



On the Structure Tensors of Almost Contact B-Metric Manifolds

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Abstract. The space of the structure $(0,3)$ -tensors of the covariant derivatives of the structure endomorphism and the metric on almost contact B-metric manifolds is considered. A known decomposition of this space in orthogonal and invariant subspaces with respect to the action of the structure group is used. We determine the corresponding components of the structure tensor and consider the case of the lowest dimension 3 of the studied manifolds. Some examples are commented.

Introduction

The idea of decomposition of the space of the structure $(0,3)$ -tensors, generated by the covariant derivative of the fundamental tensor of type $(1,1)$, is used by different authors in order to obtain classifications of manifolds with additional tensor structures. For example, let us mention the classification of almost Hermitian manifolds given in [4], of almost complex manifolds with Norden metric – in [2], of almost contact metric manifolds – in [1], of almost contact manifolds with B-metric – in [3], of Riemannian almost product manifolds – in [8], of Riemannian manifolds with traceless almost product structure – in [10], of almost paracontact metric manifolds – in [7], of almost paracontact Riemannian manifolds of type (n,n) – in [6].

The almost contact structure is well studied, especially in the case when it is equipped with a compatible Riemannian (or pseudo-Riemannian) metric, *i.e.* the almost contact endomorphism φ acts as an isometry with respect to the metric in each tangent fibre of the contact distribution $H = \ker(\eta)$, where η is the contact 1-form. An indefinite counterpart is the almost contact B-metric structure, *i.e.* φ acts as an anti-isometry with respect to B-metric in H .

The goal of this work is the description of the structure tensor generated by the covariant derivative of φ and the B-metric by its components in the different basic classes of the classification of the almost contact B-metric manifolds made by G. Ganchev, V. Mihova, K. Gribachev in [3]. The case of the lowest dimension 3 of the studied manifolds is considered and it is established that four of the basic classes are restricted to the special class with zero structure tensors.

The paper is organized as follows. In Sect. 1 we recall some facts about the almost contact manifolds with B-metric. In Sect. 2 we decompose the vector space of the structure tensors on the considered manifolds.

2010 *Mathematics Subject Classification.* Primary 53C15, 53C50; Secondary 53D15

Keywords. almost contact manifold, Norden metric, B-metric, structure tensor

Received: 28 July 2014; Accepted: 14 December 2014

Communicated by Ljubica Velimirović and Mića Stanković

Research supported by the project NI13-FMI-002 of the Scientific Research Fund at the University of Plovdiv

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In Sect. 3 we deduce the components of the structure tensor in the case of the lowest dimension 3. In Sect. 4 we comment some examples in relation with the above investigations.

1. Almost contact manifolds with B-metric

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact manifold with B-metric or an *almost contact B-metric manifold*, i.e. M is a $(2n + 1)$ -dimensional differentiable manifold with an almost contact structure (φ, ξ, η) consisting of an endomorphism φ of the tangent bundle, a vector field ξ , its dual 1-form η as well as M is equipped with a pseudo-Riemannian metric g of signature $(n + 1, n)$, such that the following algebraic relations are satisfied: [3]

$$\varphi\xi = 0, \quad \varphi^2 = -\text{Id} + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \quad g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y) \tag{1}$$

for any x, y of the algebra $\mathfrak{X}(M)$ of the smooth vector fields on M . Further x, y, z will stand for arbitrary elements of $\mathfrak{X}(M)$ or vectors in the tangent space T_pM at $p \in M$.

The associated metric \tilde{g} of g on M is defined by $\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y)$. The manifold $(M, \varphi, \xi, \eta, \tilde{g})$ is also an almost contact B-metric manifold. Both metrics g and \tilde{g} are necessarily of signature $(n + 1, n)$. The Levi-Civita connection of g and \tilde{g} will be denoted by ∇ and $\tilde{\nabla}$, respectively.

Let us denote the structure group of $(M, \varphi, \xi, \eta, g)$ by \mathcal{G} . It is determined by $\mathcal{G} = O(n; \mathbb{C}) \times \mathcal{I}$, where \mathcal{I} is the identity on $\text{span}(\xi)$ and $O(n; \mathbb{C}) = \mathcal{GL}(n; \mathbb{C}) \cap O(n, n)$, i.e. \mathcal{G} consists of the real square matrices of order $2n + 1$ of the following type

$$\left(\begin{array}{c|c|c} A & B & \vartheta^\top \\ \hline -B & A & \vartheta^\top \\ \hline \vartheta & \vartheta & 1 \end{array} \right), \quad \begin{array}{l} A^\top A - B^\top B = I_n, \\ B^\top A + A^\top B = O_n, \end{array} \quad A, B \in \mathcal{GL}(n; \mathbb{R}),$$

where ϑ and its transpose ϑ^\top are the zero row n -vector and the zero column n -vector; I_n and O_n are the unit matrix and the zero matrix of size n , respectively.

A classification of the almost contact B-metric manifolds is given in [3]. This classification, consisting of eleven basic classes $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{11}$, is made with respect to the tensor F of type $(0,3)$ defined by

$$F(x, y, z) = g((\nabla_x \varphi) y, z), \tag{2}$$

which have the following properties

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi). \tag{3}$$

The intersection of the basic classes is the special class \mathcal{F}_0 determined by the condition $F(x, y, z) = 0$. Hence \mathcal{F}_0 is the class of the almost contact B-metric manifolds with ∇ -parallel structures, i.e. $\nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = \nabla \tilde{g} = 0$.

