (2.4) consists of \( \ell_\mu \) terms.

2. Reducing (Collapsing). To decrease arity from \( n \) to \( n' = n_{\text{red}} < n \) one can use \( n_c \) distinguished elements ("constants") as additional 0-ary operations \( \mu_0^{(c)} \), \( i = 1, \ldots, n_c \), such that\(^3\) the reduced product is defined by
\[ \mu_{n'}^{(c)} = \mu_n^{(c_1, \ldots, c_{n_c})} \overset{\text{def}}{=} \mu_n \circ \left\{ \mu_0^{(c_1)} \times \ldots \times \mu_0^{(c_{n_c})} \times \text{id}^{\times (n-n_c)} \right\}, \] (2.5)

where
\[ n' = n_{\text{red}} = n - n_c, \] (2.6)

and the 0-ary operations \( \mu_0^{(c)} \) can be on any places in (2.5). For instance, if we compose \( \mu_n \) with the 0-ary operation \( \mu_0^{(c)} \), we obtain
\[ \mu_{n-1}^{(c)} [g] = \mu_n [g, c], \] (2.7)

and this reduced product is described by the commutative diagram

\[ G^{n-1} \times \{ \star \} \xrightarrow{\text{id}^{\times (n-1)} \times \mu_0^{(c)}} G^{\times n} \]

(2.8)

which can be treated as a definition of a new \((n - 1)\)-ary operation \( \mu_{n-1}^{(c)} = \mu_n \circ \mu_0^{(c)} \).

3. Mixing. Changing (increasing or decreasing) arity by combining the iterating and reducing (collapsing) methods.

\(^3\)In [23] \( \mu_n^{(c_1, \ldots, c_{n_c})} \) is called a retract, which is already a busy and widely used term in category theory for another construction.
Example 2.2. If the initial multiplication is binary $\mu_2 = (\cdot)$, and there is one 0-ary operation $\mu_0^{(c)}$, we can construct the following mixing operation
\[
\mu_n^{(c)}[g_1, \ldots, g_n] = g_1 \cdot g_2 \cdot \ldots \cdot g_n \cdot c,
\] (2.9)
which in our notation can be called a c-iterated multiplication. \footnote{According to [24] the operation (2.9) can be called $c$-derived.}

Let us recall some special elements of polyadic systems. A positive power of an element (according to Post [4]) coincides with the number of multiplications $\ell_{\mu}$ in the iteration (2.1).

Definition 2.3. A (positive) polyadic power of an element is
\[
\mu_n^{(\ell_{\mu})} = \mu_n^{(0)} [g_1^{(n-1)+1}],
\] (2.10)

Example 2.4. Let us consider a polyadic version of the binary $q$-addition which appears in study of nonextensive statistics (see, e.g., [25, 26])
\[
\mu_n^{(g)} = \sum_{i=1}^{n} g_i + h \prod_{i=1}^{n} g_i,
\] (2.11)
where $g_i \in \mathbb{C}$ and $h = 1 - q_0$, $q_0$ is a real constant (we put here $q_0 \neq 1$ or $h \neq 0$). It is obvious that $g^{(0)} = g$, and
\[
g^{(1)} = \mu_n^{(0)} [g^{(0)}, g^{(0)}] = ng + hg^n.
\] (2.12)
So we have the following recurrence formula
\[
g^{(k)} = \mu_n^{(0)} [g^{(k-1)}, g^{(k-1)}] = (n-1) g + (1 + hg^{n-1})g^{(k-1)},
\] (2.13)
Solving this for an arbitrary polyadic power we get
\[
g^{(k)} = g \left(1 + \frac{n-1}{h} g^{1-n} \right) \left(1 + hg^{n-1} \right)^k - \frac{n-1}{h} g^{2-n}.
\] (2.14)

Definition 2.5. A polyadic ($n$-ary) identity (or neutral element) of a polyadic system is a distinguished element $\varepsilon$ (and the corresponding 0-ary operation $\mu_0^{(\varepsilon)}$) such that for any element $g \in G$ we have [27]
\[
\mu_n^{(\varepsilon)} [g, \varepsilon^{n-1}] = g,
\] (2.15)
where $g$ can be on any place in the l.h.s. of (2.15).

In polyadic systems, for an element $g$ there can exist many neutral polyads $n \in \mathbb{C}^{(n-1)}$ satisfying
\[
\mu_n^{(g, n)} = g,
\] (2.16)
where $g$ may be on any place. The neutral polyads are not determined uniquely. It follows from (2.15) and (2.16) that $\varepsilon^{n-1}$ is a neutral polyad.

Definition 2.6. An element of a polyadic system $g$ is called $\ell_{\mu}$-idempotent, if there exist such $\ell_{\mu}$ that
\[
g^{(\ell_{\mu})} = g.
\] (2.17)
It is obvious that an identity is $\ell_n$-idempotent with arbitrary $\ell_n$. We define (total) associativity as invariance of the composition of two $n$-ary multiplications

$$\mu_n^2 [g, h, u] = \text{invariant} \quad (2.18)$$

under placement of the internal multiplication in the r.h.s. with a fixed order of elements in the whole polyad of $(2n - 1)$ elements $t^{(2n-1)} = (g, h, u)$. Informally, “internal brackets/multiplication can be moved on any place”, which gives

$$\mu_n \circ \left( \mu_n \times \text{id}^{\times(n-1)} \right) = \mu_n \circ \left( \text{id} \times \mu_n \times \text{id}^{\times(n-2)} \right) = \ldots = \mu_n \circ \left( \text{id}^{\times(n-1)} \times \mu_n \right), \quad (2.19)$$

where the internal $\mu_n$ can be on any place $i = 1, \ldots, n$. There are many other particular kinds of associativity which were introduced in [4, 28] and studied in [29, 30] (see, also [31]). Here we will confine ourselves to the most general, total associativity (2.18).

**Definition 2.7.** A polyadic semigroup $(n$-ary semigroup) is a $n$-ary system whose operation is associative, or $G_{\text{semigrp}} = \langle G \mid \mu_n \mid \text{associativity} (2.18) \rangle$.

In general, it is very important to find the associativity preserving conditions, when an associative initial operation $\mu_n$ leads to an associative final operation $\mu_n'$ while changing the arity (by iterating (2.1) or reducing (2.5)).

**Example 2.8.** An associativity preserving reduction can be given by the construction of a binary associative operation using a $(n - 2)$-tuple $c$ as

$$\mu_n^2 [g, h] = \mu_n [g, c, h]. \quad (2.20)$$

The associativity preserving mixing constructions with different arities and places were considered in [23, 30, 32].

In polyadic systems, there are several analogs of binary commutativity. The most straightforward one comes from commutation of the multiplication with permutations.

**Definition 2.9.** A polyadic system is $\sigma$-commutative, if $\mu_n = \mu_n \circ \sigma$, where $\sigma$ is a fixed element of $S_n$, the permutation group on $n$ elements. If this holds for all $\sigma \in S_n$, then a polyadic system is commutative.

A special type of the $\sigma$-commutativity

$$\mu_n [g, t, h] = \mu_n [h, t, g] \quad (2.21)$$

is called semicommutativity. So for a $n$-ary semicommutative system we have

$$\mu_n [g, h^{n-1}] = \mu_n [h^{n-1}, g]. \quad (2.22)$$

If a $n$-ary semigroup $G_{\text{semigrp}}$ is iterated from a commutative binary semigroup with identity, then $G_{\text{semigrp}}$ is semicommutative. Another possibility is to generalize the binary mediality in semigroups

$$(g_{11} \cdot g_{12}) \cdot (g_{21} \cdot g_{22}) = (g_{11} \cdot g_{21}) \cdot (g_{12} \cdot g_{22}), \quad (2.23)$$

which follows from the binary commutativity. For $n$-ary systems, it is seen that the mediality should contain $(n + 1)$ multiplications, that it is a relation between $n \times n$ elements, and therefore that it can be presented in a matrix from.

