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The Projective Curvature of the Tangent Bundle with Natural Diagonal Metric

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Dedicated to Academician Professor Mileva Prvanovic on her birthday

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Abstract. Our study is mainly devoted to a natural diagonal metric G on the total space TM of the tangent bundle of a Riemannian manifold (M, g). We provide the necessary and sufficient conditions under which (TM, G) is a space form, or equivalently (TM, G) is projectively Euclidean. Moreover, we classify the natural diagonal metrics G for which (TM, G) is horizontally projectively flat (resp. vertically projectively flat).

1. Introduction

The natural lifts on the total space of the tangent bundle of a (pseudo-)Riemannian manifold, introduced in [10], were intensively studied in the last decades, e.g. in [1]-[11], [16]-[20], [27].

A general natural metric on the total space TM of the tangent bundle of a Riemannian manifold (M, g) is obtained in [16] by lifting the metric g to TM, using six coefficients, which are smooth functions of the energy density t on TM. With respect to a metric of this kind, the horizontal and vertical distributions of the tangent bundle to TM are not orthogonal to each other. If the two coefficients involved in the mixed component of the metric vanish, then the metric becomes a natural diagonal metric, i.e. a metric with respect to which the horizontal and vertical distributions are orthogonal.

In [5], it was shown that TM, endowed with a general natural metric is a space form if and only if the base manifold is flat, and the metric depends on a real constant and two smooth functions of t .

In the present paper we prove that TM endowed with a natural diagonal metric G has constant sectional curvature if and only if the base manifold is flat, and the metric has a certain expression, involving a constant, two smooth functions of t, and their derivatives. Moreover, it follows that TM is flat.

We recall that two linear connections having the same system of geodesics, are obtained one from another, by a projective transformation (see [28]), generalized by the notion of geodesic mapping (see [13], [14], [24], [25], and the references therein). The projective curvature tensor field, obtained by Weyl, is an invariant of any projective transformation on a real manifold. Other invariants of Weyl type, namely the holomorphically-projective ($H-$ projective) curvature tensor fields in the context of the Kähler manifolds and resp. para-Kähler manifolds were studied e.g. in [29], [23], and resp. in [21], [22]. The notion of

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holomorphically-projective transformation was generalized by that of holomorphically–projective mapping (see e.g. in [13] and the references therein). Moreover, in the almost contact case, the $C-$projective transformations (which preserve the $C-$flat paths of an adapted connection without torsion) led to the notion of $C-$projective curvature tensor filed, which is an invariant of any $C-$projective transformation (see [12], [16]).

It is known that a connected (pseudo-)Riemannian manifold of dimension grater than 3 is a space form if and only if it is projectively Euclidean (see [23]).

We prove that there exist two classes of natural diagonal metrics G such that (TM, G) is horizontally projectively flat. Moreover, we classify the natural diagonal metrics G such that (TM, G) is vertically projectively Euclidean.

2. The sectional curvature of the tangent bundle with natural diagonal metric

On a Riemannian manifold (M, g), denote by $\dot{\nabla}$ be the Levi-Civita connection of g, by $\pi : TM \to M$ the tangent bundle of M, and by $(x^1, \dots, x^n)$ (resp. $(x^1, \dots, x^n, y^1, \dots, y^n)$) the local coordinates on M (resp. on TM).

The horizontal (resp. vertical) lift of a vector field $X = X^i \frac{\partial}{\partial x^i} \in \Gamma(TM)$ to TM has the expression $X^H = X^i \frac{\delta}{\delta x^i}$ (resp. $X^V = X^i \frac{\partial}{\partial y^i}$), where $\Gamma^h_{ki}(x)$ are the coefficients of $\dot{\nabla}$ and $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{ki} y^k \frac{\partial}{\partial y^h}$, $\forall i = \overline{1, n}$.

