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The Constants to Measure the Differences Between Birkhoff and Isosceles Orthogonalities

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Abstract. The notion of orthogonality for vectors in inner product spaces is simple, interesting and fruitful. When moving to normed spaces, we have many possibilities to extend this notion. We consider Birkhoff orthogonality and isosceles orthogonality, which are the most used notions of orthogonality. In 2006, Ji and Wu introduced a geometric constant $D(X)$ to give a quantitative characterization of the difference between these two orthogonality types. However, this constant was considered only in the unit sphere S_X of the normed space X. In this paper, we introduce a new geometric constant $IB(X)$ to measure the difference between Birkhoff and isosceles orthogonalities in the entire normed space X. To consider the difference between these orthogonalities, we also treat constant $BI(X)$.

1. Introduction

We denote by X a real normed space with the norm $\|\cdot\|$, the unit ball B_X and the unit sphere S_X. Throughout this paper, we assume that the dimension of X is at least two. In the case of that X is an inner product space, an element $x \in X$ is said to be orthogonal to $y \in X$ (denoted by $x \perp y$) if the inner product $\langle x, y\rangle$ is zero. In the general setting of normed spaces, many notions of orthogonality have been introduced by means of equivalent propositions to the usual orthogonality in inner product spaces. For example, Roberts [11] introduced *Roberts orthogonality*: for any $x, y \in X$, x is said to be Roberts orthogonal to y (denoted by $x \perp_R y$) if

$$\forall \lambda \in \mathbb{R}, \quad \|x + \lambda y\| = \|x - \lambda y\|.$$

Birkhoff [3] introduced *Birkhoff orthogonality*: x is said to be Birkhoff orthogonal to y (denoted by $x \perp_B y$) if

$$\forall \lambda \in \mathbb{R}, \quad \|x + \lambda y\| \geq \|x\|.$$

James [5] introduced *isosceles orthogonality*: x is said to be isosceles orthogonal to y (denoted by $x \perp_I y$) if

$$\|x + y\| = \|x - y\|.$$

These generalized orthogonality types have been studied in a lot of papers ([1, 6] and so on).

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Recently, quantitative studies of the difference between two orthogonality types have been performed:

$$D(X) = \inf\left\{\inf_{\lambda\in\mathbb{R}} \|x + \lambda y\| : x, y \in S_X, x \perp_I y\right\},$$

$$D'(X) = \sup\{\|x + y\| - \|x - y\| : x, y \in S_X, x \perp_B y\},$$

$$NH_X = \sup_{\alpha>0}\left\{\frac{\|x + \alpha y\| - \|x - \alpha y\|}{\alpha} : x, y \in S_X, x \perp_I y\right\},$$

$$BR(X) = \sup_{\alpha>0}\left\{\frac{\|x + \alpha y\| - \|x - \alpha y\|}{\alpha} : x, y \in S_X, x \perp_B y\right\}$$

(see [4, 7, 10]). We note that these suprema and infima are considered only in the unit sphere S_X.

Take arbitrary nonzero elements $x, y \in X$ with $x \perp_B y$. Since Birkhoff orthogonality is homogeneous, we have $\frac{x}{\|x\|} \perp_B \frac{y}{\|y\|}$ and hence

$$\frac{\|x + y\| - \|x - y\|}{\|y\|} = \frac{\left\|\frac{x}{\|x\|} + \frac{\|y\|}{\|x\|}\frac{y}{\|y\|}\right\| - \left\|\frac{x}{\|x\|} - \frac{\|y\|}{\|x\|}\frac{y}{\|y\|}\right\|}{\frac{\|y\|}{\|x\|}} \le BR(X)$$

Thus, we obtain

$$BR(X) = \sup\left\{\frac{\|x + y\| - \|x - y\|}{\|y\|} : x, y \in X, x, y \neq 0, x \perp_B y\right\}$$

and so, in a certain sense, the constant $BR(X)$ measures the difference between Birkhoff orthogonality and isosceles orthogonality in the entire space X.