**Definition 2.10.** A polyadic system is medial (or entropic), if [33, 34]

$$\mu_n \left[ \begin{array}{c} \mu_n [g_{11}, \ldots, g_{1n}] \\ \vdots \\ \mu_n [g_{n1}, \ldots, g_{nm}] \end{array} \right] = \mu_n \left[ \begin{array}{c} \mu_n [g_{11}, \ldots, g_{n1}] \\ \vdots \\ \mu_n [g_{1n}, \ldots, g_{mn}] \end{array} \right]. \quad (2.24)$$
In the case of polyadic semigroups we use the notation (2.1) and can present the mediality as follows

\[ \mu_n^\alpha [G] = \mu_n^\alpha [G^T], \]  

(2.25)

where \( G = [g_{ij}] \) is the \( n \times n \) matrix of elements and \( G^T \) is its transpose.

The semicommutative polyadic semigroups are medial, as in the binary case, but, in general (except \( n = 3 \)) not vice versa [35].

**Definition 2.11.** A polyadic system is cancellative, if

\[ \mu_n [g, t] = \mu_n [h, t] \implies g = h, \]  

(2.26)

where \( g, h \) can be on any place. This means that the mapping \( \mu_n \) is one-to-one in each variable.

If \( g, h \) are on the same \( i \)-th place on both sides of (2.26), the polyadic system is called \( i \)-cancellative. The left and right cancellativity are 1-cancellativity and \( n \)-cancellativity respectively. A right and left cancellative \( n \)-ary semigroup is cancellative (with respect to the same subset).

**Definition 2.12.** A polyadic system is called (uniquely) \( i \)-solvable, if for all polyads \( t, u \) and element \( h \), one can (uniquely) resolve the equation (with respect to \( h \)) for the fundamental operation

\[ \mu_n [u, h, t] = \]  

(2.27)

where \( h \) can be on any \( i \)-th place.

**Definition 2.13.** A polyadic system which is uniquely \( i \)-solvable for all places \( i = 1, \ldots, n \) in (2.27) is called a \( n \)-ary (or polyadic) quasigroup.

It follows, that, if (2.27) uniquely \( i \)-solvable for all places, then

\[ \mu_n^{t_i} [u, h, t] = g \]  

(2.28)

can be (uniquely) resolved with respect to \( h \) being on any place.

**Definition 2.14.** An associative polyadic quasigroup is called a \( n \)-ary (or polyadic) group.

In a polyadic group the only solution of (2.27) is called a querelement\(^5\) of \( g \) and is denoted by \( \overline{g} \) [3], such that

\[ \mu_n [h, g] = g, \]  

(2.29)

where \( g \) can be on any place. Obviously, any idempotent \( g \) coincides with its querelement \( \overline{g} = g \).

**Example 2.15.** For the \( q \)-addition (2.11) from Example 2.4, using (2.29) with \( h = g^{n-1} \) we obtain

\[ g = \frac{(n - 2) g}{1 + h g^{n-1}}. \]  

(2.30)

\(^5\)We use the original notation after [3] and do not use “skew element”, because it can be confused with the wide usage of “skew” in other, different senses.
It follows from (2.29) and (2.16), that the polyad
\[ n_{(\bar{g})} = (g^{n-2}, \bar{g}) \]  
(2.31)
is neutral for any element \( g \), where \( \bar{g} \) can be on any place. If this \( i \)-th place is important, then we write \( n_{(\bar{g})_{i}} \).

More generally, because any neutral polyad plays a role of identity (see (2.16)), for any element \( g \) we define its **polyadic inverse** (the sequence of length \((n-2)\) denoted by the same letter \( g^{-1} \) in bold) as (see [4] and by modified analogy with [15, 36])
\[ n_{(\bar{g})} = (g^{-1}, g) = (g, g^{-1}) \]  
(2.32)
which can be written in terms of the multiplication as
\[ \mu_n[g^{-1}, g] = \mu_n[h, g^{-1}, g] = h \]  
(2.33)
for all \( h \) in \( G \). It is obvious that the polyads
\[ n_{(\bar{g})} = (g^{n-2}, \bar{g}) = (g^{n-2}, g) \]  
(2.34)
are neutral as well for any \( k \geq 1 \). It follows from (2.31) that the polyadic inverse of \( g \) is \( (g^{n-3}, \bar{g}) \), and one of \( \bar{g} \) is \( (g^{n-2}) \), and in this case \( g \) is called **querable**. In a polyadic group all elements are querable [37, 38].

The number of relations in (2.29) can be reduced from \( n \) (the number of possible places) to only 2 (when \( g \) is on the first and last places [3, 39]), such that in a polyadic group the Dörnte relations
\[ \mu_n[g, n_{(\bar{g})_{i}}] = \mu_n[n_{(\bar{g})_{j}}, g] = g \]  
(2.35)
hold valid for any allowable \( i, j \), and (2.35) are analogs of \( g \cdot h \cdot h^{-1} = h \cdot h^{-1} \cdot g = g \) in binary groups. The relation (2.29) can be treated as a definition of the (unary) **queroperation** \( \mu_1 : G \rightarrow G \) by
\[ \mu_1[g] = \bar{g} \]  
(2.36)
such that the diagram
\[ \begin{array}{ccc}
G^{\times n} & \xrightarrow{\mu_n} & G \\
\downarrow{\text{id}^{\times(n-1)} \times \mu_1} & & \\
G^{\times n} & \xrightarrow{\text{Pr}_n} & G^{\times n}
\end{array} \]  
(2.37)
commutes. Then, using the queroperation (2.36) one can give a diagrammatic definition of a polyadic group (cf. [40]).

**Definition 2.16.** A polyadic group is a universal algebra
\[ G_n^{\text{grp}} = \langle G \mid \mu_n, \mu_1 \mid \text{associativity, Dörnte relations} \rangle, \]  
(2.38)
where \( \mu_n \) is \( n \)-ary associative operation and \( \mu_1 \) is the queroperation (2.36), such that the following diagram
\[ \begin{array}{ccc}
G^{\times(n)} & \xrightarrow{\text{id} \times \text{Diag}_{n-1}} & G^{\times(n)} \\
\downarrow{\text{id} \times \text{Pr}_1} & & \downarrow{\text{Pr}_2} \\
G \times G & \xrightarrow{\mu_n} & G \times G \\
\downarrow{\text{Diag}_{n-1} \times \text{id}} & & \downarrow{\text{id} \times \text{Diag}_{n-1}} \\
G \times G & \xrightarrow{\mu_1 \times \text{id}} & G \times G
\end{array} \]  
(2.39)
commutes, where \( \mu_1 \) can be only on the first and second places from the right (resp. left) on the left (resp. right) part of the diagram.
A straightforward generalization of the querpower concept and corresponding definitions can be made by substituting in the above formulas (2.29)–(2.36) the n-ary multiplication \( \mu_n \) by the iterating multiplication \( \mu_n^k \) (2.1) (cf. [41] for \( \ell_n = 2 \) and [42]).

