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Fixed Points of Generalized Contractive Mappings in Ordered Metric Spaces

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Abstract. Existence theorem for fixed point of mappings satisfying a new generalized contractive condition, involving some well-known contractive conditions of rational type, in ordered metric spaces is proved. Some examples are given which illustrate the value of the obtained results in comparison to some of the existing ones in literature.

1. Introduction and Preliminaries.

It is well known that the Banach contraction principle is one of the pivotal results of analysis. This principle have been generalized in several directions. Existence of fixed points in partially ordered sets has been considered recently by many authors [1-10], where some applications to matrix equations, ordinary differential equations and to boundary value problems are presented. The following are fixed point theorems for mappings satisfying some contractive conditions of rational type.

Theorem 1.1. *(Jaggi [11]) Let (X, d) be a complete metric space and let $T : X \to X$ be a continuous mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, y), \qquad (1.1)$$

for all $x, y \in X$, $x \neq y$. Then T has a unique fixed point.

Theorem 1.2. *(Dass and Gupta [12]) Let (X, d) be a complete metric space and $T : X \to X$ a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y), \qquad (1.2)$$

for all $x, y \in X$. Then T has a unique fixed point.

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Recently Harjani et al. [13] and Luong et al. [14] generalized Theorem 1.1 in partially ordered metric spaces and Cabrera et al. [15] presented theorem 1.2 in the context of partially ordered metric spaces.
The aim of this paper is to give a generalization of the above mentioned theorems in ordered metric spaces. To set up our main result in the next section, we need the following definition.
Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. Denote $\mathcal{S} = \mathcal{R} \cup \mathcal{R}^{-1}$. Clearly, for any $x, y \in X$, $x \,\mathcal{S}\, y \Leftrightarrow x \,\mathcal{R}\, y$ or $y \,\mathcal{R}\, x$. If $x \,\mathcal{S}\, y$ we say that x and y are comparable. Let $T : X \to X$ be a mapping. We say that $T : X \to X$ is a comparable mapping if T maps comparable elements into comparable elements, that is,

$$\text{for any } \ x, y \in X, \ \ x \,\mathcal{S}\, y \Rightarrow Tx \,\mathcal{S}\, Ty.$$

2. Fixed Point Theory

Throughout the paper, denote with Λ the family of functions $\lambda(u_1, u_2, u_3, u_4, u_5) : \mathbb{R}_+^5 \to \mathbb{R}_+$ such that λ is nondecreasing in u_4 and $\lambda(u, u, v, u + v, 0) \le v$ for each $u, v \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$.
Now, we are ready to state our main result.

Theorem 2.1. *Let (X, d) be a complete metric space and let $\mathcal{R}$ be a binary relation over X. Let $T : X \to X$ be a comparable mapping satisfying*

$$d(Tx, Ty) \le \alpha(d(x, y))N(x, y) + \beta(d(x, y))M(x, y), \qquad (2.1)$$

for each $x \,\mathcal{S}\, y, x \neq y$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\},$$

$$N(x, y) = \lambda(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)),$$

$\lambda \in \Lambda$ and $\alpha, \beta : [0, \infty) \to [0, 1)$ are mappings such that α is continuous and $\alpha(t) + \limsup_{s \to t^+} \beta(s) < 1$, for each $t \ge 0$. Assume either

(i) T is continuous or

(ii) λ is continuous at $(0, 0, u, u, 0)$ for each $u \ge 0$ and X has the property:

$$\text{if } \ x_n \,\mathcal{S}\, x_{n+1} \ \text{ for each } \ n \ge 0 \text{ and } \ x_n \to x \ \text{ then } \ x_n \,\mathcal{S}\, x.$$

If there exists $x_0 \in X$ such that $x_0 \,\mathcal{S}\, Tx_0$, then T has a fixed point.

Proof. Define the sequence $\{x_n\}$ in X inductively by $x_{n+1} = Tx_n$ for all $n \ge 0$. If $x_{n+1} = x_n$ for some n, then $Tx_n = x_n$ and so we are done. So, we may assume that $x_{n+1} \neq x_n$ for each n. Note that,

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\} \\ &\le \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}, \end{aligned}$$

and

$$\begin{gathered} N(x_{n-1}, x_n) = \\ \lambda(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0)) \\ \le \lambda(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0)) \\ \le d(x_n, x_{n+1}), \end{gathered}$$

for each $n \geq 0$. Since $x_0 \mathcal{S} Tx_0$ and T is a comparable mapping then $x_n \mathcal{S} x_{n+1}$ for each $n \geq 0$. Then by the contractive condition (2.1), we have for $n \geq 1$,

$$d(x_n, x_{n+1}) \leq \alpha(d(x_{n-1}, x_n))N(x_{n-1}, x_n) + \beta(d(x_{n-1}, x_n))M(x_{n-1}, x_n)$$

$$\leq \alpha(d(x_{n-1}, x_n))d(x_n, x_{n+1}) + \beta(d(x_{n-1}, x_n)) \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},$$

and so

$$(1 - \alpha(d(x_{n-1}, x_n)))d(x_n, x_{n+1})$$
$$\leq \beta(d(x_{n-1}, x_n)) \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \quad (2.2)$$