In this paper, we consider two constant $BI(X)$ and $IB(X)$ to measure the difference between these two orthogonalities in the entire space X:

$$BI(X) = \sup\left\{\frac{\|x + y\| - \|x - y\|}{\|x\|} : x, y \in X, x, y \neq 0, x \perp_B y\right\},$$

$$IB(X) = \inf\left\{\frac{\inf_{\lambda\in\mathbb{R}} \|x + \lambda y\|}{\|x\|} : x, y \in X, x, y \neq 0, x \perp_I y\right\}.$$

We note that, the Birkhoff orthogonality is not symmetric, that is, $x \perp_B y$ does not necessarily imply $y \perp_B x$ and hence the constants $BI(X)$ and $BR(X)$ are different from each other. In addition, one can easily see that $IB(X) \le D(X)$.

2. The Properties of the Constant $BI(X)$

For the constant $BR(X)$, one can have

$$BR(X) = \sup\left\{\frac{\|x + y\| - \|x - y\|}{\|y\|} : x, y \in X, x, y \neq 0, x \perp_B y\right\}.$$

Moreover, in the paper [10], it is noted that $BR(X)$ is reformulated as

$$BR(X) = \sup_{\alpha>0}\{\|\alpha x + y\| - \|\alpha x - y\| : x, y \in S_X, x \perp_B y\}.$$

Suppose that an element $x \in S_X$ is Birkhoff orthogonal to another element $y \in S_X$. Then, from the definition, we have $\|x + \lambda y\| \ge 1$ for all $\lambda \in \mathbb{R}$. However, it is known that $x \perp_B y$ does not necessarily imply $y \perp_B x$, and hence the norms $\|\alpha x \pm y\|$ are not necessarily greater than 1. Thus, we consider the constant $BI(X)$ defined in the above section. One has $BI(X) = \sup_{\alpha>0}\{\|x + \alpha y\| - \|x - \alpha y\| : x, y \in S_X, x \perp_B y\}$. First we obtain the following

Proposition 2.1. *Let X be a normed space. Then $0 \leq BI(X) \leq 2$ and $BI(X) = 0$ if and only if X is an inner product space.*

Proof. It is trivial that $0 \leq BI(X)$. Take arbitrary $x, y \in S_X$ with $x \perp_B y$. From the definition of Birkhoff orthogonality and the triangle inequality, one has

$$\max\{1, |1-\lambda|\} \leq \|x + \lambda y\| \leq 1 + |\lambda|$$

for all $\lambda \in \mathbb{R}$. If $0 < \alpha < 2$, then we have

$$\|x + \alpha y\| - \|x - \alpha y\| \leq 1 + \alpha - 1 = \alpha < 2.$$

On the other hand, when $2 \leq \alpha$ we have

$$\|x + \alpha y\| - \|x - \alpha y\| \leq 1 + \alpha - (\alpha - 1) = 2.$$

Hence we obtain $BI(X) \leq 2$.

Suppose that $BI(X) = 0$. Then for each pair of elements x, y with $x \perp_B y$, one has $x \perp_I y$. This is a characteristic property of an inner product space [2, page33–34] . Conversely, if X is an inner product space, then both Birkhoff and isosceles orthogonalities coincide with the usual orthogonality defined by inner product. Thus, we obtain $BI(X) = 0$. □

We consider the condition of $BI(X) = 2$.

Example 2.2. *Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. Then $BI(X) = 2$.*

Proof. Let $x = (1, 1)$ and $y = (1, 0)$. Then we have $x, y \in S_X$ and $x \perp_B y$. We note that $y \not\perp_B x$. For a real number α with $\alpha \geq 2$, one has

$$\|x + \alpha y\| - \|x - \alpha y\| = 1 + \alpha - (\alpha - 1) = 2.$$

We obtain $BI(X) \geq 2$ and hence $BI(X) = 2$. □

One can also obtain a uniformly non-square space in which $BI(X) = 2$. A normed space X said to be uniformly non-square if there exits $\delta > 0$ such that $\|x+y\| > 2(1-\delta)$ and $\|x\| = \|y\| = 1$ imply $\|x-y\| < 2(1-\delta)$

Example 2.3. *Let X be a Banach space on $\mathbb{R}^2$ whose unit circle is the polygon with $x = (1, 1)$, $y = (0, 1)$, $z = (-1, 1/2)$ and $-x$, $-y$, $-z$ as vertices. Then $BI(X) = 2$.*