Let us define the querpower \( k \) of \( g \) recursively by [43, 44]

\[
\tilde{g}^{(k)} = (\tilde{g}^{(k-1)})^k,
\]

where \( \tilde{g}^{(0)} = g \), \( \tilde{g}^{(1)} = g_0 \), \( \tilde{g}^{(2)} = g_{11} \), \( \tilde{g}^{(3)} = g_{10} \), \ldots or as the k composition \( \tilde{\mu}_n^k = \tilde{\mu}_1 \circ \tilde{\mu}_1 \circ \ldots \circ \tilde{\mu}_1 \) of the unary querpoperation (2.36). We can define the negative polyadic power of an element \( g \) by the recursive relationship

\[
\mu_n \left[ g^{(\ell_n-1)}, g^{\ell_n-2}, g^{(-\ell_n)} \right] = g,
\]

or (after the use of the positive polyadic power (2.10)) as a solution of the equation

\[
\mu_n^k \left[ g^{(\ell_n-1)}, g^{\ell_n-2}, g^{(-\ell_n)} \right] = g.
\]

The querppower (2.40) and the polyadic power (2.42) are connected [45]. We reformulate this connection using the so called Heine numbers [46] or \( q \)-deformed numbers [47]

\[
[[k]]_q = \frac{q^k - 1}{q - 1},
\]

which have the “nondeformed” limit \( q \to 1 \) as \([k]]_q \to k \) and \([0]]_q = 0 \). If \([k]]_q = 0 \), then \( q \) is a k-th root of unity. From (2.40) and (2.42) we obtain

\[
\tilde{g}^{(k)} = (\tilde{g}^{(-[[k]]]})^{-1},
\]

which can be treated as the following “deformation” statement:

**Assertion 2.17.** The querppower coincides with the negative polyadic deformed power with the “deformation” parameter \( q \) which is equal to the “deviation” \((2 - n)\) from the binary group.

**Example 2.18.** Let us consider a binary group \( G_2 = \langle G, \mu_2 \rangle \), we denote \( \mu_2 = (\cdot) \), and construct (using (2.1) and (2.5)) the reduced 4-ary product by \( \mu'_4 [g] = g_1, g_2, g_3, g_4 \cdot c \), where \( g_i \in G \) and \( c \) is in the center of the group \( G_2 \). In the 4-ary group \( G'_4 = \langle G, \mu'_4 \rangle \) we derive the following positive and negative polyadic powers (obviously \( g^{(0)} = \tilde{g}^{(0)} = g \))

\[
\tilde{g}^{(1)} = g^4 \cdot c, \quad \tilde{g}^{(2)} = g^7 \cdot c^2, \ldots, \quad \tilde{g}^{(k)} = g^{3k+1} \cdot c^k,
\]

and the querpowers

\[
\tilde{g}^{(1)} = g^2 \cdot c^{-1}, \quad \tilde{g}^{(2)} = g^{-4} \cdot c, \ldots, \quad \tilde{g}^{(k)} = g^{-2k} \cdot c^{-k}.
\]

Let \( G_n = \langle G, \mu_n \rangle \) and \( G'_n = \langle G', \mu'_n \rangle \) be two polyadic systems of any kind. If their multiplications are of the same arity \( n = n' \), then one can define the following one-place mappings from \( G_n \) to \( G'_n \) (for many-place mappings, which change arity \( n \neq n' \) and corresponding heteromorphisms, see [17]).

Suppose we have \( n + 1 \) mappings \( \Phi_i : G \to G'_i, i = 1, \ldots, n + 1 \). An ordered system of mappings \( \{\Phi_n\} \) is called a homotopy from \( G_n \) to \( G'_n \), if (see, e.g., [34])

\[
\Phi_{n+1}(\mu_n [g_1, \ldots, g_n]) = \mu'_n [\Phi_1(g_1), \ldots, \Phi_n(g_n)], \quad g_i \in G.
\]
A homomorphism from \( G_n \) to \( G'_n \) is given, if there exists a (one-place) mapping \( \Phi : G \to G' \) satisfying
\[
\Phi(\mu_n [g_1, \ldots, g_n]) = \mu'_n [\Phi(g_1), \ldots, \Phi(g_n)], \quad g_i \in G,
\]
which means that the corresponding (equiary\(^6\)) diagram is commutative
\[
\begin{array}{ccc}
G & \xrightarrow{\psi} & G' \\
\mu_n & & \mu'_n \\
G_n & \xrightarrow{\Phi} & (G')_n
\end{array}
\]  
(2.50)

It is obvious that, if a polyadic system contains distinguished elements (identities, querelements, etc.), they are also mapped by \( \psi \) correspondingly (for details and a review, see, e.g., [42, 48]). The most important application of one-place mappings is in establishing a general structure for \( n \)-ary multiplication.

3. The Hosszú-Gluskin Theorem

Let us consider possible concrete forms of polyadic multiplication in terms of lesser arity operations. Obviously, the simplest way of constructing a \( n \)-ary product \( \mu'_n \) from the binary one \( \mu_2 = (\ast) \) is \( \ell_n = n \) iteration (2.1) \[8, 49\]
\[
\mu'_n [g] = g_1 \ast g_2 \ast \ldots \ast g_n, \quad g_i \in G. 
\]
(3.1)

In [3] it was noted that not all \( n \)-ary groups have a product of this special form. The binary group \( G_2 = (G | \mu_2 = \ast, e) \) was called a covering group of the \( n \)-ary group \( G'_n = (G | \mu'_n) \) in [4] (see, also, [50]), where a theorem establishing a more general (than (3.1)) structure of \( \mu'_n [g] \) in terms of subgroup structure of the covering group was given. A manifest form of the \( n \)-ary group product \( \mu'_n [g] \) in terms of the binary one and a special mapping was found in [11, 13] and is called the Hosszú-Gluskin theorem, despite the same formulas having appeared much earlier in [4, 51] (for the relationship between the formulations, see [52]).

A simple construction of \( \mu'_n [g] \) which is present in the Hosszú-Gluskin theorem was given in [16]. Here we follow this scheme in the opposite direction, by just deriving the final formula step by step (without writing it immediately) with clear examples. Then we introduce a “deformation” to it in such a way that a generalized “\( q \)-deformed” Hosszú-Gluskin theorem can be formulated.

First, let us rewrite (3.1) in its equivalent form
\[
\mu'_n [g] = g_1 \ast g_2 \ast \ldots \ast g_n \ast e, \quad g_i, e \in G, 
\]
(3.2)

where \( e \) is a distinguished element of the binary group \( (G | \ast, e) \), that is the identity. Now we apply to (3.2) an “extended” version of the homotopy relation (2.48) with \( \Phi_i = \psi_i, i = 1, \ldots, n \), and the l.h.s. mapping \( \Phi_{n+1} = \text{id} \), but add an action \( \psi_{n+1} \) on the identity \( e \) of the binary group \( (G | \ast, e) \). Then we get (see (2.7) and (2.9))
\[
\mu_n [g] = \mu'_n [g] = \psi_1 (g_1) \ast \psi_2 (g_2) \ast \ldots \ast \psi_n (g_n) \ast \psi_{n+1} (e) = \left( \prod_{i=1}^{n} \psi_i (g_i) \right) \ast \psi_{n+1} (e).
\]
(3.3)

In this way we have obtained the most general form of polyadic multiplication in terms of \( (n + 1) \) “extended” homomorphism maps \( \psi_i, i = 1, \ldots, n + 1 \), such that the diagram
\[
\begin{array}{ccc}
G^{\times(n)} & \xrightarrow{\id^{n} \times \mu_1^{(n)}} & G^{\times(n+1)} \xrightarrow{\psi_1 \times \ldots \times \psi_{n+1}} G^{\times(n+1)} \\
\mu_n^{(n)} & & \mu'_n \\
G^{\times(n)} & \xrightarrow{\mu_n^{(n)}} & G
\end{array}
\]  
(3.4)

\(^6\)The map is equiary, if it does not change the arity of operations i.e. \( n = n' \), for nonequairly maps see [17] and refs. therein.
commutes. A natural question arises, whether all associative polyadic systems have this form of multiplication or do we have others? In general, we can correspondingly classify polyadic systems as:

1) *Homotopic* polyadic systems which can be presented in the form (3.3).

