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Conditions for the Equivalence of Power Series and Discrete Power Series Methods of Summability

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Abstract. Discrete power series methods were introduced and their regularity results were developed by Watson [Analysis (Munich), **18**(1): 97–102, 1998]. It was shown by Watson that discrete power series method (P_λ) strictly includes corresponding power series method (P). In the present work we present theorems showing when (P_λ) and (P) are equivalent methods and when two discrete power series methods are equivalent.

1. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be a series of real or complex numbers and (s_n) be its corresponding sequence of partial sums. Let (p_n) be a sequence of nonnegative numbers with $p_0 > 0$ such that

$$P_n := \sum_{k=0}^{n} p_k \to \infty, \ \ n \to \infty. \tag{1}$$

Assume that the power series

$$p(x) := \sum_{k=0}^{\infty} p_k x^k \tag{2}$$

has radius of convergence ρ and define

$$t_n := \sum_{k=1}^{n} P_{k-1} a_k. \tag{3}$$

The sequence (λ_n) is a strictly increasing sequence of real numbers such that $\lambda_0 \geq 1$ and $\lambda_n \to \infty$ as $n \to \infty$.

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Definition 1.1. *If* $\sigma_n := \dfrac{1}{P_n}\sum_{k=0}^{n} p_k s_k \to s$ *as* $n \to \infty$, *then we say that* (s_n) *is summable to* s *by the weighted mean method* (M_p) *and write* $(s_n) \to s(M_p)$.

Definition 1.2. *Suppose that* $p_s(x) := \dfrac{1}{p(x)}\sum_{k=0}^{\infty} p_k s_k x^k$ *exists for each* $x \in (0,\rho)$. *If* $\lim_{x\to\rho^-} p_s(x) = s$, *then we say that* (s_n) *is summable to* s *by the power series method* (P) *and write* $(s_n) \to s(P)$.

In the literature, the weighted mean method and the power series method are called the method (M_p) and the method (P), respectively. The summability methods (M_p) and (P) were studied by a number of authors such as Móricz and Rhoades [13], Móricz and Stadtmüller [14] and Kratz and Stadtmüller [9, 10]. Recently, Çanak and Totur [4–6], Totur and Çanak [16], Erdem and Totur [7] and Totur and Dik [17] have proved some Tauberian theorems for the methods (M_p) and (P).

The weighted mean method of summability is regular if and only if (1) is satisfied. The basic regularity results for the power series method were summarized by Borwein [3] and his result recalled here.

Theorem 1.3. *([3])*

(1) If $0 < \rho < \infty$, *then the method* (P) *is regular if and only if* $\sum\limits_{k=0}^{\infty} p_k \rho^k = \infty$.

(2) If $\rho = \infty$, *then the method* (P) *is regular.*

Furthermore, Ishuguro [8] proved that (M_p) implies (P). If $p_n = 1$ for all nonnegative integer n, then corresponding weighted mean and power series summability methods reduce to Cesàro $(C, 1)$ and Abel (A) summability methods, respectively. For Abel summability, $p(x) = \dfrac{1}{1-x}$, $\rho = 1$ and $p_s(x) = (1-x)\sum_{k=0}^{\infty} s_k x^k$. For Borel summability, $p_k = \dfrac{1}{k!}$, $p(x) = e^x$, $\rho = \infty$ and $p_s(x) = e^{-x}\sum_{k=0}^{\infty} \frac{s_k}{k!} x^k$.

Discrete summability methods were first introduced by Armitage and Maddox. Armitage and Maddox defined discrete methods corresponding to $(C, 1)$ and (A) in [1] and [2] and later Maddox [11, 12] established Tauberian results relating discrete Abel means. Moreover, discrete methods corresponding to the methods (M_p) and (P) were defined by Watson [18, 19] as follows.

Set

$$x_n := \begin{cases} \rho\left(1 - \frac{1}{\lambda_n}\right) & \text{if } 0 < \rho < \infty \\ \lambda_n & \text{if } \rho = \infty. \end{cases}$$

Definition 1.4. *We say that* (s_n) *is summable to* s *by the discrete weighted mean method,* (M_{P_λ}), *and write* $(s_n) \to s(M_{P_\lambda})$ *if* $\tau_n := \sigma_{[\lambda_n]} = \dfrac{1}{P_{[\lambda_n]}}\sum_{k=0}^{[\lambda_n]} p_k s_k \to s$ *as* $n \to \infty$, *where* $[\lambda_n]$ *denotes the integer part of* λ_n.

Definition 1.5. *Suppose that* $p_s(x_n)$ *exists for each* $n \geq 0$. *If* $p_s(x_n) := (P_\lambda s)_n \to s$ *as* $n \to \infty$, *then we say that* (s_n) *is summable to* s *by the discrete power series method* (P_λ) *and we write* $(s_n) \to s(P_\lambda)$.