If $\{e_i; \xi\}$ ($i = 1, 2, \dots, 2n$) is a basis of T_pM and (g^{ij}) is the inverse matrix of the matrix (g_{ij}) of g , then the following 1-forms are associated with F :

$$\theta(z) = g^{ij}F(e_i, e_j, z), \quad \theta^*(z) = g^{ij}F(e_i, \varphi e_j, z), \quad \omega(z) = F(\xi, \xi, z). \tag{4}$$

These 1-forms are known also as the Lee forms. Obviously, the identities $\omega(\xi) = 0$ and $\theta^* \circ \varphi = -\theta \circ \varphi^2$ are always valid.

Further we use the following characteristic conditions of the basic classes: [5]

$$\begin{aligned} \mathcal{F}_1 : & \quad F(x, y, z) = \frac{1}{2n} \{g(x, \varphi y)\theta(\varphi z) + g(\varphi x, \varphi y)\theta(\varphi^2 z) + g(x, \varphi z)\theta(\varphi y) + g(\varphi x, \varphi z)\theta(\varphi^2 y)\}; \\ \mathcal{F}_2 : & \quad F(\xi, y, z) = F(x, \xi, z) = 0, \quad F(x, y, \varphi z) + F(y, z, \varphi x) + F(z, x, \varphi y) = 0, \quad \theta = 0; \\ \mathcal{F}_3 : & \quad F(\xi, y, z) = F(x, \xi, z) = 0, \quad F(x, y, z) + F(y, z, x) + F(z, x, y) = 0; \\ \mathcal{F}_4 : & \quad F(x, y, z) = -\frac{1}{2n} \theta(\xi) \{g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y)\}; \\ \mathcal{F}_5 : & \quad F(x, y, z) = -\frac{1}{2n} \theta^*(\xi) \{g(x, \varphi y)\eta(z) + g(x, \varphi z)\eta(y)\}; \end{aligned} \tag{5a}$$

$$\begin{aligned}
 \mathcal{F}_6 : \quad & F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \\
 & F(x, y, \xi) = F(y, x, \xi) = -F(\varphi x, \varphi y, \xi), \quad \theta = \theta^* = 0; \\
 \mathcal{F}_7 : \quad & F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \quad F(x, y, \xi) = -F(y, x, \xi) = -F(\varphi x, \varphi y, \xi); \\
 \mathcal{F}_8 : \quad & F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \quad F(x, y, \xi) = F(y, x, \xi) = F(\varphi x, \varphi y, \xi); \\
 \mathcal{F}_9 : \quad & F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \quad F(x, y, \xi) = -F(y, x, \xi) = F(\varphi x, \varphi y, \xi); \\
 \mathcal{F}_{10} : \quad & F(x, y, z) = F(\xi, \varphi y, \varphi z)\eta(x); \\
 \mathcal{F}_{11} : \quad & F(x, y, z) = \eta(x) \{ \eta(y)\omega(z) + \eta(z)\omega(y) \}.
 \end{aligned}
 \tag{5b}$$

2. A decomposition of the space of the structure tensors

Let us consider T_pM at arbitrary $p \in M$ as a $(2n + 1)$ -dimensional vector space equipped with almost contact B-metric structure (φ, ξ, η, g) . Let \mathcal{F} be the vector space of all tensors F of type $(0,3)$ over T_pM having properties (3), i.e.

$$\mathcal{F} = \{ F(x, y, z) \in \mathbb{R} \mid F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi) \}.
 \tag{6}$$

The metric g induces an inner product $\langle \cdot, \cdot \rangle$ on \mathcal{F} defined by

$$\langle F', F'' \rangle = g^{iq} g^{jr} g^{ks} F'(e_i, e_j, e_k) F''(e_q, e_r, e_s)$$

for any $F', F'' \in \mathcal{F}$ and a basis $\{e_i\}$ ($i = 1, 2, \dots, 2n + 1$) of T_pM .

The standard representation of the structure group \mathcal{G} in T_pM induces a natural representation λ of \mathcal{G} in \mathcal{F} as follows

$$((\lambda a)F)(x, y, z) = F(a^{-1}x, a^{-1}y, a^{-1}z)
 \tag{7}$$

for any $a \in \mathcal{G}$ and $F \in \mathcal{F}$, so that for $F', F'' \in \mathcal{F}$

$$\langle (\lambda a)F', (\lambda a)F'' \rangle = \langle F', F'' \rangle.$$

The decomposition $x = -\varphi^2x + \eta(x)\xi$ generates the projectors h and v on T_pM determined by $h(x) = -\varphi^2x$ and $v(x) = \eta(x)\xi$, having the properties $h \circ h = h, v \circ v = v, h \circ v = v \circ h = 0$.

Therefore, we have the orthogonal decomposition $T_pM = h(T_pM) \oplus v(T_pM)$.

Bearing in mind these projectors on T_pM , we construct a partial decomposition of \mathcal{F} as follows.

At first, we define the operator $p_1 : \mathcal{F} \rightarrow \mathcal{F}$ by

$$p_1(F)(x, y, z) = -F(\varphi^2x, \varphi^2y, \varphi^2z).
 \tag{8}$$

It is easy to check the following

Lemma 2.1. *The operator p_1 has the following properties:*

- (i) $p_1 \circ p_1 = p_1$;
- (ii) $\langle p_1(F'), F'' \rangle = \langle F', p_1(F'') \rangle, \quad F', F'' \in \mathcal{F}$;
- (iii) $p_1 \circ (\lambda a) = (\lambda a) \circ p_1$.