2) *Nonhomotopic* polyadic systems with multiplication of other than (3.3) shapes.

(3.5) (3.6)

If the second class is nonempty, it would be interesting to find examples of nonhomotopic polyadic systems. The Hosszu-Gluskin theorem considers the homotopic polyadic systems and gives one of the possible choices for the “extended” homotopy maps $\psi_i$ in (3.3). We will show that this choice can be extended (“deformed”) to the infinite “q-series”.

The main idea in constructing the “automatically” associative $n$-ary operation $\mu_n$ in (3.3) is to express the binary multiplication $(\cdot)$ and the “extended” homotopy maps $\psi_i$ in terms of $\mu_n$ itself [16]. A simplest binary multiplication which can be built from $\mu_n$ is (see (2.20))

$$g \ast_t h = \mu_n [g, t, h],$$

(3.7)

where $t$ is any fixed polyad of length $(n - 2)$. If we apply here the equations for the identity $e$ in a binary group

$$g \ast_t e = g, \quad e \ast_t h = h,$$

(3.8)

then we obtain

$$\mu_n [g, t, e] = g, \quad \mu_n [e, t, h] = h.$$  

(3.9)

We observe from (3.9) that $(t, e)$ and $(e, t)$ are neutral sequences of length $(n - 1)$, and therefore using (2.32) we can take $t$ as a polyadic inverse of $e$ (the identity of the binary group) considered as an element (but not an identity) of the polyadic system $(G \mid \mu_n)$, that is $t = e^{-1}$. Then, the binary multiplication constructed from $\mu_n$ and which has the standard identity properties (3.8) can be chosen as

$$g \ast h = g \ast_t h = \mu_n [g, e^{-1}, h].$$

(3.10)

Using this construction any element of the polyadic system $(G \mid \mu_n)$ can be distinguished and may serve as the identity of the binary group, and is then denoted by $e$ (for clarity and convenience).

We recognize in (3.10) a version of the Maltsev term (see, e.g., [18]), which can be called a *polyadic Maltsev term* and is defined as

$$p (g, e, h) \overset{\text{def}}{=} \mu_n [g, e^{-1}, h]$$

(3.11)

having the standard term properties [18]

$$p (g, e, e) = g, \quad p (e, e, h) = h,$$

(3.12)

which now follow from (3.9), i.e. the polyads $(e, e^{-1})$ and $(e^{-1}, e)$ are neutral, as they should be (2.32). Denote by $g^{-1}$ the inverse element of $g$ in the binary group $(g \ast g^{-1} = g^{-1} \ast g = e)$ and $g^{-1}$ its polyadic inverse in a $n$-ary group (2.32), then it follows from (3.10) that $\mu_n [g, e^{-1}, g^{-1}] = e$. Thus, we get

$$g^{-1} = \mu_n [e, g^{-1}, e],$$

(3.13)

which can be considered as a connection between the inverse $g^{-1}$ in the binary group and the polyadic inverse in the polyadic system related to the same element $g$. For $n$-ary group we can write $g^{-1} = (g^{n-3}, g)$ and the binary group inverse $g^{-1}$ becomes

$$g^{-1} = \mu_n [e, g^{n-3}, g, e].$$

(3.14)
Comparing this with (3.16), we can exactly identify the “extended” homotopy maps which can be described by the commutative diagram

\[
\begin{array}{ccc}
[\bullet] \times G \times [\bullet] & \xrightarrow{\psi_1 \times \text{id} \times \mu_1} & G^3 \times \text{id} \times \mu_1 \times \psi_2 \times \mu_3 \times \psi_4 \times \mu_1 \times \psi_5 \times \mu_1
\
\psi & \downarrow & \mu_2
\
G & \searrow & G
\end{array}
\]

and the neutral polyads are \((e, \bar{e})\) and \((\bar{e}, e)\).

Now let us turn to build the main construction, that of the “extended” homotopy maps \(\psi_i\) (3.3) in terms of \(\mu_n\), which will lead to the Hosszu-Gluskin theorem. We start with a simple example of a ternary system (3.15), derive the Hosszu-Gluskin “chain formula”, and then it will be clear how to proceed for generic \(n\). Instead of (3.3) we write

\[
\mu_3 [g, h, u] = \psi_1 (g) \times \psi_2 (h) \times \psi_3 (u) \times \psi_4 (e)
\]

and try to construct \(\psi_i\) in terms of the ternary product \(\mu_3\) and the binary identity \(e\). We already know the structure of the binary multiplication (3.15): it contains \(\bar{e}\), and therefore we can insert between \(g, h\) and \(u\) in the l.h.s. of (3.16) a neutral ternary polyad \((e, \bar{e})\) or its powers \((e^k, \bar{e}^k)\). Thus, taking for all insertions the minimal number of neutral polyads, we get

\[
\mu_3 [g, h, u] = \mu_3^2 [g, \bar{e}, g, h, \bar{e}, \bar{e}, e, u] = \mu_3^7 [g, \bar{e}, \bar{e}, e, h, \bar{e}, \bar{e}, e, e, u] = \mu_3^7 [g, \bar{e}, \bar{e}, e, h, \bar{e}, \bar{e}, e, e, u] = \mu_3^7 [g, \bar{e}, \bar{e}, e, h, \bar{e}, \bar{e}, e, e, u].
\]

We show by arrows the binary products in special places: there should be 1, 3, 5, … \((2k - 1)\) elements in between them to form inner ternary products. Then we rewrite (3.17) as

\[
\mu_3 [g, h, u] = \mu_3^2 [g, \bar{e}, \mu_3 [e, h, \bar{e}], \bar{e}, \mu_3^2 [e, e, u, \bar{e}, \bar{e}], \mu_3^3 [e, e, e]].
\]

Comparing this with (3.16), we can exactly identify the “extended” homotopy maps \(\psi_i\) as

\[
\psi_1 (g) = g,
\]

\[
\psi_2 (g) = \varphi (g),
\]

\[
\psi_3 (g) = \varphi (\varphi (g)) = \varphi^2 (g),
\]

\[
\psi_4 (e) = \mu_3 [e, e, e],
\]

where

\[
\varphi (g) = \mu_3 [e, g, \bar{e}],
\]

which can be described by the commutative diagram

\[
\begin{array}{ccc}
[\bullet] \times G \times [\bullet] & \xrightarrow{\psi_1 \times \text{id} \times \mu_1} & G^3 \times \text{id} \times \mu_1 \times \psi_2 \times \mu_3 \times \psi_4 \times \mu_1 \times \psi_5 \times \mu_1
\
\psi & \downarrow & \mu_2
\
G & \searrow & G
\end{array}
\]
The mapping \( \psi_4 \) is the first polyadic power (2.10) of the binary identity \( e \) in the ternary system
\[
\psi_4 (e) = e^{(1)}.
\]
Thus, combining (3.18)–(3.25) we obtain the Hosszú-Gluskin “chain formula” for \( n \) = 3
\[
\mu_3 \left[ g, h, u \right] = g * \varphi (h) * \varphi^2 (u) * b, \quad b = e^{(1)},
\]
which depends on one mapping \( \varphi \) (taken in the chain of powers) only, and the first polyadic power \( e^{(1)} \) of the binary identity \( e \). The corresponding Hosszú-Gluskin diagram
\[
G \times G \times G \xrightarrow{\mu_3} G \times \bullet^3 \xrightarrow{id \times \varphi \times \varphi} G \times \bullet^8 \xrightarrow{id \times \varphi \times \varphi^2} \bullet^5 \times G \times G \xrightarrow{\mu_3} G
\]
commutes.

