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A Certain Subclass of Multivalent Functions Involving Higher-Order Derivatives

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Abstract. In this paper we introduce and study a new class of analytic and p-valent functions involving higher-order derivatives. For this p-valent function class, we derive several interesting properties including (for example) coefficient inequalities, distortion theorems, extreme points, and the radii of close-to-convexity, starlikeness and convexity. Several applications involving an integral operator are also considered. Finally, we obtain some results for the modified Hadamard product of the functions belonging to the p-valent function class which is introduced here.

1. Introduction, Definitions and Motivation

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{UST}(p, \alpha, \beta)$ of p-valent β-uniformly starlike functions of order α in $\mathbb{U}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) \geqq \beta \left|\frac{zf'(z)}{f(z)} - p\right| \qquad (z \in \mathbb{U};\ -p \leqq \alpha < p;\ \beta \geqq 0). \tag{1.2}$$

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On the other hand, a function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{UCV}(p,\alpha,\beta)$ of p-valent β-uniformly convex functions of order α in $\mathbb{U}$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right) \geqq \beta\left|1+\frac{zf''(z)}{f'(z)}-p\right| \qquad (z \in \mathbb{U};\ -p \leqq \alpha < p;\ \beta \geqq 0). \tag{1.3}$$

The above-defined function classes $\mathcal{UST}(p,\alpha,\beta)$ and $\mathcal{UCV}(p,\alpha,\beta)$ were introduced recently by Khairnar and More [10]. Various analogous classes of analytic and univalent or multivalent functions were studied in many papers (see, for example, [2], [4] and [9]).

We notice from the inequalities (1.2) and (1.3) that

$$f(z) \in \mathcal{UCV}(p,\alpha,\beta) \iff \frac{zf'(z)}{p} \in \mathcal{UST}(p,\alpha,\beta). \tag{1.4}$$

Now, for each $f(z) \in \mathcal{A}(p)$, it is easily seen upon differentiating both sides of (1.1) q times with respect to z that

$$f^{(q)}(z) = \delta(p,q)z^{p-q} + \sum_{k=p+1}^{\infty} \delta(k,q)a_k z^{k-q} \qquad (q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ p > q), \tag{1.5}$$

where, and in what follows, $\delta(p,q)$ denotes the q-permutations of p objects $(p \geqq q \geqq 0)$, that is,

$$\delta(p,q) := \frac{p!}{(p-q)!} = \begin{cases} p(p-1)\cdots(p-q+1) & (q \neq 0) \\ 1 & (q = 0), \end{cases}$$

which may also be identified with the notation $\{p\}_q$ for the *descending factorial*.

Let

$$-\delta(p-q,m) \leqq \alpha < \delta(p-q,m), \quad \beta \geqq 0 \qquad \text{and} \qquad p > q+m \qquad (p \in \mathbb{N};\ q,m \in \mathbb{N}_0).$$

We then denote by $\mathcal{US}_m(p,q;\alpha,\beta)$ the subclass of the p-valent function class $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (1.1), which also satisfy the following analytic criterion:

$$\Re\left(\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)}-\alpha\right) \geqq \beta\left|\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)}-\delta(p-q,m)\right| \qquad (z \in \mathbb{U}). \tag{1.6}$$

We also denote by $\mathcal{T}(p)$ the subclass of $\mathcal{A}(p)$ consisting of functions of the following form:

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \qquad (a_k \geqq 0;\ p \in \mathbb{N}). \tag{1.7}$$

Further, we define the class $\mathcal{UST}_m(p,q;\alpha,\beta)$ as follows:

$$\mathcal{UST}_m(p,q;\alpha,\beta) = \mathcal{US}_m(p,q;\alpha,\beta) \cap \mathcal{T}(p). \tag{1.8}$$

For suitable choices of $p,q,$ m and $\beta,$ we obtain the following known subclasses:

(i) It is easily verified that $\big($see Liu and Liu [11] (with $\gamma = 1$ and $n = 1$)$\big)$

$$\mathcal{UST}_m(p,q,\alpha,0) = \mathcal{A}^*_{1,p}(m,q,\alpha,1)$$
$$\big(0 \leqq \alpha < \delta(p-q,m);\ p \in \mathbb{N};\ m,q \in \mathbb{N}_0;\ p > q+m\big);$$