Consider a natural diagonal metric G on TM, given by:

$$\begin{cases} G(X^H_y, Y^H_y) = c_1 g_{\pi(y)}(X, Y) + d_1 g_{\pi(y)}(X, y) g_{\pi(y)}(Y, y), \\ G(X^V_y, Y^V_y) = c_2 g_{\pi(y)}(X, Y) + d_2 g_{\pi(y)}(X, y) g_{\pi(y)}(Y, y), \\ G(X^V_y, Y^H_y) = 0, \end{cases} \tag{1}$$

for all $X, Y \in \Gamma(TM)$, $y \in TM$, where c_1, c_2, d_1, d_2 are smooth functions depending on the energy density t of y, defined as

$$t = \frac{1}{2} g_{\pi(y)}(y, y). \tag{2}$$

The metric G is positive definite provided that

$$c_1,\ c_2 > 0,\ c_1 + 2td_1,\ c_2 + 2td_2 > 0.$$

The matrix of the metric G w.r.t the local adapted frame $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\}_{i,j=\overline{1,n}}$ is

$$\begin{pmatrix} G^{(1)}_{ij} & 0 \\ 0 & G^{(2)}_{ij} \end{pmatrix} = \begin{pmatrix} c_1 g_{ij} + d_1 g_{0i} g_{0j} & 0 \\ 0 & c_2 g_{ij} + d_2 g_{0i} g_{0j} \end{pmatrix}, \tag{3}$$

having the inverse

$$\begin{pmatrix} H^{ij}_{(1)} & 0 \\ 0 & H^{ij}_{(2)} \end{pmatrix} = \begin{pmatrix} \frac{1}{c_1}(g^{ij} - \frac{d_1}{c_1 + 2td_1}) y^i y^j & 0 \\ 0 & \frac{1}{c_2}(g^{ij} - \frac{d_2}{c_2 + 2d_2 t}) y^i y^j \end{pmatrix}. \tag{4}$$

From [19, Theorem 3.1], by imposing the vanishing of the mixed component of the metric and by using the expressions of the blocks $G^{(\alpha)}_{ij}$ (resp. $H^{ij}_{(\alpha)}$), $\alpha = \overline{1, 2}$, from (3) (resp. (4)), we obtain the following:

Proposition 2.1. *The Levi-Civita connection ∇ of G has the following expression in the local adapted frame $\left\{\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right\}_{i,j=\overline{1,n}}$:*

$$\begin{cases} \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = Q^h_{ij} \frac{\partial}{\partial y^h},\ \nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Gamma^h_{ij} \frac{\partial}{\partial y^h} + P^h_{ji} \frac{\delta}{\delta x^h}, \\ \nabla_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = P^h_{ij} \frac{\delta}{\delta x^h},\ \nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma^h_{ij} \frac{\delta}{\delta x^h} + S^h_{ij} \frac{\partial}{\partial y^h}, \end{cases}$$

where Γ^h_{ij} are the Christoffel symbols of $\dot\nabla$ and the coefficients involved in the above expressions are given as

$$\begin{cases} Q^h_{ij} = \frac{1}{2}(\partial_i G^{(2)}_{jk} + \partial_j G^{(2)}_{ik} - \partial_k G^{(2)}_{ij})H^{kh}_{(2)}, \\ P^h_{ij} = \frac{1}{2}(\partial_i G^{(1)}_{jk} + R^l_{0jk} G^{(2)}_{li})H^{kh}_{(1)}, \\ S^h_{ij} = -\frac{1}{2}(\partial_k G^{(2)}_{ij} + R^l_{0ij} G^{(2)}_{lk})H^{kh}_{(2)}, \end{cases}$$

where R^h_{kij} are the components of the curvature tensor field of the Levi Civita connection $\dot\nabla$ of the base manifold (M, g), and ∂_i denotes the derivative w.r.t. the tangential coordinates y^i.

The curvature tensor field K of the connection ∇, defined by the well known formula

$$K(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad X, Y, Z \in \Gamma(TM),$$

has the following components w.r.t. $\{\frac{\delta}{\delta x^i}\frac{\partial}{\partial y^j}\}_{i,j=\overline{1,n}}$:

$$\begin{aligned} K\Big(\tfrac{\delta}{\delta x^i}, \tfrac{\delta}{\delta x^j}\Big)\tfrac{\delta}{\delta x^k} &= (P^h_{li}S^l_{jk} - P^h_{lj}S^l_{ik} + R^l_{0ij}P^h_{lk} + R^h_{kij})\tfrac{\delta}{\delta x^h} \\ K\Big(\tfrac{\delta}{\delta x^i}, \tfrac{\delta}{\delta x^j}\Big)\tfrac{\partial}{\partial y^k} &= (P^l_{kj}S^h_{il} - P^l_{ki}S^h_{jl} + R^l_{0ij}Q^h_{lk} + R^h_{kij})\tfrac{\partial}{\partial y^h} \\ K\Big(\tfrac{\partial}{\partial y^i}, \tfrac{\partial}{\partial y^j}\Big)\tfrac{\delta}{\delta x^k} &= (\partial_i P^h_{jk} - \partial_j P^h_{ik} + P^l_{jk}P^h_{il} - P^l_{ik}P^h_{jl})\tfrac{\delta}{\delta x^h} \\ K\Big(\tfrac{\partial}{\partial y^i}, \tfrac{\partial}{\partial y^j}\Big)\tfrac{\partial}{\partial y^k} &= (\partial_i Q^h_{jk} - \partial_j Q^h_{ik} + Q^l_{jk}Q^h_{il} - Q^l_{ik}Q^h_{jl})\tfrac{\partial}{\partial y^h} \\ K\Big(\tfrac{\partial}{\partial y^i}, \tfrac{\delta}{\delta x^j}\Big)\tfrac{\delta}{\delta x^k} &= (\partial_i S^h_{jk} + S^l_{jk}Q^h_{il} - P^l_{ik}S^h_{jl} - \dot\nabla_j R^r_{0ik}G^{(2)}_{rl}H^{(1)}_{hl})\tfrac{\partial}{\partial y^h} \\ K\Big(\tfrac{\partial}{\partial y^i}, \tfrac{\delta}{\delta x^j}\Big)\tfrac{\partial}{\partial y^k} &= (\partial_i P^h_{kj} + P^l_{kj}P^h_{il} - Q^l_{ik}P^h_{lj})\tfrac{\delta}{\delta x^h}. \end{aligned} \tag{5}$$

In the local adapted frame $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\}_{i,j=\overline{1,n}}$, the curvature tensor field K_0 of a Riemannian manifold (TM, G) of constant sectional curvature k, given by:

$$K_0(X, Y)Z = k[G(Y, Z)X - G(X, Z)Y],$$

has the components:

$$K_0\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} = k\Big[G^{(1)}_{jk}\frac{\delta}{\delta x^i} - G^{(1)}_{ik}\frac{\delta}{\delta x^j}\Big],\ K_0\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\frac{\partial}{\partial y^k} = 0,$$

$$K_0\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big)\frac{\delta}{\delta x^k} = 0,\ K_0\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} = k\Big[G^{(2)}_{jk}\frac{\partial}{\partial y^i} - G^{(2)}_{ik}\frac{\partial}{\partial y^j}\Big],$$

$$K_0\Big(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} = kG^{(1)}_{jk}\frac{\partial}{\partial y^i},\ K_0\Big(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\Big)\frac{\partial}{\partial y^k} = -kG^{(2)}_{ik}\frac{\delta}{\delta x^j}.$$

Studying the conditions under which the difference $K - K_0$ vanishes, we prove the following results.

Proposition 2.2. *Let (M, g) be a Riemannian manifold. If the tangent bundle TM endowed with a natural diagonal metric G is a space form, then the base manifold is flat.*

Proof: Since $(K - K_0)\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\frac{\partial}{\partial y^k}$ must vanish for every $y \in TM$, it follows that it vanishes for $y = 0$, too, case when it reduces to R^h_{kij}. The curvature of the base manifold do not depend on the tangent vector y, hence $R^h_{kij} = 0$, i.e. the base manifold is flat.

Theorem 2.3. *Let TM be the total space of the tangent bundle of a Riemannian manifold (M, g), endowed with a natural diagonal metric G. Then (TM, G) is a space form if and only if the base manifold is flat and the metric G is given by (1), provided that c_1 is a real constant, $d_1 = 0$ and $d_2 = c_2'\Big(1 + t\frac{c_2'}{2c_2}\Big)$. Moreover, (TM, G) cannot have nonzero constant sectional curvature.*

Proof: If (TM, G) is a space form, it follows from Proposition 2.2 that the base manifold (M, g) is flat. On the other hand, all the components of the difference between the curvature tensors K and K_0 must vanish. By imposing $R^h_{kij} = 0$, we obtain

$$(K-K_0)\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\frac{\partial}{\partial y^k} = \frac{d_1^2 t}{2c_2(c_1+2td_1)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i)+$$

$$+\frac{d_1(c_1d_1-2c_1'd_1t+2c_1d_1't)}{4c_1c_2(c_1+2td_1)}(\delta^h_j g_{0i} - \delta^h_i g_{0j})g_{0k}+$$

$$+\frac{c_1c_2d_1^2-2c_1'c_2d_1^2t+2c_1c_2d_1d_1't-2c_1d_1^2d_2t}{4c_1c_2(c_1+2td_1)}(g_{ik}g_{0j} - g_{jk}g_{0i})y^h.$$

By applying [19, Lemma 3.2], it follows that the above expression vanishes if and only if all its coefficients vanish, i.e. if and only if $d_1 = 0$.