Proof. Let $w = (1, 0)$. Then we have $w \in S_X$ and $x \perp_B w$. For a real number α with $\alpha \geq 3$, one has

$$\|x + \alpha w\| - \|x - \alpha w\| = 1 + \alpha - (\alpha - 1) = 2.$$

Thus, as in the above example, we obtain $BI(X) = 2$. □

Theorem 2.4. *Let X be a normed space. Then the upper bound 2 of $BI(X)$ is attained by a practical pair of S_X and a practical positive number if and only if there exist elements $x, y \in S_X$ and real number $t_0 \in [1/2, 1)$ satisfying $x \perp_B y$, $[x, y] \subset S_X$ and $x \perp_B (1 - t_0)x + t_0 y$.*

Proof. Suppose that there exist $x, y \in S_X$ and $\alpha > 0$ such that $x \perp_B y$ and $\|x + \alpha y\| - \|x - \alpha y\| = 2$. Then from the inequality

$$2 = \|x + \alpha y\| - \|x - \alpha y\| \leq 1 + \alpha - |1 - \alpha| \leq 2,$$

one has $\alpha \geq 2$, $\|x + \alpha y\| = 1 + \alpha$ and $\|x - \alpha y\| = \alpha - 1$. Letting $w = (\alpha y - x)/(\alpha - 1)$, we have $w \in S_X$ and

$$y = \left(1 - \frac{\alpha - 1}{\alpha}\right)x + \frac{\alpha - 1}{\alpha} w.$$

Since $y \in S_X$, we have $[x, w] \subset S_X$.

For any nonnegative number λ we have

$$\|x + \lambda w\| = (1 + \lambda) \left\| \frac{x + \lambda w}{1 + \lambda} \right\| = 1 + \lambda \geq 1.$$

If $\lambda < 0$, then

$$\|x + \lambda w\| = \left\| x + \lambda \frac{\alpha y - x}{\alpha - 1} \right\| = \left\| \frac{\alpha - 1 - \lambda}{\alpha - 1} x + \frac{\lambda \alpha}{\alpha - 1} y \right\| \geq \frac{\alpha - 1 - \lambda}{\alpha - 1} > 1.$$

Thus we also obtain $x \perp_B w$.

Conversely, let there exist elements $x, y \in S_X$ and a real number $t_0 \in [1/2, 1)$ satisfying $x \perp_B y$, $[x, y] \subset S_X$ and $x \perp_B (1 - t_0)x + t_0 y$. Putting $z = (1 - t_0)x + t_0 y$, from the assumption, one has $z \in S_X$ and $x \perp_B z$.

For any $\alpha > 0$, we have

$$\|x + \alpha z\| = (1 + \alpha) \left\| \frac{x + \alpha z}{1 + \alpha} \right\| = 1 + \alpha.$$

On the other hand, letting $\alpha_0 = 1/(1 - t_0)$, one can see that $\alpha_0 \geq 2$ and that

$$x - \alpha_0 z = x - \frac{1}{1 - t_0} \{(1 - t_0)x + t_0 y\} = -(\alpha_0 - 1)y.$$

Thus we obtain

$$\|x + \alpha_0 z\| - \|x - \alpha_0 z\| = 1 + \alpha_0 - (\alpha_0 - 1) = 2$$

and hence the upper bound 2 of $BI(X)$ is attained by elements x, z and the number α_0. □

Remark 2.5. *Even if we connect with a parabola between the points x and z in* Example 2.3*, then we obtain that $BI(X) = 2$ is attained by x, w and $\alpha \geq 3$. Thus, the element $x \in S_X$ in the above theorem is not necessarily the common endpoint of two segment lines contained in S_X (cf.* [7, Theorem 2] and [10, Theorem 2.2]*).*

3. The Properties of the Constant *IB*(*X*) and Some Other Constants

To consider the difference between Birkhoff and isosceles orthogonalities, the following results obtained by James in [5] are important.

Proposition 3.1 ([5]).