(ii) We observe that (see Khairnar and More [10])

$$\mathcal{UST}_1(p,0;\alpha,\beta) = \mathcal{UST}(p,\alpha,\beta) \qquad (-p \leqq \alpha < p;\ \beta \geqq 0;\ p \in \mathbb{N})$$

and

$$\mathcal{UST}_1(p,1;\alpha,\beta) = \mathcal{UCV}(p,\gamma,\beta) \qquad (-p \leqq \gamma = \alpha+1 < p;\ \beta \geqq 0;\ p \in \mathbb{N});$$

(iii) It is easy to see that $\big($see Aouf [3] (with $\beta = 1$ and $n = 1$)$\big)$

$$\mathcal{UST}_1(p,q,\alpha,0) = \mathcal{S}_1(p,q,\alpha,1) \qquad (0 \leqq \alpha < p-q;\ p \in \mathbb{N};\ q \in \mathbb{N}_0;\ p > q+1)$$

and

$$\mathcal{UST}_1(p,q,\alpha,0) = C_1(p,t,\gamma,1) \qquad (0 \leqq \alpha < p-q;\ p,q \in \mathbb{N};\ p > q+1;\ t = q-1;\ \gamma = \alpha+1);$$

(iv) We notice that $\big($see Chen *et al.* [5] (with $n = 1$)$\big)$

$$\mathcal{UST}_1(p,q,\alpha,0) = \mathcal{S}_1(p,q,\alpha) \qquad (0 \leqq \alpha < p-q;\ p \in \mathbb{N};\ q \in \mathbb{N}_0;\ p > q+1)$$

and

$$\mathcal{UST}_1(p,q,\alpha,0) = C_1(p,t,\gamma) \qquad (0 \leqq \alpha < p-q;\ p,q \in \mathbb{N};\ p > q+1;\ t = q-1;\ \gamma = \alpha+1).$$

In this paper we obtain several properties (including the coefficient inequalities, distortion theorems, extreme points, and the radii of close-to-convexity, starlikeness and convexity) of the class $\mathcal{UST}_m(p,q;\alpha,\beta)$. We also consider some applications involving an integral operator. Finally, we obtain some results for the modified Hadamard product of functions in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$.

Various other papers were dedicated to the study of such aspects of analytic function theory as we have considered in this paper. For example, in the paper [1] several interesting properties of closed-to-convex functions with negative coefficients were investigated by using the familiar Sălăgean derivative operator, in the paper [8] a certain subclass of univalent functions with negative coefficients was introduced and studied by using a generalization of the Sălăgean derivative operator, and so on and so forth. Another class of analytic and multivalent functions was studied in the paper [12] by applying the Hadamard product (or convolution) and the widely-investigated Dziok-Srivastava operator, where the class was proved as being closed under convolution and some integral operators (see also the recent works [6], [14] and [15]).

2. Coefficient Estimates

Unless otherwise mentioned, we assume throughout this paper that

$$-\delta(p-q,m) \leqq \alpha < \delta(p-q,m), \quad \beta \geqq 0, \quad q,m \in \mathbb{N}_0, \quad p \in \mathbb{N} \quad \text{and} \quad p > q+m.$$

Our first result (Theorem 1 below) provides the coefficient inequalities for functions in the class $\mathcal{US}_m(p,q;\alpha,\beta)$.