The mapping \( \varphi \) is an automorphism of the binary group \( \langle G \mid *, e \rangle \), because it follows from (3.15) and (3.23) that
\[
\varphi (g) * \varphi (h) = \mu_3 \left[ \mu_3 [e, g, e, \varepsilon, \mu_3 [e, h, e]] \right] = \mu_3 \left[ e, g, e, (e, e), h, e \right],
\]
\[
\varphi (e) = \mu_3 [e, e, \varepsilon, \mu_3 [e, e, \varepsilon, e]] = \mu_3 \left[ e, e, \varepsilon, e \right] = e.
\]
It is important to note that not only the binary identity \( e \), but also its first polyadic power \( e^{(1)} \) is a fixed point of the automorphism \( \varphi \), because
\[
\varphi \left( e^{(1)} \right) = \mu_3 [e, e^{(1)}, \varepsilon] = \mu_3 \left[ e, e, \varepsilon, \mu_3 [e, e, \varepsilon] \right] = \mu_3 [e, e, \varepsilon] = e^{(1)}.
\]
Moreover, taking into account that in the binary group (see (3.15))
\[
\left( e^{(1)} \right)^{-1} = \mu_3 [e, \mu_3 [e, e, \varepsilon], e] = \mu_3 \left[ e, e, \varepsilon, e \right] = \mu_3 [e, e, \varepsilon],
\]
we get
\[
\varphi^2 (g) = \mu_3 [e, e, g, e, \varepsilon] = \mu_3 \left[ e, e, \varepsilon, e, g, e, \varepsilon \right] = e^{(1)} * g * \left( e^{(1)} \right)^{-1}.
\]
The higher polyadic powers \( e^{(k)} \) of the binary identity \( e \) are obviously also fixed points
\[
\varphi \left( e^{(k)} \right) = e^{(k)}.
\]
The elements \( e^{(k)} \) form a subgroup \( H \) of the binary group \( \langle G \mid *, e \rangle \), because
\[
e^{(k)} * e^{(l)} = e^{(k+l)},
\]
\[
e^{(k)} * e = e * e^{(k)} = e^{(k)}.
\]
We can express the even powers of the automorphism $\varphi$ through the polyadic powers $\varepsilon^{(k)}$ in the following way

$$\varphi^{2k}(g) = \varepsilon^{(k)} \ast g \ast (\varepsilon^{(k)})^{-1}. \quad (3.37)$$

This gives a manifest connection between the Hosszú-Gluskin “chain formula” and the sequence of cosets (see [4]) for the particular case $n = 3$.

**Example 3.1.** Let us consider the ternary copula associative multiplication [54, 55]

$$\mu_3 [g, h, u] = \frac{g(1 - h)u}{g(1 - h)u + (1 - g)h(1 - u)}, \quad (3.38)$$

where $g_i \in G = [0, 1]$ and $0/0 = 0$ is assumed\(^7\). It is associative and cannot be iterated from any binary group. Obviously, $\mu_3 [g^3] = g$, and therefore this polyadic system is $\ell_\nu$-idempotent (2.17) $g^{(\ell)} = g$. The querelement is $\bar{g} = \mu_3 [g] = g$. Because each element is quereable, then $(G \mid \mu_3, \mu_1)$ is a ternary group. Take a fixed element $\varepsilon \in [0, 1]$. We define the binary multiplication as $g \ast h = \mu_3 [g, \varepsilon, h]$ and the automorphism

$$\varphi (g) = \mu_3 [\varepsilon, g, e] = e^2 \frac{1 - g}{e^2 - 2ge + g}, \quad (3.39)$$

which has the property $\varphi^{2k} = \text{id}$ and $\varphi^{2k+1} = \varphi$, where $k \in \mathbb{N}$. Obviously, in (3.39) $g$ can be on any place in the product $\mu_3 [\varepsilon, g, e] = \mu_3 [\varepsilon, e, g] = \mu_3 [\varepsilon, e, g]$. Now we can check the Hosszú-Gluskin “chain formula” (3.26) for the ternary copula

$$\mu_3 [g, h, u] = (((g \ast \varphi (h)) \ast u) \ast e) = \mu_3^* [g, e, e^2 \frac{1 - h}{e^2 - 2he + g}, e, (u, e, e)]$$

$$= \mu_3^* [g, e, e^2 \frac{1 - h}{e^2 - 2he + g}, e, u] = \mu_3 [g, \varphi^2 (h), u] = \mu_3 [g, h, u]. \quad (3.40)$$

The language of polyadic inverses allows us to generalize the Hosszú-Gluskin “chain formula” from $n = 3$ (3.26) to arbitrary $n$ in a clear way. The derivation coincides with (3.18) using the multiplication (3.10) (with substitution $\varepsilon \rightarrow e^{-1}$), neutral polyads $(e^{-1}, e)$ or their powers $\left((e^{-1})^k, e^k\right)$, but contains $n$ terms

$$\mu_n [g_1, \ldots, g_n] = \mu_n^* [g_1, e^{-1}, e, g_2, \ldots, e_n] = \mu_n^* [g_1, e^{-1}, e, g_2, e^{-1}, e, e, g_3, \ldots, e_n] = \ldots$$

$$= \mu_n^* [g_1, e^{-1}, e, g_2, e^{-1}, e, e, g_3, \ldots, e_n, e^{-1}, e, \ldots, e_n, e^{-1}, e, \ldots, e]. \quad (3.41)$$

We observe from (3.41) that the mapping $\varphi$ in the $n$-ary case is

$$\varphi (g) = \mu_n [\varepsilon, g, e^{-1}], \quad (3.42)$$

\(^7\)In this example all denominators are supposed nonzero.
and the last product of the binary identities $\mu_n[e, \ldots, e]$ is also the first $n$-ary power $e^{(1)}$ (2.10). It follows from (3.42) and (3.10), that

$$q^{n-1}(g) = e^{(1)} \ast g \ast (e^{(1)})^{-1}. \quad (3.43)$$

In this way, we obtain the Hosszú-Gluskin “chain formula” for arbitrary $n$

$$\mu_n[g_1, \ldots, g_n] = g_1 \ast q(g_2) \ast q^2(g_3) \ast \ldots \ast q^{n-2}(g_{n-1}) \ast q^{n-1}(g_n) \ast e^{(1)} = \left(\prod_{i=1}^{n} q^{i-1}(g_i)\right) \ast e^{(1)}. \quad (3.44)$$

Thus, we have found the “extended” homotopy maps $\psi_i$ from (3.3) as

$$\psi_i(g) = q^{i-1}(g), \quad i = 1, \ldots, n, \quad (3.45)$$
$$\psi_{n+1}(g) = g^{(1)}, \quad (3.46)$$

where we put by definition $q^0(g) = g$. Using (3.31) and (3.44) we can formulate the Hosszú-Gluskin theorem in the language of polyadic powers.

**Theorem 3.2.** On a polyadic group $G_n = \langle G \mid \mu_n, \mu_1 \rangle$ one can define a binary group $G_2 = \langle G \mid \mu_2 = \ast, e \rangle$ and its automorphism $q$ such that the Hosszú-Gluskin “chain formula” (3.44) is valid, where the polyadic powers of the identity $e$ are fixed points of $q$ (3.34), form a subgroup $H$ of $G_2$, and the $(n-1)$ power of $q$ is a conjugation (3.43) with respect to $H$.