(i) *If x $(\neq 0)$ and y are isosceles orthogonal elements in a normed space, then $\|x + ky\| > \frac{1}{2}\|x\|$ for all k.*

(ii) *If x $(\neq 0)$ and y are isosceles orthogonal elements in a normed space, and $\|y\| \leq \|x\|$, then $\|x + ky\| \geq 2(\sqrt{2} - 1)\|x\|$ for all k.*

Theorem 3.2. *Let X be a normed space. Then $1/2 \leq IB(X) \leq 1$ and $IB(X) = 1$ if and only if X is an inner product space.*

Proof. For all $x, y \in X$ with $x \perp_I y$, we apparently have $\inf_{\lambda \in \mathbb{R}} \|x + \lambda y\| \leq \|x\|$. Hence we obtain $IB(X) \leq 1$. On the other hand, from Proposition 3.1 (i), we obtain $1/2 \leq IB(X)$.

Under the condition $IB(X) = 1$, for all $x, y \in X$ with $x \perp_I y$, we always have $\inf_{\lambda \in \mathbb{R}} \|x + \lambda y\| = \|x\|$, that is, isosceles orthogonality implies Birkhoff orthogonality. Thus from [2, page33–34], X is an inner product space. Because those two orthogonalities coincide in inner product spaces, the converse is also true. □

The upper bound of $D(X)$ is also 1, and $D(X) = 1$ characterize the inner product spaces, too. On the other hand, by Proposition 3.1 (ii), the lower bound of $D(X)$ is $2(\sqrt{2}-1)$. If $D(X) > 2(\sqrt{2}-1)$, then X is uniformly non-square. However, uniform non-squareness does not imply $D(X) > 2(\sqrt{2}-1)$ (see [7, 10]).

We consider the condition of that $IB(X)$ attains the lower bound $1/2$. From [5, Example 4.1], we obtain that $IB(X) = 1/2$ when $X = (\mathbb{R}^2, \|\cdot\|_\infty)$.

Example 3.3 ([5, Example 4.1]**).** *Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. Then $IB(X) = 1/2$.*

Proof. Let $x = (1,0)$ and $y_n = (n-1, n)$. Then $x + y_n = (n,n)$ and $x - y_n = (2-n, -n)$. If $n \geq 1$, then we have $\|x+y_n\|_\infty = \|x-y_n\|_\infty = n$ and hence $x \perp_I y_n$.

Moreover, we have

$$x - \frac{1}{2n}y_n = \left(\frac{n+1}{2n}, -\frac{1}{2}\right) \quad \text{and} \quad \left\|x - \frac{1}{2n}y_n\right\|_\infty = \frac{n+1}{2n}.$$

Thus we obtain $IB(X) \leq (n+1)/2n$ for all $n \in \mathbb{N}$ and hence $IB(X) = 1/2$ □

We obtain $IB(X) = 1/2$ also in the space $X = (\mathbb{R}^2, \|\cdot\|_1)$.

Example 3.4. *Let $X = (\mathbb{R}^2, \|\cdot\|_1)$. Then $IB(X) = 1/2$.*

Proof. Let $x = (1,-1)$ and $y_n = (1, 2n-1)$. Then $x + y_n = (2, 2n-2)$ and $x - y_n = (0, -2n)$. If $n \geq 1$, then we have $\|x+y_n\|_1 = \|x-y_n\|_1 = 2n$ and hence $x \perp_I y_n$.

In this situation, one has

$$x + \frac{1}{2n-1}y_n = \left(\frac{2n}{2n-1}, 0\right) \quad \text{and} \quad \left\|x + \frac{1}{2n-1}y_n\right\|_1 = \frac{2n}{2n-1}.$$

Thus we have

$$\frac{\left\|x + \frac{1}{2n-1}y_n\right\|_1}{\|x\|_1} = \frac{n}{2n-1}$$

and hence $IB(X) \leq n/(2n-1)$ for all $n \in \mathbb{N}$. Therefore we obtain $IB(X) = 1/2$. □

Let T be a operator from $(\mathbb{R}^2, \|\cdot\|_1)$ onto $(\mathbb{R}^2, \|\cdot\|_\infty)$ defined by $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$. Then the operator T is a linear isometry. From this fact, one can have Example 3.4, too. However, the above fundamental proof helps us to obtain a proposition in section 4.