According to Lemma 2.1 we have the following orthogonal decomposition of \mathcal{F} by the image and the kernel of p_1 :

$$\mathcal{W}_1 = \text{im}(p_1) = \{ F \in \mathcal{F} \mid p_1(F) = F \}, \quad \mathcal{W}_1^\perp = \text{ker}(p_1) = \{ F \in \mathcal{F} \mid p_1(F) = 0 \}.
 \tag{9}$$

Further, we consider the operator $p_2 : \mathcal{W}_1^\perp \rightarrow \mathcal{W}_1^\perp$ defined by

$$p_2(F)(x, y, z) = \eta(y)F(\varphi^2x, \xi, \varphi^2z) + \eta(z)F(\varphi^2x, \varphi^2y, \xi)
 \tag{10}$$

for which we obtain

Lemma 2.2. *The operator p_2 has the following properties:*

- (i) $p_2 \circ p_2 = p_2$;
- (ii) $\langle p_2(F'), F'' \rangle = \langle F', p_2(F'') \rangle, \quad F', F'' \in \mathcal{W}_1^\perp$;
- (iii) $p_2 \circ (\lambda a) = (\lambda a) \circ p_2$.

Then, bearing in mind Lemma 2.2, we obtain

$$\mathcal{W}_2 = \text{im}(p_2) = \{F \in \mathcal{W}_1^\perp \mid p_2(F) = F\}, \quad \mathcal{W}_2^\perp = \ker(p_2) = \{F \in \mathcal{W}_1^\perp \mid p_2(F) = 0\}. \tag{11}$$

Finally, we consider the operator $p_3 : \mathcal{W}_2^\perp \rightarrow \mathcal{W}_2^\perp$ defined by

$$p_3(F)(x, y, z) = \eta(x)F(\xi, \varphi^2 y, \varphi^2 z)$$

and we get the following

Lemma 2.3. *The operator p_3 has the properties:*

- (i) $p_3 \circ p_3 = p_3$;
- (ii) $\langle p_3(F'), F'' \rangle = \langle F', p_3(F'') \rangle, \quad F', F'' \in \mathcal{W}_2^\perp$;
- (iii) $p_3 \circ (\lambda a) = (\lambda a) \circ p_3$.

By virtue of Lemma 2.3 we have

$$\mathcal{W}_3 = \text{im}(p_3) = \{F \in \mathcal{W}_2^\perp \mid p_3(F) = F\}, \quad \mathcal{W}_4 = \ker(p_3) = \{F \in \mathcal{W}_2^\perp \mid p_3(F) = 0\}. \tag{12}$$

From Lemma 2.1, Lemma 2.2 and Lemma 2.3 we have immediately

Theorem 2.4. *The decomposition $\mathcal{F} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ is orthogonal and invariant under the action of \mathcal{G} . The subspaces \mathcal{W}_i ($i = 1, 2, 3, 4$) are determined by*

$$\begin{aligned} \mathcal{W}_1 : F(x, y, z) &= -F(\varphi^2 x, \varphi^2 y, \varphi^2 z), \\ \mathcal{W}_2 : F(x, y, z) &= \eta(y)F(\varphi^2 x, \xi, \varphi^2 z) + \eta(z)F(\varphi^2 x, \varphi^2 y, \xi), \\ \mathcal{W}_3 : F(x, y, z) &= \eta(x)F(\xi, \varphi^2 y, \varphi^2 z), \\ \mathcal{W}_4 : F(x, y, z) &= -\eta(x) \{ \eta(y)F(\xi, \xi, \varphi^2 z) + \eta(z)F(\xi, \varphi^2 y, \xi) \}. \end{aligned} \tag{13}$$

Obviously, we have

$$F = p_1(F) + p_2(F) + p_3(F) + p_4(F). \tag{14}$$

Corollary 2.5. *The subspaces \mathcal{W}_i ($i = 1, 2, 3, 4$) are characterized as follows:*

$$\begin{aligned} \mathcal{W}_1 &= \{F \in \mathcal{F} \mid F(v(x), y, z) = F(x, v(y), z) = F(x, y, v(z)) = 0\}, \\ \mathcal{W}_2 &= \{F \in \mathcal{F} \mid F(v(x), y, z) = F(x, h(y), h(z)) = 0\}, \\ \mathcal{W}_3 &= \{F \in \mathcal{F} \mid F(h(x), y, z) = F(x, v(y), z) = F(x, y, v(z)) = 0\}, \\ \mathcal{W}_4 &= \{F \in \mathcal{F} \mid F(h(x), y, z) = F(x, h(y), h(z)) = 0\}. \end{aligned} \tag{15}$$

According to (13), (15) and (4) we obtain the following

Corollary 2.6. *The Lee forms of F have the following properties in each of the subspaces \mathcal{W}_i ($i = 1, 2, 3, 4$):*

- (i) If $F \in \mathcal{W}_1$, then $\theta \circ v = \theta^* \circ v = \omega = 0$;
- (ii) If $F \in \mathcal{W}_2$, then $\theta \circ h = \theta^* \circ h = \omega = 0$;
- (iii) If $F \in \mathcal{W}_3$, then $\theta = \theta^* = \omega = 0$;
- (iv) If $F \in \mathcal{W}_4$, then $\theta = \theta^* = 0$.

Further we continue the decomposition of the subspaces \mathcal{W}_i ($i = 1, 2, 3, 4$) of \mathcal{F} .