The following reverse Hosszú-Gluskin theorem holds.

**Theorem 3.3.** If in a binary group $G_2 = \langle G \mid \mu_2 = \ast, e \rangle$ one can define an automorphism $q$ such that

$$q^{n-1}(g) = b \ast g \ast b^{-1}, \quad (3.47)$$
$$q(b) = b, \quad (3.48)$$

where $b \in G$ is a distinguished element, then the “chain formula”

$$\mu_n[g_1, \ldots, g_n] = \left(\prod_{i=1}^{n} q^{i-1}(g_i)\right) \ast b \quad (3.49)$$

determines a $n$-ary group, in which the distinguished element is the first polyadic power of the binary identity

$$b = e^{(1)}. \quad (3.50)$$

4. “Deformation” of Hosszú-Gluskin Chain Formula

Let us raise the question: can the choice (3.45)-(3.46) of the “extended” homotopy maps (3.3) be generalized? Before answering this question positively we consider some preliminary statements.

First, we note that we keep the general idea of inserting neutral sequences into a polyadic product (see (3.17) and (3.41)), because this is the only way to obtain “automatic” associativity. Second, the number of the inserted neutral polyads can be chosen arbitrarily, not only minimally, as in (3.17) and (3.41) (as they are neutral). Nevertheless, we can show that this arbitrariness is somewhat restricted.

Indeed, let us consider a polyadic group $G_n = \langle G \mid \mu_n, \mu_1 \rangle$ in the particular case $n = 3$, where for any $e_0 \in G$ and natural $k$ the sequence $(\mu^k_0, e^k_0)$ is neutral, then we can write

$$\mu_3 [g, h, u] = \mu_3^* \left[g, e^k_0, e^k_0, h, e^k_0, e^k_0, u, e^{mk}_0, e^{mk}_0\right]. \quad (4.1)$$
If we make the change of variables \( e_0^q = e \), then we obtain

\[
\mu_3 [g, h, u] = \mu_3^* \left[ g, e, h, e', e^{+1}, u, e^{m}, e^m \right].
\]  

(4.2)

Because this should reproduce the formula (3.16), we immediately conclude that \( \psi_1 (g) = \text{id} \), and the multiplication is the same as in (3.15), and \( e \) is again the identity of the binary group \( G^* = (G, \ast, e) \). Moreover, if we put \( \psi_2 (g) = q (g) \), as in the standard case, then we have a first “half” of the mapping \( q \), that is \( q (g) = \mu_3 [e, h, \text{something}] \). Now we are in a position to find this “something” and other “extended” homotopy maps \( \psi_i \) from (3.16), but without the requirement of a minimal number of inserted neutral polyads, as it was in (3.17). By analogy, we rewrite (4.2) as

\[
\mu_3 [g, h, u] = \mu_3^* \left[ g, e, h, e^{+1}, u, e^{m}, e^m \right],
\]  

(4.3)

where we put \( l = q + 1 \). So we have found the “something”, and the map \( q \) is

\[
q (g) = \mu_3^* (g) \left[ e, g, e^m \right],
\]  

(4.4)

where the number of multiplications

\[
\ell_q (g) = \frac{q + 1}{2}
\]  

(4.5)

is an integer \( \ell_q (g) = 1, 2, 3, \ldots \), while \( q = 1, 3, 5, 7, \ldots \). The diagram defined \( \varphi_q \) (e.g., for \( q = 3 \) and \( \ell_q (g) = 2 \))

\[
\begin{align*}
\bullet \times G \times \{ \bullet \}^3 & \\ & \leftarrow \mu_0^* (\mu_0^0 \times \text{id} \times (\mu_0^0)^0) \xrightarrow{\varphi} \mu_3^* (\mu_0^1 \times \text{id} \times (\mu_0^1)^0) \xrightarrow{\varphi} G \times G \times G
\end{align*}
\]  

(4.6)

commutes (cf. (3.24)). Then, we can find power \( m \) in (4.3)

\[
\mu_3 [g, h, u] = \mu_3^* \left[ g, e, h, e^m \right],
\]  

(4.7)

and therefore \( m = q (q + 1) + 1 \). Thus, we have obtained the “\( q \)-deformed” maps \( \psi_i \) (cf. (3.19)–(3.22))

\[
\begin{align*}
\psi_1 (g) &= q_0^{[1]} (g) = q_0^0 (g) = g, \\
\psi_2 (g) &= q_2 (g) = q_2^{[1]} (g), \\
\psi_3 (g) &= q_3^{[1]} (g) = q_3^{[2]} (g), \\
\psi_4 (g) &= q_4^{[2]} (g) = q_4^{[3]} (g),
\end{align*}
\]  

(4.8)–(4.11)

where \( q \) is defined by (4.4) and \([k]_q \) is the \( q \)-deformed number (2.43), and we put \( q_0^0 = \text{id} \). The corresponding “\( q \)-deformed” chain formula (for \( n = 3 \) can be written as (cf. (3.26)–(3.27) for “nondeformed” case)

\[
\begin{align*}
\mu_3 [g, h, u] &= g * q_1^{[1]} (h) * q_2^{[2]} (u) * b_q, \\
b_q &= e^{(q \ell_q (g))},
\end{align*}
\]  

(4.12)–(4.13)

where the degree of the binary identity polyadic power

\[
\ell_q (g) = \frac{q [2]_q}{2} = \ell_q (q) \left( 2 \ell_q (q) + 1 \right)
\]  

(4.14)
is an integer. The corresponding “deformed” chain diagram (e.g., for \( q = 3 \))

\[
G^3 \times \{ Mult \} \xrightarrow{\text{id} \times \varphi \times \varphi \times (\varphi^{(3)})^{13}} G^{16} \xrightarrow{\text{id} \times \varphi} G^{13} \xrightarrow{\varphi} G \times G \times G \xrightarrow{\varphi} G
\]

\[(4.15)\]

commutes (cf. the Hosszu-Gluskin diagram (3.28)). In the “deformed” case the polyadic power \( e^{(\ell_q)} \) is not a fixed point of \( \varphi_q \) and satisfies

\[
\varphi_q \left( e^{(\ell_q)} \right) = \varphi_q \left( \mu^*_n \left[ e^{(q+1)} \right] \right) = \mu^*_n \left[ e^{(q+1)} \right] = e^{(\ell_q)} \star \varphi_q (e)
\]

or

\[
\varphi_q (b_q) = b_q \star \varphi_q (e).
\]

Instead of (3.33) we have

\[
\varphi_q^{q+1} (g) \cdot e^{(\ell_q)} = \mu^*_n \left[ e^{(q+1)} \right] = \mu^*_n \left[ e^{(q+1)} \right] \star g = e^{(\ell_q)} \star \varphi_q^{q+1} (e) \star g
\]

or

\[
\varphi_q^{q+1} (g) \cdot b_q = b_q \star \varphi_q^{q+1} (e) \star g.
\]

The “nondeformed” limit \( q \to 1 \) of (4.12) gives the Hosszu-Gluskin chain formula (3.26) for \( n = 3 \). Now let us turn to arbitrary \( n \) and write the \( n \)-ary multiplication using neutral polyads analogously to (4.3). By the same arguments, as in (4.2), we insert only one neutral polyad \((e^{-1}, e)\) between the first and second elements in the multiplication, but in other places we insert powers \((e^{-1})^k, e^k\) (allowed by the chain properties), and obtain

\[
\mu_n [g_1, \ldots, g_n] = \mu^*_n [g_1, e^{-1}, e, g_2, \ldots, g_n] = \mu^*_n [g_1, e^{-1}, e, g_2, (e^{-1})^q, e^{-1}, e^{q+1}, g_3, \ldots, g_n] = \ldots
\]

\[
= \mu_n \begin{bmatrix} g_1, e^{-1}, e, g_2, (e^{-1})^q, e^{-1}, e^{q+1}, g_3, e^{-1}, e^{q+1}, g_4, \ldots, e^{-1}, e^{q+1}, g_n \end{bmatrix}
\]

\[
\begin{bmatrix} q^{q+1} & q^{q+1} & \ldots & q^{q+1} \\ q^{q+1} & q^{q+1} & \ldots & q^{q+1} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ q^{q+1} & q^{q+1} & \ldots & q^{q+1} \end{bmatrix}
\]

\[(4.20)\]