We show that $IB(X) > 1/2$ if and only if the space X is uniformly non-square. To do this, we need to recall the Dunkl-Williams constant defined in [8]:

$$\begin{aligned} DW(X) &= \sup\left\{\frac{\|x\|+\|y\|}{\|x-y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| : x, y \in X, x, y \neq 0, x \neq y\right\} \\ &= \sup\left\{\frac{\|u+v\|}{\|(1-t)u + tv\|} : u, v \in S_X, 0 \leq t \leq 1\right\}. \end{aligned}$$

For any normed space X, we have $2 \leq DW(X) \leq 4$. It is known that a normed space X is uniformly non-square if and only if $DW(X) < 4$. In [9], a calculation method of this constant can be found.

We shall prove an equality concerning the constants $IB(X)$ and $DW(X)$. We note that for nonzero $x, y \in X$, the function $\lambda \mapsto \|x + \lambda y\|$ is continuous and so there exists a real number λ_0 such that $\|x + \lambda_0 y\| = \inf_{\lambda \in \mathbb{R}} \|x + \lambda y\|$.

Theorem 3.5. *For any normed space X, $IB(X)DW(X) = 2$.*

Proof. Take arbitrary nonzero elements $x, y \in X$ with $x \perp_I y$. Then there exist a number $\lambda_0 \in [-1, 1]$ such that $\|x + \lambda_0 y\| = \inf_{\lambda\in\mathbb{R}} \|x + \lambda y\|$. We may assume that $\lambda_0 \in [0, 1]$. Let

$$u = \frac{x+y}{\|x+y\|}, \quad v = \frac{x-y}{\|x-y\|}, \quad \text{and} \quad t_0 = \frac{1-\lambda_0}{2}.$$

Then from $\|x + y\| = \|x - y\|$, we have

$$(1-t_0)u + t_0 v = \frac{(1+\lambda_0)(x+y)}{2\|x+y\|} + \frac{(1-\lambda_0)(x-y)}{2\|x-y\|} = \frac{x+\lambda_0 y}{\|x+y\|}$$

and hence

$$\frac{\|x+\lambda_0 y\|}{\|x\|} = 2\frac{\|x+y\|}{\|2x\|} \cdot \frac{\|x+\lambda_0 y\|}{\|x+y\|} = 2\frac{\|(1-t_0)u + t_0 v\|}{\|u+v\|} \geq \frac{2}{DW(X)}.$$

Thus we obtain $IB(X)DW(X) \geq 2$.

Take any $u, v \in S_X$ with $u + v \neq 0$. Then we have $u + v \perp_I u - v$. On the other hand, we have a real number $t_1 \in [0, 1]$ such that $\|(1 - t_1)u + t_1 v\| = \min_{t\in[0,1]} \|(1 - t)u + tv\|$. We may assume that $t_1 \in [0, 1/2]$. Letting $\lambda_1 = 1 - 2t_1$, we have

$$u + v + \lambda_1(u - v) = (1 + \lambda_1)u + (1 - \lambda_1)v = 2\{(1 - t_1)u + t_1 v\}.$$

Hence, from the fact that $u + v$ is isosceles orthogonal to $u - v$, we have

$$\frac{\|u+v\|}{\|(1-t_1)u + t_1 v\|} = 2\frac{\|u+v\|}{\|u+v+\lambda_1(u-v)\|} \leq \frac{2}{IB(X)}.$$

Thus, we have $IB(X)DW(X) \leq 2$.

Therefore we obtain $IB(X)DW(X) = 2$. □

From this theorem, we have the following

Corollary 3.6. *A normed space X is uniformly non-square if and only if $IB(X) > 1/2$.*

In addition, one has that in a certain sense the Dunkl-Williams constant measures the difference between Birkhoff and Isosceles orthogonalities, too.