In [20] and [21], Watson also proved some Tauberian theorems for the methods (M_{P_λ}) and (P_λ). Note that, trivially, (M_{P_λ}) includes (M_p) and (P_λ) includes (P) in the sense that $(s_n) \to s(M_p)$ or $(s_n) \to s(P)$ implies $(s_n) \to s(M_{P_\lambda})$ or $(s_n) \to s(P_\lambda)$, respectively. Eventually, the methods (M_{P_λ}) and (P_λ) inherit regularity from the underlying methods (M_p) or (P).

The aim of this paper is to present equivalence relations between both (P_λ) and (P) and two discrete power series methods.

2. Auxilary Results

We require following lemmas for the proofs of the theorems in the next section.

Lemma 2.1. *The identity*

$$t_n = \sum_{k=0}^{n} p_k(s_n - s_k) \tag{4}$$

is valid.

Proof. By (3), we have

$$\begin{aligned} t_n &= \sum_{k=1}^{n} P_{k-1}a_k \\ &= p_0a_1 + (p_0 + p_1)a_2 + ... + (p_0 + p_1 + ... + p_{n-2})a_{n-1} + (p_0 + p_1 + ... + p_{n-1})a_n \\ &= p_0(a_1 + a_2 + ... + a_n) + p_1(a_2 + a_3 + ... + a_n) + ... + p_{n-2}(a_{n-1} + a_n) + p_{n-1}a_n \\ &= p_0(s_n - s_0) + p_1(s_n - s_1) + ... + p_{n-2}(s_n - s_{n-2}) + p_{n-1}(s_n - s_{n-1}) \\ &= \sum_{k=0}^{n} p_k(s_n - s_k). \end{aligned}$$

□

Lemma 2.2. *The identity*

$$\frac{1}{p(x)} \sum_{k=0}^{\infty} p_k s_k x^k = \sum_{k=0}^{\infty} p_k a_k x^k \tag{5}$$

holds if and only if

$$p_0 = 1, \quad p_n = {p_1}^n \quad (n \geq 1)\cdot \tag{6}$$

Proof. By simple calculations, it is easy to show that the identity (5) holds if and only if $\sum_{v=0}^{k} p_v p_{k-v} a_v = p_k \sum_{v=0}^{k} a_v$ for all $k \geq 0$. Thus, (5) holds for each sequence (a_k) if and only if $p_k = p_v p_{k-v}$ for all $0 \leq v \leq k$. Since $p_0 \neq 0$, this yields $p_0 = 1$ and $p_n = {p_1}^n$ for all $n \geq 1$. □

In addition, to ensure the convergence of $p(x)$ we must choose $\rho = \frac{1}{p_1}$, that is, $p(x) = \frac{1}{1 - p_1 x}$. In this case the regularity condition (1) in Theorem 1.3 is also satisfied.

In the remainder of this paper we assume that $p_0 = 1,\ p_n = {p_1}^n$ for every positive integers n and $\rho = \frac{1}{p_1}$.

Lemma 2.3. *If $\sum_{k=0}^{\infty} p_k a_k x^k$ converges for all $x \in (0, \rho)$, then*

$$\sum_{k=1}^{\infty} p_k a_k x^k = \sum_{k=1}^{\infty} t_k \Delta\left(\frac{p_k x^k}{P_{k-1}}\right) \quad (0 < x < \rho).$$

Proof. We have

$$\sum_{k=1}^{n} p_k a_k x^k = \sum_{k=1}^{n} P_{k-1} a_k \left(\frac{p_k x^k}{P_{k-1}}\right).$$

Applying Abel's partial summation formula, we get

$$\sum_{k=1}^{n} p_k a_k x^k = \frac{p_n x^n}{P_{n-1}} t_n + \sum_{k=1}^{n-1} t_k \Delta\left(\frac{p_k x^k}{P_{k-1}}\right).$$

Hence it is enough to show that $p_n x^n t_n = o(P_{n-1})$ for $x \in (0,\rho)$. Fix $x \in (0,\rho)$ and choose $y \in (x,\rho)$. Since $\sum_{k=0}^{\infty} p_k a_k y^k$ converges, $\left|p_k a_k y^k\right| \leq M$ for $k \geq 1$. Therefore

$$\begin{aligned}
\left|p_n x^n t_n\right| &\leq p_n x^n \sum_{k=1}^{n} P_{k-1} |a_k| \\
&\leq p_n x^n \sum_{k=1}^{n} P_{k-1} M y^{-k} p_k{}^{-1} \\
&\leq p_n M P_{n-1} x^n \sum_{k=1}^{n} (p_1 y)^{-k} \\
&= p_n M P_{n-1} x^n \frac{(p_1 y)^{-n} - 1}{1 - p_1 y} \\
&\leq P_{n-1} M \frac{(x/y)^n}{1 - p_1 y} \\
&= o(P_{n-1}).
\end{aligned}$$

□

Lemma 2.4. *If (s_n) is a bounded sequence, then $t_n = O(P_n)$.*

Proof. Suppose that (s_n) is bounded. By (4),

$$\begin{aligned}
|t_n| = \left|\sum_{k=0}^{n} p_k (s_n - s_k)\right| &= \left|s_n \sum_{k=0}^{n} p_k - \sum_{k=0}^{n} p_k s_k\right| \\
&\leq |s_n| \sum_{k=0}^{n} p_k + \sum_{k=0}^{n} p_k |s_k| \\
&\leq M \sum_{k=0}^{n} p_k + M \sum_{k=0}^{n} p_k \\
&= O(P_n).
\end{aligned}$$

□

3. Equivalance Results

Theorem 3.1. *If the condition*

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1 \tag{7}$$

is satisfied, then the method (P_λ) is equivalent to the method (P) for bounded sequences.