Theorem 1. *A function $f(z)$ of the form* (1.1) *is in the class $\mathcal{US}_m(p,q;\alpha,\beta)$ if*

$$\sum_{k=p+1}^{\infty} \Big[(1+\beta)[\delta(k-q,m) - \delta(p-q,m)] + [\delta(p-q,m) - \alpha]\Big]\delta(k,q)a_k \leqq [\delta(p-q,m) - \alpha]\,\delta(p,q). \tag{2.1}$$

Proof. It is easy to show that

$$\beta\left|\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \delta(p-q,m)\right| - \Re\left(\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \delta(p-q,m)\right) \leqq [\delta(p-q,m)-\alpha],$$

which implies the result (2.1) asserted by Theorem 1. □

Theorem 2. *A necessary and sufficient condition for* $f(z)$ *of the form* (1.7) *to be in the class* $\mathcal{UST}_m(p,q;\alpha,\beta)$ *is that*

$$\sum_{k=p+1}^{\infty}\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(k,q)a_k \leqq [\delta(p-q,m)-\alpha]\,\delta(p,q). \tag{2.2}$$

Proof. In view of Theorem 1, we need only to prove the necessity. If $f(z) \in \mathcal{UST}_m(p,q;\alpha,\beta)$ and z is a real number, then

$$\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \alpha \geqq \beta\left|\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \delta(p-q,m)\right|.$$

By making some calculations and letting $z \to 1-$ along the real axis, we have the desired inequality (2.2). □

Remark 1.

(i) Taking $\beta = 0$, Theorem 2 extends the result for the coefficient estimates related to the class $\mathcal{A}^*_{1,p}(m,q,\alpha,1)$, which is due to Liu and Liu [11] (with $\gamma = 1$ and $n = 1$);

(ii) Taking $q = 0$ and $p = m = 1$, Theorem 2 extends the result for the coefficient estimates related to the class $\mathcal{ST}_0(\alpha,\beta)$, which is due to Frasin [7] (with $a_1 = 1$);

(iii) Taking $p = m = 1$, $q = t+1$, $t = 0$ and $\alpha = \gamma - 1$, Theorem 2 extends the result for the coefficient estimates related to the class $\mathcal{UCT}_0(\gamma,\beta)$, which is due to Frasin [7] (with $a_1 = 1$);

(iv) Taking $\beta = 0$ and $m = 1$, Theorem 2 extends the result for the coefficient estimates related to the class $\mathcal{S}_1(p,q,\alpha,1)$, which is due to Aouf [3] (with $\beta = 1$ and $n = 1$);

(v) Taking $\beta = 0$, $m = 1$, $q = t+1$ and $\alpha = \gamma - 1$, Theorem 2 extends the result for the coefficient estimates related to the class $C_1(p,t,\gamma,1)$, which is due to Aouf [3] (with $\beta = 1$ and $n = 1$);

(vi) Taking $\beta = 0$ and $m = 1$, Theorem 2 extends the result for the coefficient estimates related to the class $\mathcal{S}(p,q,\alpha)$, which is due to Chen *et al.* [5] (with $n = 1$);

(vii) Taking $\beta = 0$, $m = 1$, $q = t+1$ and $\alpha = \gamma - 1$, Theorem 2 extends the result for the coefficient estimates related to the class $C(p,t,\gamma)$, which is due to Chen *et al.* [5] (with $n = 1$).

Corollary 1. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $\mathcal{UST}_m(p,q;\alpha,\beta)$. *Then*

$$a_k \leqq \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(k,q)} \qquad (k \geqq p+1). \tag{2.3}$$

The result is sharp for the functions $f_k(z)$ given by

$$f_k(z) = z^p - \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(k,q)} z^k \qquad (k \geqq p+1). \tag{2.4}$$

3. Distortion Theorems

Theorem 3. *Let the function $f(z)$ defined by* (1.7) *be in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$. Then, for $|z| = r < 1$,*

$$\big|f(z)\big| \geqq r^p - \frac{[\delta(p-q,m)-\alpha]\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(p+1,q)}\, r^{p+1} \tag{3.1}$$

and

$$\big|f(z)\big| \leqq r^p + \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(p+1,q)}\, r^{p+1}, \tag{3.2}$$

The equalities in (3.1) *and* (3.2) *are attained for the function $f(z)$ given by*

$$f(z) = z^p - \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(p+1,q)}\, z^{p+1} \tag{3.3}$$

at $z = r$ and $z = re^{i(2s+1)\pi}$ $(s \in \mathbb{Z})$.