Then, the component $K - K_0$ corresponding to all horizontal arguments becomes

$$(K-K_0)\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} = \Big[\frac{c_1'^2 t}{2c_1(c_2+2td_2)} + kc_1\Big](g_{ik}\delta^h_j - g_{jk}\delta^h_i),$$

and from its vanishing condition it follows that

$$k = -\frac{c_1'^2 t}{2c_1^2(c_2+2d_2t)}. \tag{6}$$

Replacing the obtained value of k into the expression of $(K-K_0)\Big(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k}$, this component takes the form

$$(K-K_0)\Big(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} = -\frac{c_1c_1'}{2(c_1c_2+c_1'c_2t+c_1c_2't)}g_{jk}\delta^h_i - \alpha g_{jk}g_{0i}y^h, \tag{7}$$

where α is a rational function depending on c_1, c_2, their first two order derivatives, and the energy density t.

Since all the terms of α contain c_1' or c_1'', the expression (7) is zero if and only if c_1 is a real constant. Then, after replacing k from (6), the component of $K - K_0$ corresponding to all vertical arguments becomes

$$(K-K_0)\Big(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} = \frac{c_2'(2c_2+c_2't)-2c_2d_2}{2c_2(c_2+2td_2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i)+$$

$$+\beta(g_{0j}\delta^h_i - g_{0i}\delta^h_j)g_{0k} + \gamma(g_{ik}g_{0j} - g_{jk}g_{0i})y^h,$$

where β, γ are two rational functions depending on $c_1, c_2, c_2', c_2'', d_2, d_2'$ and the energy density t.

The above expression is zero if and only if

$$d_2 = c_2' + \frac{c_2'^2 t}{2c_2}, \tag{8}$$

and then all the components of the difference $K - K_0$ vanish, hence the proof is completed.

3. The projective curvature of (TM, G)

On a differentiable manifold, the projective curvature tensor field associated to a linear connection ∇ is invariant under a projective transformation of ∇, i.e. a transformation which preserves the geodesics (see [23]). In the particular situation of a connected (pseudo-)Riemannian manifold of dimension $n \le 3$, the manifold has constant sectional curvature if and only if it is projectively flat.

Definition 3.1. *On an n-dimensional differentiable manifold M, the projective curvature tensor field associated to a linear connection* ∇*, is a* $(1,3)-$*tensor field P defined by:*

$$P(X,Y)Z = R(X,Y)Z - L(Y,Z)X + L(X,Z)Y + [L(X,Y) - L(Y,X)]Z, \ \forall X,Y,Z \in \Gamma(TM),$$

where R and Ric are respectively the curvature tensor field and the Ricci tensor field of ∇*, and L is the Brinkman tensor field, given by:*

$$L(X,Y) = \frac{1}{n^2-1}[Ric(X,Y) + nRic(Y,X)], \ \forall X,Y \in \Gamma(TM).$$

Since the Ricci tensor associated to the Levi-Civita connection is symmetric, it follows that the projective curvature tensor field associated to the Levi-Civita connection has the expression:

$$P(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[Ric(X,Z)Y - Ric(Y,Z)X], \ \forall X,Y,Z \in \Gamma(TM). \tag{9}$$

Remark: Let (M,g) be a Riemannian manifold, and TM the total space of its tangent bundle, endowed with a natural diagonal metric G. Then (TM,G) is a space form if and only if it is projectively flat w.r.t. the projective curvature tensor field associated to the Levi-Civita connection of G.

Definition 3.2. *The Riemannian manifold* (TM,G) *is called horizontally (resp. vertically) projectively flat if the projective curvature tensor field associated to the Levi-Civita connection of G vanishes on the horizontal (resp. vertical) distribution of TTM.*

By using Theorem 2.3 and the expression (9) of the projective curvature tensor field, we prove the following results.