Now we recall the definitions of $D(X)$ and $IB(X)$:

$$D(X) = \inf\left\{\inf_{\lambda\in\mathbb{R}} \|x + \lambda y\| : x, y \in S_X, x \perp_I y\right\},$$

$$IB(X) = \inf\left\{\frac{\inf_{\lambda\in\mathbb{R}} \|x+\lambda y\|}{\|x\|} : x, y \in X, x \neq 0, x \perp_I y\right\} = \inf\left\{\inf_{\lambda\in\mathbb{R}} \|x + \lambda y\| : x \in S_X, y \in X, x \perp_I y\right\}.$$

On the other hand, from Proposition 3.1 (ii), we have that, if x ($\neq 0$) and y are isosceles orthogonal elements in a normed space, and $\|y\| \leq \|x\|$, then $\|x + ky\| \geq 2(\sqrt{2} - 1)\|x\|$ for all k. Thus, it is natural for us to consider the following constant:

$$IB'(X) = \inf\left\{\frac{\inf_{\lambda\in\mathbb{R}} \|x+\lambda y\|}{\|x\|} : x, y \in X, x, y \neq 0, \|y\| \leq \|x\|, x \perp_I y\right\}$$

$$= \inf\left\{\inf_{\lambda\in\mathbb{R}} \|x + \lambda y\| : x \in S_X, y \in B_X, x \perp_I y\right\}.$$

From the definitions, we clearly have $IB(X) \leq IB'(X) \leq D(X)$ for any normed space X. In addition, we easily obtain the following

Proposition 3.7. *Let X be a normed linear space. Then*

(i) $2(\sqrt{2}-1) \le IB'(X) \le 1$.

(ii) $IB'(X) = 1$ *if and only if X is an inner product space.*

(iii) $IB'(X)$ *does not necessarily coincide with* $IB(X)$.

Proof. From Proposition 3.1 (ii), the assertion (i) is clear.

(ii) If X is an inner product space, then Birkhoff and isosceles orthogonalities coincide with each other. Hence we have $IB'(X) = 1$. Conversely, we suppose that $IB'(X) = 1$. Then we have

$$1 = IB'(X) \le D(X) \le 1$$

and so $D(X) = 1$. Hence the space X is an inner product space (cf. [7]).

(iii) Suppose that a normed space X is not uniformly non-square. Then we have $IB(X) = 1/2$. On the other hand, from

$$2(\sqrt{2}-1) \le IB'(X) \le D(X) = 2(\sqrt{2}-1),$$

one has $IB'(X) = 2(\sqrt{2}-1)$. Thus we obtain $IB(X) < IB'(X)$. □

From Theorem 3.5, one has

$$IB(X) = 2\inf\left\{\frac{\|(1-t)u+tv\|}{\|u+v\|} : u, v \in S_X, 0 \le t \le 1\right\}.$$

Following the proof of Theorem 3.5, we obtain similar results on $D(X)$ and $IB'(X)$:

Proposition 3.8. *Let X be a normed linear space. Then*

$$D(X) = 2\inf\left\{\frac{\|(1-t)u+tv\|}{\|u+v\|} : u, v \in S_X, u \perp_I v, 0 \le t \le 1\right\}$$

and

$$IB'(X) = 2\inf\left\{\frac{\|(1-t)u+tv\|}{\|u+v\|} : u, v \in S_X, \|u+v\| \ge \|u-v\|, 0 \le t \le 1\right\}.$$

We already have that the constants $IB'(X)$ and $D(X)$ do not necessarily coincide with $IB(X)$. Thus we have the following

Corollary 3.9. *For a normed space X, the Dunkl-Williams constant*

$$DW(X) = \sup\left\{\frac{\|u+v\|}{\|(1-t)u+tv\|} : u, v \in S_X, 0 \le t \le 1\right\}$$

does not necessarily coincide with the suprema

$$\sup\left\{\frac{\|u+v\|}{\|(1-t)u+tv\|} : u, v \in S_X, u \perp_I v, 0 \le t \le 1\right\}$$

and

$$\sup\left\{\frac{\|u+v\|}{\|(1-t)u+tv\|} : u, v \in S_X, \|u+v\| \ge \|u-v\|, 0 \le t \le 1\right\}.$$

4. The Constant $IB(X)$ of Some Day-James Spaces

By following Example 3.3, we consider $IB(\ell_\infty\text{-}\ell_p)$ of the Day-James space $\ell_\infty\text{-}\ell_p$. The Day-James space $\ell_\infty\text{-}\ell_p$ for $1 \le p \le \infty$ is defined as $\mathbb{R}^2$ with the norm

$$\|(x_1, x_2)\|_{\infty,p} = \begin{cases} \|(x_1, x_2)\|_\infty & if\ x_1x_2 \ge 0, \\ \|(x_1, x_2)\|_p & if\ x_1x_2 \le 0. \end{cases}$$