2.1. The decomposition of \mathcal{W}_1

Let us consider the $2n$ -dimensional distribution $H = \ker(\eta)$ of the tangent bundle of $(M, \varphi, \xi, \eta, g)$, the endomorphism $J = \varphi|_H$ and the metric $h = g|_H$, where $\varphi|_H$ and $g|_H$ are the restrictions of φ and g on H , respectively. Let us remark that J and h are an almost complex structure and a Norden metric on H , respectively, i.e.

$$J^2 = -\text{Id}, \quad h(Jx, Jy) = -h(x, y). \tag{16}$$

Then (H, J, h) can be considered as an almost complex manifold with Norden metric.

Moreover, the subspace \mathcal{W}_1 coincides with the restriction of \mathcal{F} on H . By this reason the decomposition of \mathcal{W}_1 is made as the decomposition, known from [2], of the corresponding space of \mathcal{F} for an almost complex manifold with Norden metric. Then we obtain the following

Proposition 2.7. *Let $F \in \mathcal{F}$ and F_i ($i = 1, 2, 3$) be the projections of F on the subspaces \mathcal{F}_i , respectively. Then*

$$\begin{aligned} F_1(x, y, z) &= \frac{1}{2n} \{g(\varphi x, \varphi y)\theta(\varphi^2 z) + g(x, \varphi y)\theta(\varphi z) + g(\varphi x, \varphi z)\theta(\varphi^2 y) + g(x, \varphi z)\theta(\varphi y)\}; \\ F_2(x, y, z) &= -\frac{1}{4} \{F(\varphi^2 x, \varphi^2 y, \varphi^2 z) + F(\varphi^2 y, \varphi^2 z, \varphi^2 x) - F(\varphi y, \varphi^2 z, \varphi x) + F(\varphi^2 x, \varphi^2 z, \varphi^2 y) \\ &\quad + F(\varphi^2 z, \varphi^2 y, \varphi^2 x) - F(\varphi z, \varphi^2 y, \varphi x)\} - \frac{1}{2n} \{g(\varphi x, \varphi y)\theta(\varphi^2 z) + g(x, \varphi y)\theta(\varphi z) \\ &\quad + g(\varphi x, \varphi z)\theta(\varphi^2 y) + g(x, \varphi z)\theta(\varphi y)\}; \\ F_3(x, y, z) &= -\frac{1}{4} \{F(\varphi^2 x, \varphi^2 y, \varphi^2 z) - F(\varphi^2 y, \varphi^2 z, \varphi^2 x) + F(\varphi y, \varphi^2 z, \varphi x) + F(\varphi^2 x, \varphi^2 z, \varphi^2 y) \\ &\quad - F(\varphi^2 z, \varphi^2 y, \varphi^2 x) + F(\varphi z, \varphi^2 y, \varphi x)\}. \end{aligned} \tag{17}$$

Therefore, the component of F on \mathcal{W}_1 is

$$p_1(F) = F_1 + F_2 + F_3. \tag{18}$$

2.2. The decomposition of \mathcal{W}_2

Let us consider linear operators $L_j : \mathcal{W}_2 \rightarrow \mathcal{W}_2$ ($j = 1, 2$) defined by

$$\begin{aligned} L_1(F)(x, y, z) &= F(\varphi x, \varphi y, \xi)\eta(z) + F(\varphi x, \varphi z, \xi)\eta(y), \\ L_2(F)(x, y, z) &= F(\varphi^2 y, \varphi^2 x, \xi)\eta(z) + F(\varphi^2 z, \varphi^2 x, \xi)\eta(y). \end{aligned} \tag{19}$$

It is easy to check the following

Lemma 2.8. *The linear operator L_j ($j = 1, 2$) is an involutive isometry on \mathcal{W}_2 and it is invariant with respect to the group \mathcal{G} , i.e.*

$$L_j \circ L_j = \text{Id}_{\mathcal{W}_2}, \quad \langle L_j(F'), L_j(F'') \rangle = \langle F', F'' \rangle, \quad L_j((\lambda a)F) = (\lambda a)(L_j(F)),$$

where $F', F'' \in \mathcal{W}_2$ and λa is determined by (7).

Therefore, L_1 has two eigenvalues $+1$ and -1 , and the corresponding eigenspaces

$$\mathcal{W}_2^+ = \{F \in \mathcal{W}_2 \mid L_1(F) = F\}, \quad \mathcal{W}_2^- = \{F \in \mathcal{W}_2 \mid L_1(F) = -F\}$$

are invariant orthogonal subspaces of \mathcal{W}_2 .

In order to decompose \mathcal{W}_2^+ and \mathcal{W}_2^- , we use the operator L_2 on \mathcal{W}_2^+ and \mathcal{W}_2^- , respectively. Let us denote the corresponding eigenspaces $\mathcal{W}_{2,k}$ ($k = 1, 2, 3, 4$) by

$$\begin{aligned} \mathcal{W}_{2,1} &= \{\mathcal{W}_2^+ \mid L_2(F) = -F\}, & \mathcal{W}_{2,2} &= \{\mathcal{W}_2^- \mid L_2(F) = -F\}, \\ \mathcal{W}_{2,3} &= \{\mathcal{W}_2^+ \mid L_2(F) = F\}, & \mathcal{W}_{2,4} &= \{\mathcal{W}_2^- \mid L_2(F) = F\}. \end{aligned}$$

Thus we establish the truthfulness of the following

Theorem 2.9. *The decomposition $\mathcal{W}_2 = \mathcal{W}_{2,1} \oplus \mathcal{W}_{2,2} \oplus \mathcal{W}_{2,3} \oplus \mathcal{W}_{2,4}$ is orthogonal and invariant with respect to the structure group.*