Theorem 3.3. *Let* (M,g) *be a Riemannian space form. The total space TM of the tangent bundle of M, endowed with a natural diagonal metric G, is horizontally projectively flat if and only if the base manifold is flat and G is given by* (1), *provided that its coefficients satisfy one of the following cases:*

Case I) c_1 *is an arbitrary real constant,* $d_1 = 0$*,* c_2 *and* d_2 *are two arbitrary smooth functions of the energy density;*

Case II) On the nonzero tangent bundle of (M,g)*,*

$$c_2 = \frac{c_0}{t}, \ d_1 = 0, \ d_2' = \frac{2c_1c_1'k - c_1'^2kt + 2c_1c_1''kt - 2c_1'^2d_2t^3 + 4c_1c_1''d_2t^3}{2c_1c_1't^3},$$

and c_1 *is an arbitrary smooth function of the energy density.*

Proof: On (TM,G), consider the projective tensor field P associated to the Levi-Civita connection ∇. The component of P corresponding to all horizontal arguments is given by:

$$P\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} = K\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} + \frac{1}{n-1}\Big[Ric\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\Big)\frac{\delta}{\delta x^j} - Ric\Big(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\Big)\frac{\delta}{\delta x^i}\Big],$$

where K is the curvature tensor field of ∇ and Ric is the corresponding Ricci tensor, obtained by the contraction of the components of K as follows:

$$Ric\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^k}\Big) = K^h_{ihk} + K^{\bar h}_{i\bar hk}, \ \forall i,j,k,h,\bar h = \overline{1,n},$$

where the indices i,j,k,h correspond to the horizontal arguments and $\bar h$ to the vertical argument.

Now we study the conditions under which (TM,G) is horizontally projectively flat, i.e. $P\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k}$ vanishes.

By replacing into (9) the component $K\Big(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k}$ and the components of the curvature involved in the expression of the Ricci tensor, we obtain the following expression:

$$P\Big(\tfrac{\delta}{\delta x^i},\tfrac{\delta}{\delta x^j}\Big)\tfrac{\delta}{\delta x^k} = (A_1 + B_1n)(g_{jk}\delta^h_i - g_{ik}\delta^h_j) +$$
$$+(A_2 + B_2n)(\delta^h_j g_{0i} - \delta^h_i g_{0j})g_{0k} + A_3(g_{ik}g_{0j} - g_{jk}g_{0i})y^h,$$

where $A_\alpha,\ \alpha = \overline{1,3},\ B_\alpha,\ \alpha = \overline{1,2}$ are some quite long functions, depending on the coefficients of the metric G, their first two order derivatives, the constant sectional curvature c of the base manifold, and the energy density t of $y \in TM$.

According to [19, Lemma 3.2], the above expression vanishes if and only if $A_\alpha n + B_\alpha = 0,\ \alpha = \overline{1,3}$. Moreover, since we study the conditions of vanishing of the expression of $P\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k}$ for the tangent bundle of a Riemannian manifold of arbitrary dimension n, it follows that $A_\alpha n + B_\alpha = 0,\ \alpha = \overline{1,3}$ for every $n > 1$, i.e. if and only if $A_\alpha = B_\alpha = 0,\ \alpha = \overline{1,3}$.

The numerator of the coefficient B_1 has the form

$$c_1 c_1'(c_2 + c_2' t)(c_1 + 2td_1)(c_2 + 2td_2),$$

and therefore the condition $B_1 = 0$ yields two cases: Case I, when the function c_1 is a real constant, and Case II, when $c_2 = \frac{c_0}{t}$, where $c_0 \in \mathbb{R}$. We mention that the second case holds good only on the nonzero section of the tangent bundle.

In Case I, the conditions of vanishing of A_1, B_2 and A_3 lead to the following system of equations:

$$\begin{cases} c_1 d_1 + c^2 c_2^2 t + d_1^2 t = 0 \\ (c_1 + 2td_1)(c_2 + 2td_2)(c^2 c_2^3 - 2cc_2^2 d_1 - 2c_1 c_2' d_1 + c_2 d_1^2 - 2c_1 c_2 d_1' + \\ +4c_1 d_1 d_2 - 2c_1 c_2' d_1' t + 2c^2 c_2^2 d_2 t - 4cc_2 d_1 d_2 t + 2d_1^2 d_2 t) = 0 \\ 3c^2 c_2^2 + 2cc_2 d_1 - d_1^2 = 0. \end{cases} \tag{10}$$

If the curvature of the base manifold is $c \neq 0$, it follows that (10) has the solution

$$d_1 = -\frac{9c_1}{10t},\ c_2 = -\frac{3c_1}{10ct},\ d_2 = \frac{-27c_1^2 + 450cc_1 c_2' t^2 + 150c_1 d_1' t^2 - 500cc_2' d_1' t^4}{720cc_1 t^2}.$$

By replacing the above values and $c_1' = 0$ into the expression of $P\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k}$, this reduces to

$$P\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k} = \frac{6c}{5t}(\delta_j^h g_{0i} - \delta_i^h g_{0j})g_{0k},$$

which is nonzero since $c \neq 0$.