If $p = 2$, then from [9, Theorem 4.10], we have that the Dunkl-Williams constant of $\ell_2\text{-}\ell_\infty$ is equal to $2\sqrt{2}$. Hence, by Theorem 3.5, we have the following

Corollary 4.1. *$IB(\ell_\infty\text{-}\ell_2) = IB(\ell_2\text{-}\ell_\infty) = 1/\sqrt{2}$.*

In the case of $p \ne 2$, it will be very difficult to calculate $IB(\ell_\infty\text{-}\ell_p)$. However, one can obtain an upper bound of $IB(\ell_\infty\text{-}\ell_p)$.

Proposition 4.2. *Let $1 \le p \le \infty$. Then*

$$IB(\ell_\infty\text{-}\ell_p) \le \min\left\{2^{-1/q}, 8/9\right\}, \quad IB'(\ell_\infty\text{-}\ell_p) \le 8/9$$

and hence

$$DW(\ell_\infty\text{-}\ell_p) \ge \max\left\{2^{1+1/q}, 9/4\right\},$$

where q is the positive number such that $1/p + 1/q = 1$.

Proof. As in Example 3.3, let $x = (1, 0)$ and $y_n = (n - 1, n)$. Then $x + y_n = (n, n)$ and $x - y_n = (2 - n, -n)$. If $n \ge 2$, then we have

$$\|x + y_n\|_{\infty,p} = \|x + y_n\|_\infty = n = \|x - y_n\|_\infty = \|x - y_n\|_{\infty,p}$$

and hence $x \perp_I y_n$ in $\ell_\infty\text{-}\ell_p$.

Letting

$$\lambda_n = -\frac{(n-1)^{q/p}}{n^q + (n-1)^q},$$

from the equality $1/p + 1/q = 1$, one has

$$\begin{aligned} \|x + \lambda_n y_n\|_{\infty,p}^p = \|x + \lambda_n y_n\|_p^p &= \left\{1 - \frac{(n-1)^{1+q/p}}{n^q + (n-1)^q}\right\}^p + \left\{\frac{n(n-1)^{q/p}}{n^q + (n-1)^q}\right\}^p \\ &= \frac{n^p\{n^q + (n-1)^q\}}{\{n^q + (n-1)^q\}^p} \\ &= \frac{n^p}{\{n^q + (n-1)^q\}^{p-1}} \end{aligned}$$

and so

$$\|x + \lambda_n y_n\|_{\infty,p} = \left\{\frac{n^p}{\{n^q + (n-1)^q\}^{p-1}}\right\}^{1/p} = \frac{n}{\{n^q + (n-1)^q\}^{1/q}} = \frac{n}{\|(n, n-1)\|_q}.$$

Hence, from the definition, $IB(\ell_\infty\text{-}\ell_p) \le n/\|(n, n-1)\|_q$ for all $n \in \mathbb{N}$. Thus we obtain $IB(\ell_\infty\text{-}\ell_p) \le 2^{-1/q}$. We note that $\inf_{\lambda\in\mathbb{R}} \|y_n + \lambda x\|_{\infty,p} = 1$ for all $n \in \mathbb{N}$.

For $t \in [1/2, 1]$, let $z_t = (1, t)$ and $w_t = (2t-1, -t)$. Then $\|z_t\|_{\infty,p} = 1$, $z_t + w_t = (2t, 0)$ and $z_t - w_t = (2(1-t), 2t)$. We have

$$\|z_t + w_t\|_{\infty,p} = 2t = \|z_t - w_t\|_\infty = \|z_t - w_t\|_{\infty,p}$$

and hence $z_t \perp_I w_t$ in ℓ_∞-ℓ_p. Letting $\lambda_t = (t-1)/(3t-1)$, one has

$$\|z_t + \lambda_t w_t\|_{\infty,p} = \|z_t + \lambda_t w_t\|_\infty = \frac{2t^2}{3t-1}.$$

The function $t \to 2t^2/(3t-1)$ attains the minimum 8/9 at $t = 2/3$ and hence $IB(\ell_\infty\text{-}\ell_p) \leq 8/9$. For $t = 2/3$, we have $\|w_t\|_{\infty,p} = \|w_t\|_p \leq 1 = \|z_t\|_{\infty,p}$. Thus we also have $IB'(\ell_\infty\text{-}\ell_p) \leq 8/9$.