Proposition 2.10. *Let $F \in \mathcal{F}$ and F_j ($j = 4, \dots, 9$) be the projections of \mathcal{F} in the classes \mathcal{F}_j . Then we have*

$$\begin{aligned} F_4(x, y, z) &= -\frac{\theta(\xi)}{2n} \{g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y)\}; \\ F_5(x, y, z) &= -\frac{\theta^*(\xi)}{2n} \{g(x, \varphi y)\eta(z) + g(x, \varphi z)\eta(y)\}; \end{aligned} \tag{20a}$$

$$\begin{aligned} F_6(x, y, z) &= \frac{\theta(\xi)}{2n} \{g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y)\} + \frac{\theta^*(\xi)}{2n} \{g(x, \varphi y)\eta(z) + g(x, \varphi z)\eta(y)\} \\ &\quad + \frac{1}{4} [F(\varphi^2 x, \varphi^2 y, \xi) + F(\varphi^2 y, \varphi^2 x, \xi) - F(\varphi x, \varphi y, \xi) - F(\varphi y, \varphi x, \xi)] \eta(z) \\ &\quad + \frac{1}{4} [F(\varphi^2 x, \varphi^2 z, \xi) + F(\varphi^2 z, \varphi^2 x, \xi) - F(\varphi x, \varphi z, \xi) - F(\varphi z, \varphi x, \xi)] \eta(y); \\ F_7(x, y, z) &= \frac{1}{4} [F(\varphi^2 x, \varphi^2 y, \xi) - F(\varphi^2 y, \varphi^2 x, \xi) - F(\varphi x, \varphi y, \xi) + F(\varphi y, \varphi x, \xi)] \eta(z) \\ &\quad + \frac{1}{4} [F(\varphi^2 x, \varphi^2 z, \xi) - F(\varphi^2 z, \varphi^2 x, \xi) - F(\varphi x, \varphi z, \xi) + F(\varphi z, \varphi x, \xi)] \eta(y); \end{aligned} \tag{20b}$$

$$\begin{aligned} F_8(x, y, z) &= \frac{1}{4} [F(\varphi^2 x, \varphi^2 y, \xi) + F(\varphi^2 y, \varphi^2 x, \xi) + F(\varphi x, \varphi y, \xi) + F(\varphi y, \varphi x, \xi)] \eta(z) \\ &\quad + \frac{1}{4} [F(\varphi^2 x, \varphi^2 z, \xi) + F(\varphi^2 z, \varphi^2 x, \xi) + F(\varphi x, \varphi z, \xi) + F(\varphi z, \varphi x, \xi)] \eta(y); \\ F_9(x, y, z) &= \frac{1}{4} [F(\varphi^2 x, \varphi^2 y, \xi) - F(\varphi^2 y, \varphi^2 x, \xi) + F(\varphi x, \varphi y, \xi) - F(\varphi y, \varphi x, \xi)] \eta(z) \\ &\quad + \frac{1}{4} [F(\varphi^2 x, \varphi^2 z, \xi) - F(\varphi^2 z, \varphi^2 x, \xi) + F(\varphi x, \varphi z, \xi) - F(\varphi z, \varphi x, \xi)] \eta(y). \end{aligned}$$

Proof. Lemma 2.8 implies that the tensor $\frac{1}{2} \{F + L_1(F)\}$ is the projection of $F \in \mathcal{W}_2$ in $\mathcal{W}_2^+ = \mathcal{W}_{2,1} \oplus \mathcal{W}_{2,3}$ and moreover $\frac{1}{2} \{F - L_2(F)\}$ is the projection of $F \in \mathcal{W}_2^+$ in $\mathcal{W}_{2,1}$. Thus, we find the expression of the projection $p_{2,1}$ of F from \mathcal{W}_2 to $\mathcal{W}_{2,1}$ in terms of L_1 and L_2 , namely

$$p_{2,1}(F) = \frac{1}{4} \{F - L_1(F) + L_2(F) - L_2 \circ L_1(F)\}.$$

In a similar way we treat with the projections $p_{2,k}(F)$ in $\mathcal{W}_{2,k}$ ($k = 2, 3, 4$). After that, using (19), we get the

following expressions

$$\begin{aligned}
 p_{2,1}(F)(x, y, z) &= \frac{1}{4} [F(\varphi^2x, \varphi^2y, \xi) + F(\varphi^2y, \varphi^2x, \xi) - F(\varphi x, \varphi y, \xi) - F(\varphi y, \varphi x, \xi)] \eta(z) \\
 &\quad + \frac{1}{4} [F(\varphi^2x, \varphi^2z, \xi) + F(\varphi^2z, \varphi^2x, \xi) - F(\varphi x, \varphi z, \xi) - F(\varphi z, \varphi x, \xi)] \eta(y); \\
 p_{2,2}(F)(x, y, z) &= \frac{1}{4} [F(\varphi^2x, \varphi^2y, \xi) - F(\varphi^2y, \varphi^2x, \xi) - F(\varphi x, \varphi y, \xi) + F(\varphi y, \varphi x, \xi)] \eta(z) \\
 &\quad + \frac{1}{4} [F(\varphi^2x, \varphi^2z, \xi) - F(\varphi^2z, \varphi^2x, \xi) - F(\varphi x, \varphi z, \xi) + F(\varphi z, \varphi x, \xi)] \eta(y); \\
 p_{2,3}(F)(x, y, z) &= \frac{1}{4} [F(\varphi^2x, \varphi^2y, \xi) + F(\varphi^2y, \varphi^2x, \xi) + F(\varphi x, \varphi y, \xi) + F(\varphi y, \varphi x, \xi)] \eta(z) \\
 &\quad + \frac{1}{4} [F(\varphi^2x, \varphi^2z, \xi) + F(\varphi^2z, \varphi^2x, \xi) + F(\varphi x, \varphi z, \xi) + F(\varphi z, \varphi x, \xi)] \eta(y); \\
 p_{2,4}(F)(x, y, z) &= \frac{1}{4} [F(\varphi^2x, \varphi^2y, \xi) - F(\varphi^2y, \varphi^2x, \xi) + F(\varphi x, \varphi y, \xi) - F(\varphi y, \varphi x, \xi)] \eta(z) \\
 &\quad + \frac{1}{4} [F(\varphi^2x, \varphi^2z, \xi) - F(\varphi^2z, \varphi^2x, \xi) + F(\varphi x, \varphi z, \xi) - F(\varphi z, \varphi x, \xi)] \eta(y).
 \end{aligned} \tag{21}$$