Proof. Note that, trivially, (P_λ) includes (P) in the sense that $(s_n) \to L(P)$ implies $(s_n) \to L(P_\lambda)$. Let (s_n) be (P_λ) summable to L and let $x_n = \rho\left(1 - \dfrac{1}{\lambda_n}\right)$. Then, for a given $x \in (x_0, \rho)$, there exists an n such that $x_n \le x \le x_{n+1}$. By Lemma 2.2,

$$\begin{aligned}
\left|p_s(x) - (P_\lambda s)_n\right| &= \left|\frac{1}{p(x)}\sum_{k=0}^{\infty} p_k s_k x^k - \frac{1}{p(x_n)}\sum_{k=0}^{\infty} p_k s_k {x_n}^k\right| \\
&= \left|\sum_{k=0}^{\infty} p_k a_k x^k - \sum_{k=0}^{\infty} p_k a_k {x_n}^k\right|.
\end{aligned}$$

By Lemma 2.3, we get the following

$$\begin{aligned}
\left|p_s(x) - (P_\lambda s)_n\right| &= \left|\sum_{k=1}^{\infty} t_k \Delta\left(\frac{p_k x^k}{P_{k-1}}\right) - \sum_{k=1}^{\infty} t_k \Delta\left(\frac{p_k {x_n}^k}{P_{k-1}}\right)\right| \\
&= \left|\sum_{k=1}^{\infty} t_k \int_{x_n}^{x} k\frac{p_k}{P_{k-1}} t^{k-1} - (k+1)\frac{p_{k+1}}{P_k} t^k dt\right| \\
&\le \sum_{k=1}^{\infty} |t_k| \int_{x_n}^{x_{n+1}} \left|k\frac{p_k}{P_{k-1}} t^{k-1} - (k+1)\frac{p_{k+1}}{P_k} t^k\right| dt.
\end{aligned}$$

By Lemma 2.4, we have $t_n = O(P_n)$. Hence

$$\begin{aligned}
\left|p_s(x) - (P_\lambda s)_n\right| &= O(1)\sum_{k=1}^{\infty} P_k \int_{x_n}^{x_{n+1}} \left|k\frac{p_k}{P_{k-1}} - (k+1)\frac{p_{k+1}}{P_k} t\right| t^{k-1} dt \\
&= O(1)\int_{x_n}^{x_{n+1}} \sum_{k=1}^{\infty} \left|k\frac{p_k}{P_{k-1}} - (k+1)\frac{p_{k+1}}{P_k} t\right| P_k t^{k-1} dt \\
&= O(1)\int_{x_n}^{x_{n+1}} \sum_{k=1}^{\infty} \left|k p_1^k - (k+1) p_1^{k+1} t\right| t^{k-1} dt \\
&= O(1)\int_{x_n}^{x_{n+1}} \sum_{k=1}^{\infty} k p_1^{k-1} \left|p_1 - \frac{k+1}{k} p_1^2 t\right| t^{k-1} dt \\
&= O(1)\int_{x_n}^{x_{n+1}} (1 - p_1 t) \sum_{k=1}^{\infty} k p_1^{k-1} t^{k-1} dt. \\
&= O(1)\int_{x_n}^{x_{n+1}} (1 - p_1 t)\frac{1}{(1 - p_1 t)^2} dt \\
&= O(1)\int_{x_n}^{x_{n+1}} \frac{1}{1 - p_1 t} dt \\
&= O(1)\log\left(\frac{1 - p_1 x_n}{1 - p_1 x_{n+1}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\log\left(\frac{\lambda_{n+1}}{\lambda_n}\right)\\
&= O(1)o(1)\\
&= o(1).
\end{aligned}
$$

Since $(P_\lambda s)_n \to L$, it now follows that $p_s(x) \to L$ $(x \to \rho^-)$. That is, (s_n) is summable to L by the method (P). Therefore, the method (P_λ) is equivalent to the method (P) for bounded sequences. □

Note that Theorem 3.1 for $p_n = 1$ for all nonnegative integer n was proved in [15].

Remark. Let λ and μ be strictly increasing sequence of real numbers such that $\lambda_0 \geq 1$, $\mu_0 \geq 1$, $\lambda_n \to \infty$, $\mu_n \to \infty$, $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$, $\lim_{n\to\infty} \frac{\mu_{n+1}}{\mu_n} = 1$ as $n \to \infty$. We deduce from Theorem 3.1 that discrete power series methods (P_λ), (P_μ) and the power series method (P) are all equivalent for bounded sequences.

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