Proof. For $k \geqq p+1$, we have

$$\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(p+1,q)$$
$$\leqq \Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(k,q).$$

Now, using the hypothesis of Theorem 2, we get

$$\sum_{k=p+1}^{\infty} a_k \leqq \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(p+1,q)}. \tag{3.4}$$

Lastly, by using the form (1.7) of the function, the proof of Theorem 3 is completed. □

Theorem 4. *Let the function $f(z)$ defined by* (1.7) *be in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$. Then, for $|z| = r < 1$,*

$$\big|f'(z)\big| \geqq pr^{p-1} - \frac{(p+1)\,[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(p+1,q)}\, r^p \tag{3.5}$$

and

$$\big|f'(z)\big| \leqq pr^{p-1} + \frac{(p+1)\,[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(p+1,q)}\, r^p. \tag{3.6}$$

The result is sharp for the function $f(z)$ given by (3.3).

Proof. Using similar techniques as in our demonstration of Theorem 3, we get

$$\sum_{k=p+1}^{\infty} ka_k \leqq \frac{(p+1)\left[\delta(p-q,m)-\alpha\right]\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(p+1,q)},$$

which leads us to the completion of the proof of Theorem 4. □

Remark 2. Taking $\beta = 0$, in the above theorems, we obtain results similar to those obtained by Liu and Liu [11] (with $\gamma = 1$ and $n = 1$).

4. Convex Linear Combinations

By applying Theorem 2, we can prove that our class is closed under convex linear combinations as a corollary of the next result.

Theorem 5. *Let $\mu_\nu \geqq 0$ for $\nu = 1, 2, \cdots, l$ and*

$$\sum_{\nu=1}^{l} \mu_\nu \leqq 1.$$

If the functions $f_\nu(z)$ defined by

$$f_\nu(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,\nu} z^k \qquad (a_{k,\nu} \geqq 0;\ \nu = 1, 2, \cdots, l), \tag{4.1}$$

are in the class $\mathcal{UST}_m(p, q; \alpha, \beta)$ for every $\nu = 1, 2, \cdots, l$, then the function $f(z)$ given by

$$f(z) = z^p - \sum_{k=p+1}^{\infty} \left(\sum_{\nu=1}^{l} \mu_\nu a_{k,\nu} \right) z^k,$$

is also in the class $\mathcal{UST}_m(p, q; \alpha, \beta)$.

Proof. In order to proof this result, the assertion of Theorem 2 is used. □

Theorem 6. *Let $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{\left[\delta(p-q,m)-\alpha\right]\delta(p,q)}{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(k,q)}\, z^k \qquad (k \geqq p+1). \tag{4.2}$$

Then $f(z)$ is in the class $\mathcal{UST}_m(p, q; \alpha, \beta)$ if and only if it can be expressed in the following form:

$$f(z) = \sum_{k=p}^{\infty} \mu_k f_k(z), \tag{4.3}$$

where

$$\mu_k \geqq 0, \quad k \geqq p \qquad \text{and} \qquad \sum_{k=p}^{\infty} \mu_k = 1.$$

Proof. The part related to sufficiency is easily proved by using again the assertion of Theorem 2. For the necessity condition, we can see that the function $f(z)$ can be expressed in the form (4.3) if we set

$$\mu_k = \frac{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(k,q)a_k}{[\delta(p-q,m)-\alpha]\,\delta(p,q)} \qquad (k \geqq p+1)$$

and

$$\mu_p = 1 - \sum_{k=p+1}^{\infty} \mu_k,$$

such that $\mu_p \geqq 0$. This is already assured by Corollary 1. □

Corollary 2. *The extreme points of the class* $\mathcal{UST}_m(p,q;\alpha,\beta)$ *are the functions* $f_p(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(k,q)}\, z^k \qquad (k \geqq p+1).$$

5. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 7. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $\mathcal{UST}_m(p,q;\alpha,\beta)$. *Then* $f(z)$ *is a p-valent close-to-convex function of order* ξ $(0 \leqq \xi < p)$ *for* $|z| \leqq r_1(p,q;\alpha,\beta;\xi)$, *where*

$$r_1 = \inf_{k \geqq p+1} \left\{ \frac{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(k,q)}{[\delta(p-q,m)-\alpha]\,\delta(p,q)} \left(\frac{p-\xi}{k}\right) \right\}^{\frac{1}{k-p}}. \tag{5.1}$$

The result is sharp and the extremal function is given by (2.4).