By virtue of Corollary 2.6, (4) and (21), we establish that the Lee forms θ and θ^* of F are zero in $\mathcal{W}_{2,k}$ ($k = 2, 3, 4$). We have no additional conditions for θ and θ^* in $\mathcal{W}_{2,1}$. Then, $\mathcal{W}_{2,1}$ can be additionally decomposed to three subspaces determined by the conditions $\theta = 0$, $\theta^* = 0$ and $\theta = \theta^* = 0$, respectively, *i.e.*

$$\mathcal{W}_{2,1} = \mathcal{W}_{2,1,1} \oplus \mathcal{W}_{2,1,2} \oplus \mathcal{W}_{2,1,3},$$

where

$$\begin{aligned}
 \mathcal{W}_{2,1,1} &= \{F \in \mathcal{W}_{2,1} \mid \theta^* = 0\}, & \mathcal{W}_{2,1,2} &= \{F \in \mathcal{W}_{2,1} \mid \theta = 0\}, \\
 \mathcal{W}_{2,1,3} &= \{F \in \mathcal{W}_{2,1} \mid \theta = 0, \theta^* = 0\}.
 \end{aligned}$$

According to (5), the classes \mathcal{F}_4 and \mathcal{F}_5 are defined by explicit expressions of F , which have the form of $p_{2,1}(F)$ in (21). Hence we conclude that \mathcal{F}_4 and \mathcal{F}_5 are the subspaces $\mathcal{W}_{2,1,1}$ and $\mathcal{W}_{2,1,2}$ of $\mathcal{W}_{2,1}$, respectively, and the projections of F on them have the form given in the first line of (20). Therefore, the equality for F_6 in (20) follows from the form of $p_{2,1}(F)$ in (21) and the fact that \mathcal{F}_6 coincides with $\mathcal{W}_{2,1,3}$.

The form of $p_{2,2}(F)$, $p_{2,3}(F)$ and $p_{2,4}(F)$ in (21) satisfies the conditions in (5) for the subspace \mathcal{F}_7 , \mathcal{F}_8 and \mathcal{F}_9 , respectively.

Thus, the subspaces $\mathcal{W}_{2,1,1}$, $\mathcal{W}_{2,1,2}$, $\mathcal{W}_{2,1,3}$, $\mathcal{W}_{2,2}$, $\mathcal{W}_{2,3}$, $\mathcal{W}_{2,4}$ correspond to the classes \mathcal{F}_4 , \mathcal{F}_5 , \mathcal{F}_6 , \mathcal{F}_7 , \mathcal{F}_8 , \mathcal{F}_9 , respectively. \square

Therefore, the component of F on \mathcal{W}_2 is

$$p_2(F) = F_4 + F_5 + F_6 + F_7 + F_8 + F_9. \tag{22}$$

2.3. The decomposition of \mathcal{W}_3 and \mathcal{W}_4

Finally, since \mathcal{F}_{10} and \mathcal{F}_{11} are determined in (5) by an expression of F which coincide with the conditions in (13) for \mathcal{W}_3 and \mathcal{W}_4 , respectively, we have the following

Proposition 2.11. *Let $F \in \mathcal{F}$ and F_l ($l = 10, 11$) be the projections of \mathcal{F} in the subspaces \mathcal{F}_l . Then we have*

$$\begin{aligned}
 F_{10}(x, y, z) &= \eta(x)F(\xi, \varphi^2y, \varphi^2z); \\
 F_{11}(x, y, z) &= -\eta(x) \{ \eta(y)F(\xi, \xi, \varphi^2z) + \eta(z)F(\xi, \varphi^2y, \xi) \}.
 \end{aligned} \tag{23}$$

Therefore, the components of F on \mathcal{W}_3 and \mathcal{W}_4 are

$$p_3(F) = F_{10}, \quad p_4(F) = F_{11}, \tag{24}$$

respectively. Then, bearing in mind (18), (22) and (24), we obtain that

$$F(x, y, z) = \sum_{i=1}^{11} F_i(x, y, z),$$

where the components F_i of F in the corresponding subspaces \mathcal{F}_i ($i = 1, \dots, 11$) of \mathcal{F} are determined in Proposition 2.7, Proposition 2.10 and Proposition 2.11.

In conclusion we give

Theorem 2.12. *The almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$ belongs to the basic class \mathcal{F}_i ($i = 1, \dots, 11$) if and only if the structure tensor F satisfies the condition $F = F_i$, where the components F_i of F are given in (17), (20) and (23).*

It is easy to conclude that an almost contact B-metric manifold belongs to a direct sum of two or more basic classes, i.e. $(M, \varphi, \xi, \eta, g) \in \mathcal{F}_i \oplus \mathcal{F}_j \oplus \dots$, if and only if the structure tensor F on $(M, \varphi, \xi, \eta, g)$ is the sum of the corresponding components F_i, F_j, \dots of F , i.e. the following condition is satisfied $F = F_i + F_j + \dots$.