If the base manifold is flat, we obtain that A_1, A_3, B_2 vanish simultaneously if and only if:

$$d_1(c_1 + td_1 t) = 0,\ d_1 = 0,$$
$$-2c_1 c_2' d_1 + c_2 d_1^2 - 2c_1 c_2 d_1' + 4c_1 d_1 d_2 - 2c_1 c_2' d_1' t + 2d_1^2 d_2 t = 0,$$

i.e. $d_1 = 0$.

In Case II, the numerator of B_2 is

$$(c_1 + 2d_1 t)(c^2 c_0^2 + 2c_1 d_1 t - 2cd_1 c_0 t + d_1^2 t^2)(c_0 + 2d_2 t^2)^2.$$

If $d_1 = 0$, it follows that $B_2 = 0$ is equivalent to the flatness of the base manifold: $c = 0$, and then the condition of vanishing of $P\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big)\frac{\delta}{\delta x^k}$ becomes

$$\frac{c_1 c_0 t(2c_1 c_1' c_0 - tc_1'^2 c_0 + 2tc_1 c_1'' c_0 - 2t^3 c_1'^2 d_2 + 4t^3 c_1 c_1'' d_2 - 2t^3 c_1 c_1' d_2')}{2c_1^2 c_0 (c_0 + 2t^2 d_2)^2 (n-1)}(g_{jk}\delta_i^h - g_{jk}\delta_i^h) = 0.$$

Since in Case II the coefficient c_1 is non-constant, the above relation is equivalent to

$$d_2' = \frac{2c_1 c_1' c_0 - c_1'^2 c_0 t + 2c_1 c_1'' c_0 t - 2c_1'^2 d_2 t^3 + 4c_1 c_1'' d_2 t^3}{2c_1 c_1' t^3}.$$

If $d_1 \neq 0$, then the numerator of B_2 is zero if and only if

$$(c_1 + 2d_1 t)(c^2 c_0^2 + 2c_1 d_1 t - 2cd_1 c_0 t + d_1^2 t^2)(c_0 + 2d_2 t^2)^2 = 0,$$

i.e. if and only if

$$c_1 = \frac{-c^2c_0^2 + 2cc_0td_1 - t^2d_1^2}{2td_1},$$

which yields the following expression of the numerator of A_3:

$$5cd_1^2c_0^2 - td_1^3c_0 + 4tcc_0^2d_1d_1' + 6t^2cc_0d_1^2d_2 + 2t^2cc_0^2d_1'^2 - 2t^3d_1^3d_2.$$

In this case, we have that $A_3 = 0$ is equivalent to

$$d_2 = \frac{5cc_0^2d_1^2 - tc_0d_1^3 + 4tcc_0^2d_1d_1' + 2t^2cc_0^2d_1'^2}{2t^2d_1^2(-3cc_0 + td_1)}. \tag{11}$$

After replacing (11), the condition of vanishing of the numerator of A_2 becomes:

$$d_1(19c^3c_0^3 - 12tc^2d_1c_0^2 - 9t^2cd_1^2c_0 + 6t^3d_1^3) = 0.$$

Solving the above equation w.r.t. d_1, we obtain that its only one real solution is of the form:

$$d_1 = \frac{cc_0}{t}\Big[\frac{1}{2} + \frac{11}{2\cdot 3^{1/3}(6\sqrt{3} - 69)^{1/3}} + \frac{(16\sqrt{3} - 69)^{1/3}}{3^{2/3}}\Big]. \tag{12}$$

The expression $c_2 + 2td_2$ vanishes when c_2 is replaced by $\frac{c_0}{t}$, d_2 by its value from (11), and d_1 from (12). Hence the subcase $d_1 \neq 0$ is not a valid subcase of Case II, and then in Case II, the coefficients of the metric have the expressions mentioned in the statement.