Proof. By applying Corollary 1 and the form (1.7), we see that, for $|z| \leqq r_1$, we have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leqq p - \xi \; for \, |z| \leqq r_1(p,q;\alpha,\beta;\xi), \tag{5.2}$$

which completes the proof of Theorem 7. □

Theorem 8. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $\mathcal{UST}_m(p,q;\alpha,\beta)$. *Then* $f(z)$ *is a p-valent starlike function of order* ξ $(0 \leqq \xi < p)$ *for* $|z| \leqq r_2(p,q,\alpha,\beta,\xi)$, *where*

$$r_2 = \inf_{k \geqq p+1} \left\{ \frac{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(k,q)}{[\delta(p-q,m)-\alpha]\,\delta(p,q)} \left(\frac{p-\xi}{k-\xi}\right) \right\}^{\frac{1}{k-p}}. \tag{5.3}$$

The result is sharp and the extremal function is given by (2.4).

Proof. Using the same steps as in the proof of Theorem 7, it is seen that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leqq p - \xi \qquad \Big(|z| \leqq r_2(p,q,\alpha,\beta,\xi)\Big), \tag{5.4}$$

which evidently proves Theorem 8. □

Corollary 3. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $\mathcal{UST}_m(p,q;\alpha,\beta)$. *Then* $f(z)$ *is a p-valent convex function of order* ξ $(0 \leqq \xi < p)$ *for* $|z| \leqq r_3(p,q,\alpha,\beta,\xi)$, *where*

$$r_3 = \inf_{k \geqq p+1} \left\{ \frac{\big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)] + [\delta(p-q,m)-\alpha]\big]\delta(k,q)}{[\delta(p-q,m)-\alpha]\,\delta(p,q)} \left(\frac{p(p-\xi)}{k(k-\xi)}\right)\right\}^{\frac{1}{k-p}}. \tag{5.5}$$

The result is sharp and the extremal function is given by (2.4).

6. Integral Operators

In view of Theorem 2, we see that the function:

$$z^p - \sum_{k=p+1}^{\infty} d_k z^k$$

is in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$ as long as $0 \leqq d_k \leqq a_k$ for all $k \geqq p+1$, where a_k is the coefficient corresponding to a function which is in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$. We are thus led to the next theorem.

Theorem 9. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $\mathcal{UST}_m(p,q;\alpha,\beta)$. *Also let* c *be a real number such that* $c > -p$. *Then the function* $F(z)$ *defined by*

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \qquad (c > -p) \tag{6.1}$$

also belongs to the class $\mathcal{UST}_m(p,q;\alpha,\beta)$.

Proof. From the representation (6.1) of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=p+1}^{\infty} d_k z^k,$$

where

$$d_k = \left(\frac{c+p}{k+c}\right) a_k \leqq a_k \qquad (k \geqq p+1),$$

which completes the proof of Theorem 9. □

Putting $c = 1-p$ in Theorem 9, we get the following corollary.

Corollary 4. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $\mathcal{UST}_m(p,q;\alpha,\beta)$. *Also let* $F(z)$ *be defined by*

$$F(z) = \frac{1}{z^{1-p}}\int_0^z \frac{f(t)}{t^p}dt. \tag{6.2}$$

Then $F(z) \in \mathcal{UST}_m(p,q;\alpha,\beta)$.

Remark 3. The converse of Theorem 9 is not true. This observation leads to the following result involving the radius of p-valence.

Theorem 10. *Let the function*

$$F(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \qquad (a_k \geqq 0)$$

be in the class $\mathcal{UST}_m(p,q;\alpha,\beta)$. *Also let c be a real number such that* $c > -p$. *Then the function* $f(z)$ *given by* (6.1) *is p-valent in* $|z| < R_p^*$, *where*

$$R_p^* = \inf_{k \geqq p+1} \left\{ \frac{\Big[(1+\beta)[\delta(k-q,m) - \delta(p-q,m)] + [\delta(p-q,m) - \alpha]\Big]\delta(k,q)}{[\delta(p-q,m) - \alpha]\,\delta(p,q)} \left(\frac{p(c+p)}{k(c+k)} \right) \right\}^{\frac{1}{k-p}}. \tag{6.3}$$

The result is sharp.