3. The components of the structure tensor for dimension 3

In this section we are interesting in the lowest dimension of the manifolds under consideration, i.e. we consider the case of $\dim M = 3$ for $(M, \varphi, \xi, \eta, g)$.

Let us denote the components $F_{ijk} = F(e_i, e_j, e_k)$ of the structure tensor F with respect to a φ -basis $\{e_i\}_{i=0}^2 = \{e_0 = \xi, e_1 = e, e_2 = \varphi e\}$, which satisfies the following conditions

$$g(e_0, e_0) = g(e_1, e_1) = -g(e_2, e_2) = 1, \quad g(e_i, e_j) = 0, \quad i \neq j \in \{0, 1, 2\}. \tag{25}$$

Then, using (4) and (25), we obtain the components of the Lee forms with respect to the basis $\{e_i\}_{i=0}^2$ as follows

$$\begin{aligned} \theta_0 &= F_{110} - F_{220}, & \theta_1 &= F_{111} - F_{221}, & \theta_2 &= F_{112} - F_{211}, \\ \theta_0^* &= F_{120} + F_{210}, & \theta_1^* &= F_{112} + F_{211}, & \theta_2^* &= F_{111} + F_{221}, \\ \omega_0 &= 0, & \omega_1 &= F_{001}, & \omega_2 &= F_{002}. \end{aligned} \tag{26}$$

Let us consider arbitrary vectors $x, y, z \in T_p M$. Therefore we have $x = x^i e_i, y = y^j e_j, z = z^k e_k$ with respect to $\{e_i\}_{i=0}^2$.

By direct computations we obtain

Proposition 3.1. *The components F_i ($i = 1, 2, \dots, 11$) of the structure tensor F in the corresponding basic classes \mathcal{F}_i are the following*

$$\begin{aligned} F_1(x, y, z) &= (x^1 \theta_1 - x^2 \theta_2)(y^1 z^1 + y^2 z^2), & \theta_1 &= F_{111} = F_{122}, & \theta_2 &= -F_{211} = -F_{222}; \\ F_2(x, y, z) &= F_3(x, y, z) = 0; \\ F_4(x, y, z) &= \frac{1}{2} \theta_0 \{x^1 (y^0 z^1 + y^1 z^0) - x^2 (y^0 z^2 + y^2 z^0)\}, & \frac{1}{2} \theta_0 &= F_{101} = F_{110} = -F_{202} = -F_{220}; \\ F_5(x, y, z) &= \frac{1}{2} \theta_0^* \{x^1 (y^0 z^2 + y^2 z^0) + x^2 (y^0 z^1 + y^1 z^0)\}, & \frac{1}{2} \theta_0^* &= F_{102} = F_{120} = F_{201} = F_{210}; \\ F_6(x, y, z) &= F_7(x, y, z) = 0; \\ F_8(x, y, z) &= \lambda \{x^1 (y^0 z^1 + y^1 z^0) + x^2 (y^0 z^2 + y^2 z^0)\}, & \lambda &= F_{101} = F_{110} = F_{202} = F_{220}; \\ F_9(x, y, z) &= \mu \{x^1 (y^0 z^2 + y^2 z^0) - x^2 (y^0 z^1 + y^1 z^0)\}, & \mu &= F_{102} = F_{120} = -F_{201} = -F_{210}; \\ F_{10}(x, y, z) &= \nu x^0 (y^1 z^1 + y^2 z^2), & \nu &= F_{011} = F_{022}; \\ F_{11}(x, y, z) &= x^0 \{(y^1 z^0 + y^0 z^1) \omega_1 + (y^2 z^0 + y^0 z^2) \omega_2\}, & \omega_1 &= F_{010} = F_{001}, & \omega_2 &= F_{020} = F_{002}. \end{aligned} \tag{27}$$

Proof. Using Theorem 2.12 and the expressions of the components F_i of F for the corresponding classes \mathcal{F}_i ($i = 1, \dots, 11$), determined by (17), (20) and (23), the equalities (25), (26) and the properties (3) of F , we obtain the corresponding explicit expression of F_i for dimension 3. \square

According to Theorem 2.12 and Proposition 3.1, we obtain

Theorem 3.2. *The class of almost contact B-metric manifolds of dimension 3 is*

$$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11},$$

i.e. the basic classes $\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_6, \mathcal{F}_7$ are restricted to the special class \mathcal{F}_0 .

4. Some examples

4.1. Time-like sphere as a manifold from the class $\mathcal{F}_4 \oplus \mathcal{F}_5$

In [3], an example of an almost contact manifold with B-metric is given. It is constructed as a time-like sphere of \mathbb{R}^{2n+2} with complex structure and Norden metric. Namely, let $\mathbb{R}^{2n+2} = \{(u^1, \dots, u^{n+1}; v^1, \dots, v^{n+1}) \mid u^i, v^i \in \mathbb{R}\}$ be considered as a complex Riemannian manifold with the canonical complex structure J and the metric g defined by

$$J \frac{\partial}{\partial u^i} = \frac{\partial}{\partial v^i}, \quad J \frac{\partial}{\partial v^i} = -\frac{\partial}{\partial u^i}, \quad g(x, x) = -\delta_{ij} \lambda^i \lambda^j + \delta_{ij} \mu^i \mu^j,$$

where $x = \lambda \frac{\partial}{\partial u^i} + \mu \frac{\partial}{\partial v^i}$. Identifying the point p in \mathbb{R}^{2n+2} with its position vector Z , we define the unit time-like sphere