So we observe that the binary product is now the same as in the “nondeformed” case (3.10), while the map \( \varphi \) is

\[
\varphi_q (g) = \mu^*_n \left[ e \cdot g, (e^{-1})^q \right],
\]

where the number of multiplications

\[
\ell_q = \frac{q (n - 2) + 1}{n - 1}
\]

\[(4.21)\]
is an integer and $\ell_\psi(n) \to q$, as $n \to \infty$, in the nondeformed case $\ell_\psi(1) = 1$, as in (3.42). Note that the "deformed" map $\varphi_q$ is the $a$-quasi-endomorphism [56] of the binary group $G_2$, because from (4.21) we get

$$
\varphi_q(g) \ast \varphi_q(h) = \mu_q^* [e, g, (e^{-1})^q, e^{-1}, e, h, (e^{-1})^q] = \mu_q^* [e, g, e^{-1}, (e, e, (e^{-1})^q), e^{-1}, h, (e^{-1})^q] = \varphi_q(g \ast a \ast h),
$$

(4.23)

where

$$
a = \mu_q^* [e, e, (e^{-1})^q] = \varphi_q(e).
$$

(4.24)

In general, a quasi-endomorphism can be defined by

$$
\varphi_q(g) \ast \varphi_q(h) = \varphi_q(g \ast \varphi_q(e) \ast h).
$$

(4.25)

The corresponding diagram

$$
\begin{array}{c}
G \times G \xrightarrow{\mu_2} G \xrightarrow{\psi_q} G \\
\mu_2 \times \mu_2 \xrightarrow{\mu_2} G \xrightarrow{id \times \id} G \xrightarrow{id \times \id} G \xrightarrow{id \times \id} G \times G
\end{array}
$$

(4.26)

commutes. If $q = 1$, then $\varphi_q(e) = e$, and the distinguished element $a$ turns to the binary identity $a = e$, such that the $a$-quasi-endomorphism $\varphi_q$ becomes an automorphism of $G_2$.

**Remark 4.1.** The choice (4.21) of the $a$-quasi-endomorphism $\varphi_q$ is different from [56], the latter (in our notation) is $\varphi_q(g) = \mu_q^* [a^{-1}, g, a^{n-k}]$, $k = 1, \ldots, n - 1$, it has only one multiplication and leads to the "nondeformed" chain formula (3.44) (for semigroup case).

It follows from (4.20), that the "extended" homotopy maps $\psi_i$ (3.3) are (cf. (4.8)–(4.11))

$$
\psi_1(g) = \varphi_q^{[1]}(g) = \varphi_q^0(g) = g,
$$

(4.27)

$$
\psi_2(g) = \varphi_q^2(g) = \varphi_q^{[1]}(g),
$$

(4.28)

$$
\psi_3(g) = \varphi_q^{[1]}(g) = \varphi_q^{[2]}(g),
$$

(4.29)

$$
\vdots
$$

$$
\psi_{n-1}(g) = \varphi_q^{[n-3] \cdots q + 1}(g) = \varphi_q^{[n-2]}(g),
$$

(4.30)

$$
\psi_n(g) = \varphi_q^{[n-2] \cdots q + 1}(g) = \varphi_q^{[n-1]}(g),
$$

(4.31)

$$
\psi_{n+1}(g) = \mu_n^* [g^{n+1} \cdots q + 1] = \mu_n^* [g^{[n+1]}].
$$

(4.32)

In terms of the polyadic power (2.10), the last map is

$$
\psi_{n+1}(g) = g^{(e)},
$$

(4.33)

where (cf. (4.22))

$$
\ell_e(q) = q \frac{[n - 1]q}{n - 1}
$$

(4.34)

is an integer. Thus the "$q$-deformed" $n$-ary chain formula is (cf. (3.44))

$$
\mu_n [g_1, \ldots, g_n] = g_1 \ast \varphi_q^{[1]}(g_2) \ast \varphi_q^{[1]}(g_3) \ast \cdots \ast \varphi_q^{[n-2]}(g_{n-1}) \ast \varphi_q^{[n-1]}(g_n) \ast e^{(\ell_e(q))}.
$$

(4.35)
In the “nondeformed” limit $q \to 1$ (4.35) reproduces the Hosszú-Gluskin chain formula (3.44). Let us obtain the “deformed” analogs of the distinguished element relations (3.47)–(3.48) for arbitrary $n$ (the case $n = 3$ is in (4.16)–(4.18)). Instead of the fixed point relation (3.48) we now have from (4.21), (4.34) and (4.32) the quasi-fixed point

$$
q_q(b_q) = b_q \ast q_q(e),
$$
(4.36)

where the “deformed” distinguished element $b_q$ is (cf. (3.50))

$$
b_q = \mu_q^n[e]\cdot e^\langle(b_q)\rangle.
$$
(4.37)

The conjugation relation (3.47) in the “deformed” case becomes the quasi-conjugation

$$
q_q^{[n-1]}(g) \ast b_q = b_q \ast q_q^{[n-1]}(e) \ast g.
$$
(4.38)

This allows us to rewrite the “deformed” chain formula (4.35) as

$$
\mu_n[g_1, \ldots, g_n] = g_1 \ast q_q^{[1]}(g_2) \ast q_q^{[2]}(g_3) \ast \ldots \ast q_q^{[n-2]}(g_{n-1}) \ast b_q \ast q_q^{[n-1]}(e) \ast g_n.
$$
(4.39)

Using the above proof sketch, we formulate the following “$q$-deformed” analog of the Hosszú-Gluskin theorem:

**Theorem 4.2.** On a polyadic group $G_n = \langle G \mid \mu_n, \mu_1 \rangle$ one can define a binary group $G^n_2 = \langle G \mid \mu_2 = \ast, e \rangle$ and (the infinite “$q$-series” of) its automorphism $q_q$ such that the “deformed” chain formula (4.35) is valid

$$
\mu_n[g_1, \ldots, g_n] = \left(\prod_{i=1}^n q_q^{[i-1]}(g_i)\right) \ast b_q;
$$
(4.40)

where (the infinite “$q$-series” of) the “deformed” distinguished element $b_q$ (being a polyadic power of the binary identity (4.37)) is the quasi-fixed point of $q_q$ (4.36) and satisfies the quasi-conjugation (4.38) in the form

$$
q_q^{[n-1]}(g) = b_q \ast q_q^{[n-1]}(e) \ast g \ast b_q^{-1}.
$$
(4.41)

In the “nondeformed” case $q = 1$ we obtain the Hosszú-Gluskin chain formula (3.44) and the corresponding Theorem 3.2.

**Example 4.3.** Let us have a binary group $G = \langle (G \mid (,) \rangle$ and a distinguished element $e \in G$, $e \neq 1$, then we can define a binary group $G_2^e = \langle G \mid (\ast, e) \rangle$ by the product

$$
g \ast h = g \cdot e^{-1} \cdot h.
$$
(4.42)

The quasi-endomorphism

$$
q_q(g) = e \cdot g \cdot e^{-q}
$$
(4.43)

satisfies (4.25) with $q_q(e) = e^{2^{-q}}$, and we take

$$
b_q = e^{[n]}.
$$
(4.44)

Then we can obtain the “$q$-deformed” chain formula (4.40) (for $q = 1$ see, e.g., [52]).