Proof. From the definition (6.1), we have

$$f(z) = \frac{z^{1-c}[z^c\, F(z)]'}{c+p} = z^p - \sum_{k=p+1}^{\infty} \frac{k+c}{c+p} a_k z^k \qquad (c > -p).$$

In order to obtain the required result, it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leqq p \quad \text{for} \quad |z| < R_p^*,$$

where R_p^* is given by (6.3). Making use of Theorem 2, we get that the required inequality is true if

$$|z| \leqq \left(\frac{\Big[(1+\beta)[\delta(k-q,m) - \delta(p-q,m)] + [\delta(p-q,m) - \alpha]\Big]\delta(k,q)}{[\delta(p-q,m) - \alpha]\,\delta(p,q)} \left(\frac{p(c+p)}{k(c+k)} \right) \right)^{\frac{1}{k-p}} \tag{6.4}$$

$$(k \geqq p+1).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(c+k)\,[\delta(p-q,m) - \alpha]\,\delta(p,q)}{(c+p)\Big[(1+\beta)[\delta(k-q,m) - \delta(p-q,m)] + [\delta(p-q,m) - \alpha]\Big]\delta(k,q)}\, z^k \qquad (k \geqq p+1). \tag{6.5}$$

□

7. Modified Hadamard Products

Let the functions $f_\nu(z)$ $(\nu = 1,2)$ be defined by (4.1). The *modified* Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k. \tag{7.1}$$

Theorem 11. *Let the functions* $f_\nu(z)$ $(\nu = 1,2)$, *defined by* (4.1) *be in the class* $\mathcal{UST}_m(p,q;\alpha,\beta)$. *Then* $(f_1 * f_2)(z) \in \mathcal{UST}_m(p,q;\eta,\beta)$, *where*

$$\eta = \delta(p-q,m) - \frac{[\delta(p-q,m)-\alpha]^2(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]^2\delta(p+1,q)-[\delta(p-q,m)-\alpha]^2\,\delta(p,q)}. \tag{7.2}$$

The result is sharp when

$$f_1(z) = f_2(z) = f(z),$$

where the function $f(z)$ *is given by*

$$f(z) = z^p - \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]\delta(p+1,q)}\,z^{p+1}. \tag{7.3}$$

Proof. Employing the technique used earlier by Schild and Silverman [13], we need to find the largest η such that

$$\sum_{k=p+1}^{\infty} \frac{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\eta]\Big]\delta(k,q)}{[\delta(p-q,m)-\eta]\,\delta(p,q)} a_{k,1}a_{k,2} \leqq 1. \tag{7.4}$$

Using the inequalities for the coefficients of the functions in the class $\mathcal{UST}_m(p,q;\eta,\beta)$, and by applying the Cauchy-Schwarz inequality, it is sufficient to show that

$$\eta \leqq \delta(p-q,m) - \frac{[\delta(p-q,m)-\alpha]^2\,(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]\delta(p,q)}{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]^2\delta(k,q)-[\delta(p-q,m)-\alpha]^2\;\delta(p,q)}. \tag{7.5}$$

Now, defining the function $G(k)$ by

$$G(k) = \delta(p-q,m) - \frac{[\delta(p-q,m)-\alpha]^2\,(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]\delta(p,q)}{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\Big]^2\delta(k,q)-[\delta(p-q,m)-\alpha]^2\;\delta(p,q)}, \tag{7.6}$$

we see that $G(k)$ is an increasing function of k $(k \geqq p+1)$, which obviously completes the proof. □

Using arguments similar to those used in the proof of Theorem 11, we obtain the following result.