From Theorem 3.5, we obtain

$$DW(\ell_\infty\text{-}\ell_p) = 2/IB(\ell_\infty\text{-}\ell_p) \geq \frac{2}{\min\{2^{-1/q}, 8/9\}} = \max\left\{2^{1+1/q}, 9/4\right\},$$

too. □

By following Example 3.4, one can also estimate $IB(\ell_1\text{-}\ell_p)$ of the Day-James space ℓ_1-ℓ_p. For $1 \leq p \leq \infty$ the Day-James space ℓ_1-ℓ_p is defined as $\mathbb{R}^2$ with the norm

$$\|(x_1, x_2)\|_{1,p} = \begin{cases} \|(x_1, x_2)\|_1 & if\ x_1 x_2 \geq 0, \\ \|(x_1, x_2)\|_p & if\ x_1 x_2 \leq 0. \end{cases}$$

Proposition 4.3. *Let $1 \leq p \leq \infty$. Then*

$$IB(\ell_1\text{-}\ell_p) \leq 2^{-1/p} \quad \text{and hence} \quad DW(\ell_1\text{-}\ell_p) \geq 2^{1+1/p}.$$

Proof. As in Example 3.4, let $x = (1, -1)$ and $y_n = (1, 2^{1/p}n - 1)$. Then $x + y_n = (2, 2^{1/p}n - 2)$ and $x - y_n = (0, -2^{1/p}n)$. If $n \geq 2$, then we have

$$\|x + y_n\|_{1,p} = \|x + y_n\|_1 = 2^{1/p}n = \|x - y_n\|_{1,p}$$

and hence $x \perp_I y_n$ in ℓ_1-ℓ_p.

Letting $\lambda_n = (2^{1/p}n - 1)^{-1}$, one has

$$x + \lambda_n y_n = \left(\frac{2^{1/p}n}{2^{1/p}n - 1}, 0\right) \quad \text{and hence} \quad \left\|x + \lambda_n y_n\right\|_{1,p} = \frac{2^{1/p}n}{2^{1/p}n - 1}.$$

Thus we have

$$\frac{\left\|x + \lambda_n y_n\right\|_{1,p}}{\|x\|_{1,p}} = \frac{n}{2^{1/p}n - 1}$$

and hence $IB(\ell_1\text{-}\ell_p) \leq n/(2^{1/p}n - 1)$ for all $n \in \mathbb{N}$. Therefore we obtain $IB(\ell_1\text{-}\ell_p) \leq 2^{-1/p}$.

We also obtain

$$DW(\ell_1\text{-}\ell_p) = 2/IB(\ell_1\text{-}\ell_p) \geq 2/(2^{-1/p}) = 2^{1+1/p}$$

by Theorem 3.5. □

Remark 4.4. *If a normed space X is not uniformly non-square, then we have $IB(X) = 1/2 < 2(\sqrt{2} - 1) = IB'(X) = D(X)$. On the other hand, from the above propositions, we have that*

$$IB(\ell_1\text{-}\ell_p) \leq 2^{-1/p} < 2(\sqrt{2} - 1) \leq IB'(\ell_1\text{-}\ell_p) \leq D(\ell_1\text{-}\ell_p)$$

for p with $2^{1/p} > (\sqrt{2} + 1)/2$, and that

$$IB(\ell_\infty\text{-}\ell_p) \leq 2^{-1/q} < 2(\sqrt{2} - 1) \leq IB'(\ell_\infty\text{-}\ell_p) \leq D(\ell_\infty\text{-}\ell_p)$$

for p with $2^{1/p} < 4(\sqrt{2} - 1)$. These results imply that the inequality $IB(X) < IB'(X) \leq D(X)$ occurs even in a uniformly non-square normed space X.

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