We observe that the chain formula is the “$q$-series” of equivalence relations (4.40), which can be formulated as an invariance. Indeed, let us denote the r.h.s. of (4.40) by $M_q(g_1, \ldots, g_n)$, and the l.h.s. as $M_0(g_1, \ldots, g_n)$, then the chain formula can be written as some invariance (cf. associativity as an invariance (2.18)).
Theorem 4.4. On a polyadic group \( G_n = \langle G \mid \mu_n, \mu_1 \rangle \) we can define a binary group \( G^* = \langle G \mid \mu_2 = *, e \rangle \) such that the following invariance is valid

\[
\mathcal{M}_q(g_1, \ldots, g_n) = \text{invariant, } q = 0, 1, \ldots,
\]

where

\[
\mathcal{M}_q(g_1, \ldots, g_n) = \left\{ \begin{array}{ll}
\mu_n g_1, \ldots, g_n, & q = 0, \\
\prod_{i=1}^{n} q^{[i-1]l_b} (g_i) \ast b_q, & q > 0,
\end{array} \right.
\]

and the distinguished element \( b_q \in G \) and the quasi-endomorphism \( \varphi_q \) of \( G^*_2 \) are defined in (4.37) and (4.21) respectively.

Example 4.5. Let us consider the ternary \( q \)-product used in the nonextensive statistics [26]

\[
\mu_3[g, t, u] = (g^h + t^h + u^h - 3)^\frac{1}{h},
\]

where \( h = 1 - q_0 \) and \( g, t, u \in G = \mathbb{R}_+, \ 0 < q_0 < 1 \), and also \( g^h + t^h + u^h - 3 > 0 \) (as for other terms inside brackets with power \( \frac{1}{h} \) below). In case \( h \to 0 \) the \( q \)-product becomes an iterated product in \( \mathbb{R}_+ \) as \( \mu_3(g, t, u) \to gtu. \) The quermap \( \mu_1 \) is given by

\[
g = \left(3 - g^h\right)^\frac{1}{h}.
\]

The polyadic system \( G_n = \langle G \mid \mu_3, \mu_1 \rangle \) is a ternary group, because each element is querable. Take a distinguished element \( e \in G \) and use (3.15), (4.47) and (4.48) to define the product

\[
g \ast t = \left(g^h - e^h + t^h\right)^\frac{1}{h},
\]

of the binary group \( G^*_2 = \langle G \mid \mu_2 = (\ast), e \rangle. \)

1) The Hosszú-Gluskin chain formula \( (q = 1) \). The automorphism (3.23) of \( G^* \) is now the identity map \( \varphi = \text{id} \). The first polyadic power of the distinguished element \( e \) is

\[
b = e^{(1)} = \mu_3[e^3] = \left(3e^h - 3\right)^\frac{1}{h}.
\]

The chain formula (3.26) can be checked as follows

\[
\mu_3[g, t, u] = \left((g \ast t) \ast u \right) = \left(g^h - e^h + t^h\right) - e^h + b^h = \left(g^h - e^h + t^h - e^h + b^h\right)^\frac{1}{h}
\]

\[
= \left(g^h - e^h + t^h - e^h + u^h - e^h + 3e^h - 3\right)^\frac{1}{h} = \left(g^h + t^h + u^h - 3\right)^\frac{1}{h}.
\]

2) The “\( q \)-deformed” chain formula (for conciseness we consider only the case \( q = 3 \)). Now the quasi-endomorphism \( \varphi_q (4.4) \) is not the identity, but is

\[
\varphi_{q=3}[g] = \left(g^h - 2e^h + 3\right)^\frac{1}{h}.
\]

In case \( q = 3 \) we need its 4th (\( = q + 1 \)) power (4.12)

\[
\varphi_{q=3}^4[g] = \left(g^h - 8e^h + 12\right)^\frac{1}{h}.
\]
The deformed polyadic power $e^{q(\tilde{\tau})}$ from (4.12) is (see, also, (4.11))

$$b_{q=3} = e^{q(5)} = \mu_5 [\ell^{13}] = \left(13e^h - 18\right)^{\frac{1}{e^h}}. \quad (4.54)$$

Now we check the “$q$-deformed” chain formula (4.12) as

$$\mu_3 [g, t, u] = g \ast q_{q=3} (t) \ast q_{q=3}^4 (u) \ast b_{q=3} = \left(\left(\left(\left(\left(g \ast q_{q=3} (t) \ast q_{q=3}^4 (u)\right) \ast b_{q=3}\right)\right)\right)\right)$$

$$= \left(g^h - e^h + (e^h - 2e^h + 3) - e^h + \left(u^h - 8e^h + 12\right) - e^h + \left(13e^h - 18\right)\right)$$

$$= \left(g^h + t^h + u^h - 3\right)^{\frac{1}{e^h}}. \quad (4.57)$$

In a similar way, one can check the “$q$-deformed” chain formula for any allowed $q$ (determined by (4.22) and (4.34)) to obtain an infinite $q$-series of the chain representation of the same $n$-ary multiplication.

5. Generalized “Deformed” Version of the Homomorphism Theorem

Let us consider a homomorphism of the binary groups entering into the “deformed” chain formula (4.40) as $\Phi^* : G_2^\circ \rightarrow G_2^\circ$, where $G_2^\circ = \langle G' \mid s', e' \rangle$. We observe that, because $\Phi^*$ commutes with the binary multiplication, we need its commutation also with the automorphisms $q_{p_{j}}$ in each term of (4.40) (which fixes equality of the “deformation” parameters $q = q'$) and its homomorphic action on $b_q$. Indeed, if

$$\Phi^* \left( q_{p_{j}} (g) \right) = q'_{p_{j}} (\Phi^* (g)),$$  \hspace{1cm} (5.1)

$$\Phi^* \left( b_q \right) = b'_{q'}, \quad (5.2)$$

then we get from (4.40)

$$\Phi^* \left( \mu_n [g_1, \ldots, g_n] \right) = \Phi^* (g_1) \ast \Phi^* (q_{p_{j}} [\ell^{1}] (g_2)) \ast \cdots \ast \Phi^* (q_{p_{j}} [\ell^{n-1}] (g_n)) \ast \Phi^* \left( b_q \right)$$

$$= \Phi^* (g_1) \ast q'_{p_{j}} [\ell^{1}] (\Phi^* (g_2)) \ast \cdots \ast q'_{p_{j}} [\ell^{n-1}] (\Phi^* (g_n)) \ast \Phi^* \left( b_q \right)$$

$$= \mu_n^* \left[ \Phi^* (g_1), \ldots, \Phi^* (g_n) \right]. \quad (5.3)$$

where $g' \ast h' = \mu_n^* [g', e'^{-1}, h'], \Phi^* (g') = \mu_n^* [e', g', (e'^{-1})']$, $b'_{q'} = \mu_n^* [e' [\ell^{n}] ]$. Comparison of (5.3) and (2.49) leads to

Theorem 5.1. A homomorphism $\Phi^*$ of the binary group $G_2^\circ$ gives rise to a homomorphism $\Phi$ of the corresponding $n$-ary group $G_n$, if $\Phi^*$ satisfies the “deformed” compatibility conditions (5.1)–(5.2).

The “nondeformed” version ($q = 1$) of this theorem and the case of $\Phi^*$ being an isomorphism was considered in [23].

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