Theorem 3.4. *Let TM be the total space of the tangent bundle of a Riemannian space form (M, g), and let G be a natural diagonal metric on TM. Then (TM, G) is vertically projectively flat if and only if one of the following cases hold good:*

Case I.1) The base manifold is flat, and the coefficients of G satisfy the following conditions: c_1 is a real constant, $d_1 = 0$, c_2 is an arbitrary smooth real function of the energy density t, and

$$d_2' = \frac{-3c_2c_2'^2 + 2c_2^2c_2'' + 4c_2d_2^2 - 4c_2'^2d_2t + 4c_2c_2''d_2t}{2c_2(c_2 + c_2't)}.$$

Case I.2) $c_1' = \frac{2cc_2(c_2+2td_2)}{c_2+tc_2'}$, $d_1 = -cc_2$, $d_2 = c_2'\big(1 + t\frac{c_2'}{2c_2}\big)$, and c_2 is an arbitrary smooth real function of the energy density t.

Case I.3) On the nonzero section of TM, $c_1 = 2tcc_2$, $d_1 = -cc_2$, and $d_2 = \frac{3c_1c_2^2 - 6tcc_2^3 + 4tc_1c_2c_2' - 4t^2cc_2^2c_2' + 2t^2c_1c_2'^2 - 2t^3cc_2c_2'^2}{2tc_2(-c_1+4tcc_2)}$, where $c_2 = \frac{1}{t}(k_1 + e^tk_2)$, with k_1 and k_2 two arbitrary real constants.

Case II) $c_1 = (cc_2 - d_1)t$, $\frac{d_1'}{d_1} = \frac{c_2'}{c_2}$, $d_2 = c_2' + \frac{c_2'^2t}{2c_2}$ and c_2 is an arbitrary smooth real function of the energy density t.

Proof: On the vertical distribution of TTM, the component of the projective curvature tensor corresponding to the Levi-Civita connection of G is:

$$P\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} = K\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} + \frac{1}{n-1}\Big[Ric\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^k}\Big)\frac{\partial}{\partial x^j} - Ric\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\Big)\frac{\partial}{\partial y^i}\Big],$$

where K is the curvature tensor field of ∇ and Ric is the corresponding Ricci tensor, whose component on the vertical distribution is given as:

$$Ric\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^k}\Big) = K^h_{\bar{i}h\bar{k}} + K^{\bar{h}}_{\bar{i}\bar{h}\bar{k}}, \ \forall i, k, h, \bar{i}, \bar{k}, \bar{h} = \overline{1, n},$$

where the indices i, k, h correspond to the horizontal arguments and $\bar{i}, \bar{k}, \bar{h}$ to the vertical arguments.

Let (TM, G) be vertically projectively flat, i.e. $P\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} = 0, \forall i, j, k = \overline{1, n}$.

By similar computations to those in the proof of Theorem 3.3, we obtain:

$$P\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} = (\bar{A}_1 + \bar{B}_1 n)(g_{jk}\delta_i^h - g_{ik}\delta_j^h)+$$
$$+(\bar{A}_2 + \bar{B}_2 n)(\delta_j^h g_{0i} - \delta_i^h g_{0j})g_{0k} + \bar{A}_3(g_{jk}g_{0i} - g_{ik}g_{0j})y^h\Big],$$

where $\bar{A}_\alpha,\ \alpha = \overline{1,3},\ \bar{B}_\alpha,\ \alpha = \overline{1,2}$ are some quite long functions, depending on the coefficients of the metric G, their first two order derivatives, the constant sectional curvature c of the base manifold, and the energy density t of $y \in TM$.

In the same way as in the previous proof, we have that $P\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} = 0, \forall i, j, k = \overline{1, n}$, if and only if $\bar{A}_\alpha = \bar{B}_\alpha = 0,\ \alpha = \overline{1,3}$.

After the computations, the numerator of the coefficient $\bar{B}_1$ has the form:

$$c_2(cc_2 + d_1)(c_1 - cc_2 t + d_1 t)(c_1 + 2d_1 t)(c_2 + 2d_2 t)^2.$$

The condition of vanishing of the above expression lead to the following cases: Case I) $d_1 = -cc_2$, and Case II) $c_1 = (cc_2 - d_1)t$.

In both cases we have that $\bar{A}_3 = 0$ if and only if

$$3c_2c_2'^2 - 2c_2^2c_2'' - 4c_2d_2^2 + 2c_2^2d_2' + 4c_2'^2d_2t - 4c_2c_2''d_2t + 2c_2c_2'd_2't = 0.$$

Notice that $c_2 + c_2't \neq 0$, since $c_2 = \frac{c_0}{t}$, with c_0 an arbitrary real constant, would lead to $\bar{A}_3 = \frac{1}{4t^2} \neq 0$.