Theorem 12. *Let the function* $f_1(z)$ *defined by* (4.1) *be in the class* $\mathcal{UST}_m(p,q;\alpha,\beta)$. *Suppose also that the function* $f_2(z)$ *defined by* (4.1) *be in the class* $\mathcal{UST}_m(p,q;\varphi,\beta)$. *Then*

$$(f_1 * f_2)(z) \in \mathcal{UST}_m(p,q;\zeta,\beta),$$

where

$$\zeta = \delta(p-q,m) - \frac{\big[\delta(p-q,m)-\alpha\big]\big[\delta(p-q,m)-\varphi\big](1+\beta)\big[\delta(p-q+1,m)-\delta(p-q,m)\big]\delta(p,q)}{\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\alpha]\big]\big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]+[\delta(p-q,m)-\varphi]\big]\delta(p+1,q)-\Omega} \tag{7.7}$$

with

$$\Omega = [\delta(p-q,m)-\alpha][\delta(p-q,m)-\varphi]\delta(p,q).$$

The result is sharp for the functions $f_\nu(z)$ $(\nu = 1, 2)$ *given by*

$$f_1(z) = z^p - \frac{[\delta(p-q,m)-\alpha]\,\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)] + [\delta(p-q,m)-\alpha]\Big]\delta(p+1,q)}\, z^{p+1} \tag{7.8}$$

and

$$f_2(z) = z^p - \frac{[\delta(p-q,m)-\varphi]\,\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)] + [\delta(p-q,m)-\varphi]\Big]\delta(p+1,q)}\, z^{p+1}. \tag{7.9}$$

Theorem 13. *Let the functions* $f_\nu(z)$ $(\nu = 1, 2)$ *defined by* (4.1) *be in the class* $\mathcal{UST}_m(p, q; \alpha, \beta)$*. Then the function* $h(z)$ *given by*

$$h(z) = z^p - \sum_{k=p+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{7.10}$$

belongs to the class $\mathcal{UST}_m(p, q; \kappa, \beta)$*, where*

$$\kappa = \delta(p-q,m) - \frac{2[\delta(p-q,m)-\alpha]^2(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)]\delta(p,q)}{\Big[(1+\beta)[\delta(p-q+1,m)-\delta(p-q,m)] + [\delta(p-q,m)-\alpha]\Big]^2\delta(p+1,q) - 2[\delta(p-q,m)-\alpha]^2\delta(p,q)}. \tag{7.11}$$

The result is sharp for

$$f_1(z) = f_2(z) = f(z),$$

where the function $f(z)$ *is given by* (7.3).

Proof. If we combine the assertions of Theorem 2 for both of the functions $f_1(z)$ and $f_2(z)$, we get

$$\sum_{k=p+1}^{\infty} \frac{1}{2} \left(\frac{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)] + [\delta(p-q,m)-\alpha]\Big]\delta(k,q)}{[\delta(p-q,m)-\alpha]\delta(p,q)} \right)^2 (a_{k,1}^2 + a_{k,2}^2) \leqq 1. \tag{7.12}$$

Therefore, we need to find the largest $\kappa = \kappa(p, q, \alpha, \beta)$ such that

$$\frac{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)] + [\delta(p-q,m)-\kappa]\Big]\delta(k,q)}{[\delta(p-q,m)-\kappa]\delta(p,q)} \leqq \frac{1}{2} \left(\frac{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)] + [\delta(p-q,m)-\alpha]\Big]\delta(k,q)}{[\delta(p-q,m)-\alpha]\delta(p,q)} \right)^2. \tag{7.13}$$

Since $D(k)$ given by

$$D(k) = \delta(p-q,m) - \frac{2[\delta(p-q,m)-\alpha]^2(1+\beta)[\delta(k-q,m)-\delta(p-q,m)]\delta(p,q)}{\Big[(1+\beta)[\delta(k-q,m)-\delta(p-q,m)] + [\delta(p-q,m)-\alpha]\Big]^2\delta(k,q) - 2[\delta(p-q,m)-\alpha]^2\delta(p,q)}$$

is an increasing function of k $(k \geqq p+1)$, we obtain $\kappa \leqq D(p+1)$. The proof of Theorem 13 is thus completed. $\square$

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