$$S^{2n+1} : g(Z, Z) = -1.$$

The almost contact structure is determined by

$$\xi = \sin t \cdot Z + \cos t \cdot JZ, \quad Jx = \varphi x + \eta(x)J\xi,$$

where $t = \arctan(g(z, Jz)) \in (-\frac{\pi}{2}; \frac{\pi}{2})$ and $x, \varphi x \in T_p S^{2n+1}$. The metric on the hypersurface is the restriction of g and it is denote by the same letter. Then $(S^{2n+1}, \varphi, \xi, \eta, g)$ is an almost contact B-metric manifold. It belongs to the class $\mathcal{F}_4 \oplus \mathcal{F}_5$ because the structure tensor has the following form

$$F(x, y, z) = -\cos t \{g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y)\} - \sin t \{g(x, \varphi y)\eta(z) + g(x, \varphi z)\eta(y)\},$$

where $\cos t = \frac{\theta(\xi)}{2n}$, $\sin t = \frac{\theta^*(\xi)}{2n}$ and then we obtain the following expression of F , bearing in mind (20):

$$F = F_4 + F_5.$$

If we consider the 3-dimensional unit time-like sphere $(S^3, \varphi, \xi, \eta, g)$ then we have the following form of the structure tensor, using (27):

$$F(x, y, z) = \frac{1}{2} \{(\theta_0 x^1 + \theta_0^* x^2)(y^0 z^1 + y^1 z^0) + (\theta_0^* x^1 - \theta_0 x^2)(y^0 z^2 + y^2 z^0)\}$$

where $\frac{1}{2}\theta_0 = F_{101} = F_{110} = -F_{202} = -F_{220}$ and $\frac{1}{2}\theta_0^* = F_{102} = F_{120} = F_{201} = F_{210}$ with respect to the orthonormal φ -basis $\{e_i\}_{i=0}^2$.

4.2. Lie group as a manifold from the class $\mathcal{F}_9 \oplus \mathcal{F}_{10}$

Let L be a $(2n + 1)$ -dimensional real connected Lie group and its associated Lie algebra with a global basis $\{E_0, E_1, \dots, E_{2n}\}$ of left invariant vector fields on L defined by

$$[E_0, E_i] = -a_i E_i - a_{n+i} E_{n+i}, \quad [E_0, E_{n+i}] = -a_{n+i} E_i + a_i E_{n+i}, \tag{28}$$

where a_1, \dots, a_{2n} are real constants and $[E_j, E_k] = 0$ in other cases. Let an invariant almost contact structure be defined for $i \in \{1, \dots, n\}$ by

$$\varphi E_0 = 0, \quad \varphi E_i = E_{n+i}, \quad \varphi E_{n+i} = -E_i, \quad \xi = E_0, \quad \eta(E_0) = 1, \quad \eta(E_i) = \eta(E_{n+i}) = 0. \tag{29}$$

Let g be a pseudo-Riemannian metric such that for $i \in \{1, \dots, n\}$ and $j, k \in \{1, \dots, 2n\}$, $j \neq k$ the following equalities are valid

$$g(E_0, E_0) = g(E_i, E_i) = -g(E_{n+i}, E_{n+i}) = 1, \quad g(E_0, E_j) = g(E_j, E_k) = 0. \tag{30}$$

Thus, because of (1), the induced $(2n + 1)$ -dimensional manifold $(L, \varphi, \xi, \eta, g)$ is an almost contact B-metric manifold.

Let us remark that in [9] the same Lie group with the same almost contact structure but equipped with a compatible Riemannian metric is studied as an almost cosymplectic manifold.

Let us consider the constructed almost contact B-metric manifold $(L, \varphi, \xi, \eta, g)$ in dimension 3, i.e. for $n = 1$.

By virtue of (28) and (30) for $n = 1$, and using the Koszul equality

$$2g(\nabla_{E_i} E_j, E_k) = g([E_i, E_j], E_k) + g([E_k, E_i], E_j) + g([E_k, E_j], E_i) \tag{31}$$

for the Levi-Civita connection ∇ of g , we obtain

$$\nabla_{E_1} E_1 = \nabla_{E_2} E_2 = -a_1 E_0, \quad \nabla_{E_0} E_1 = -a_2 E_2, \quad \nabla_{E_0} E_2 = -a_2 E_1, \quad \nabla_{E_1} E_0 = a_1 E_1, \quad \nabla_{E_2} E_0 = -a_1 E_2. \tag{32}$$

Then, using the latter equalities, (29) and (2), we get the following nonzero components $F_{ijk} = F(E_i, E_j, E_k)$ of the structure tensor:

$$F_{011} = F_{022} = -2a_2, \quad F_{102} = F_{120} = -F_{201} = -F_{210} = a_1.$$

Thus, we establish the following form of F for arbitrary vectors $x = x^i E_i, y = y^j E_j, z = z^k E_k$

$$F(x, y, z) = -2a_2 x^0 (y^1 z^1 + y^2 z^2) + a_1 \{z^0 (x^1 y^2 - x^2 y^1) + y^0 (x^1 z^2 - x^2 z^1)\}.$$

The latter equality implies that F is represented in the form

$$F(x, y, z) = F_9(x, y, z) + F_{10}(x, y, z),$$

bearing in mind (27) for $\mu = a_1, \nu = -2a_2$; or alternatively, the corresponding equalities from (20) and (23). Therefore, we prove that the constructed 3-dimensional manifold belongs to the class $\mathcal{F}_9 \oplus \mathcal{F}_{10}$.

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