Hence $\bar{A}_3$ vanishes if and only if

$$d_2' = \frac{-3c_2c_2'^2 + 2c_2^2c_2'' + 4c_2d_2^2 - 4c_2'^2d_2t + 4c_2c_2''d_2t}{2c_2(c_2 + c_2't)}. \tag{13}$$

In the sequel we shall study the two cases, separately.

Case I) $d_1 = -cc_2$ implies that $\bar{A}_1 = 0$ if and only if

$$c_1'c_2 - 2cc_2^2 + c_1'c_2't - 4cc_2d_2t = 0,$$

which is equivalent to

$$c_1' = \frac{2(cc_2^2 + 2cc_2d_2t)}{c_2 + c_2't}. \tag{14}$$

Replacing the above value of c_1' into the expression of $P\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k}, \forall i, j, k = \overline{1, n}$, this becomes of the form

$$P\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big)\frac{\partial}{\partial y^k} = \frac{c(2c_2c_2' - 2c_2d_2 + c_2'^2 t)}{2c_2(c_1 - 2cc_2t)^2(c_2 + c_2't)^2(n-1)}(-3c_1c_2^2 + 6cc_2^3t - 4c_1c_2c_2't -$$
$$-2c_1c_2d_2t + 4cc_2^2c_2't^2 - 2c_1c_2'^2t^2 + 8cc_2^2d_2t^2 + 2cc_2c_2'^2t^3)(\delta_j^h g_{0i} - \delta_i^h g_{0j}),$$

and it vanishes if and only if one of the following subcases holds good:

Case I.1) $c = 0$ leads to $c_1' = 0$, $d_1 = 0$ and the expression of d_2 remains (13).

On the nonzero section of the tangent bundle we have also other two subcases.

Case I.2) $d_2 = c_2'(1 + \frac{c_2'}{2c_2}t)$, c_1' has the expression (14), and $d_1 = -cc_2$.

Case I.3) $d_2 = \frac{3c_1c_2^2 - 6tcc_2^3 + 4tc_1c_2c_2' - 4t^2cc_2^2c_2' + 2t^2c_1c_2'^2 - 2t^3cc_2c_2'^2}{2tc_2(-c_1 + 4tcc_2)}$ and then the value of d_2' is the one given by the relation (13) if and only if

$$\frac{3(c_2 + c_2't)^3(c_1^3 - 6cc_1^2c_2t + 14c^2c_1c_2^2t^2 - 12c^3c_2^3t^3)}{c_2^2t^2(-c_1 + 4cc_2t)^3} = 0.$$

Since we proved that $c_2 + c_2' \neq 0$, it follows that the above relation is equivalent to the equation

$$c_1^3 - 6cc_1^2c_2t + 14c^2c_1c_2^2t^2 - 12c^3c_2^3t^3 = 0,$$

which solved w.r.t. c_1 has only one real solution:

$$c_1 = 2cc_2t.$$

By imposing the condition (14), we obtain

$$2c(c_2 - 2c_2' + c_2't - c_2''t) = 0. \tag{15}$$

The subcase when $c = 0$ leads to Case I.1), which was already treated.

If $c \neq 0$, (15) is equivalent to the second order differential equation:

$$c_2'' = \frac{c_2 - 2c_2' + c_2't}{t},$$

which has the solution

$$c_2 = \frac{1}{t}(k_1 + k_2e^t),\ k_1,\ k_2 \in \mathbb{R}.$$

Case II) $c_1 = (cc_2 - d_1)t$ yields

$$\bar{B}_2 = (cc_2 + d_1)^2(c_2d_1' - c_2'd_1)(c_2 + c_2't)(c_2 + 2d_2t).$$

Since $c_2 + c_2't \neq 0$ and the case $d_1 = -cc_2$ was studied at Case I.2), the condition of vanishing of $\bar{B}_2$ is

$$d_1' = \frac{c_2'}{c_2}d_1. \tag{16}$$

Replacing the value of d_1' from (16) into $\bar{A}_1$, the relation $\bar{A}_1 = 0$ becomes

$$2c_2c_2' - 2c_2d_2 + c_2'^2t = 0.$$

Hence (TM, G) is vertically projectively flat if and only if the coefficients of the metric G satisfy one of the cases in the above theorem.

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