If for some n, $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then from (2.2) we get $1 \leq (\alpha + \beta)(d(x_{n-1}, x_n))$, a contradiction. Thus

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n) \text{ for each } n,$$

and so from (2.2), we get

$$(1 - \alpha(d(x_{n-1}, x_n)))d(x_n, x_{n+1})) \leq \beta(d(x_{n-1}, x_n))d(x_{n-1}, x_n), \quad (2.3)$$

for each $n \in \mathbb{N}$. Let $\gamma(t) = \frac{\beta(t)}{1-\alpha(t)}$ for each $t \in \mathbb{R}_+$. Then by our assumptions,

$$\limsup_{s \to t^+} \gamma(s) = \frac{\limsup_{s \to t^+} \beta(s)}{1 - \alpha(t)} < 1 \text{ for each } t \geq 0. \quad (2.4)$$

From (2.3), we have

$$d(x_n, x_{n+1}) \leq \gamma(d(x_{n-1}, x_n))d(x_{n-1}, x_n), \quad (2.5)$$

for all $n \geq 1$. Then $\{d(x_n, x_{n+1})\}$ is a non-increasing non-negative sequence, so $\lim_{n\to\infty} d(x_n, x_{n+1}) = r \geq 0$ exists. Assume $r > 0$. Then from (2.5), we have

$$\limsup_{s \to r^+} \gamma(s) \geq \limsup_{n\to\infty} \gamma(d(x_n, x_{n+1})) \geq \limsup_{n\to\infty} \frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} = 1,$$

which contradicts (2.4). Then

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \quad (2.6)$$

We show that $\{x_n\}$ is a Cauchy sequence. Let $\limsup_{s\to 0^+} \gamma(s) < c < 1$. Since $\limsup_{n\to\infty} \gamma(d(x_{n-1}, x_n) \leq \limsup_{t\to 0^+} \gamma(t) < 1$ then there exists $N > 0$ such that $\gamma(d(x_{n-1}, x_n) < c$ for $n \geq N$. Then from (2.5), we have $d(x_n, x_{n+1}) \leq cd(x_{n-1}, x_n)$ for $n \geq N$. Hence $\Sigma_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ and so

$$d(x_m, x_n)$$

$$\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \to 0 \text{ as } m, n \to \infty.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, $\lim_{n\to\infty} x_n = x$ for some $x \in X$. Assume first that T is continuous. Then,

$$Tx = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

Therefore, x is a fixed point of T. Now, suppose that T satisfies condition (ii). Then $x_n \ \mathcal{S} \ x$ for each n and so from (2.1), we get

$$d(x_{n+1}, Tx) = d(Tx_n, Tx) \leq \alpha(d(x_n, x))N(x_n, x) + \beta(d(x_n, x))M(x_n, x).$$

Then

$$d(x, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx)$$

$$\leq \limsup_{n\to\infty}[\alpha(d(x_n,x))N(x_n,x)] + \limsup_{n\to\infty}[\beta(d(x_n,x))M(x_n,x)]$$
$$\leq \alpha(0)\lambda(0,0,d(x,Tx),d(x,Tx),0) + \limsup_{t\to 0^+}\beta(t)d(x,Tx)$$
$$\leq (\alpha(0) + \limsup_{t\to 0^+}\beta(t))d(x,Tx),$$

and so $Tx = x$ (note that $\alpha(0) + \limsup_{t\to 0^+}\beta(t) < 1$). $\square$

Letting

$$\lambda(u_1,u_2,u_3,u_4,u_5) = \begin{cases} \frac{u_2u_3}{u_1}, & u_1 > 0 \\ 0. & u_1 = 0 \end{cases}$$

we get the following improvement of the main results of [11] and [13].

Corollary 2.2. *Let (X,d) be a complete metric space and let $\mathcal{R}$ be a binary relation over X. Let $T : X \to X$ be a continuous comparable mapping satisfying*

$$d(Tx,Ty) \leq \alpha(d(x,y))\frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta(d(x,y))M(x,y),$$

for each $x\,\mathcal{S}\,y, x \neq y$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\},$$

$\alpha,\beta : [0,\infty) \to [0,1)$ are mappings such that α is continuous and $\alpha(t) + \limsup_{s\to t^+}\beta(s) < 1$, for each $t \geq 0$. If there exists $x_0 \in X$ such that $x_0\,\mathcal{S}\,Tx_0$, then T has a fixed point.

Letting $\lambda(u_1,u_2,u_3,u_4,u_5) = \frac{u_3(1+u_2)}{1+u_1}$, we get the following improvement of the main results of [12] and [15].

Corollary 2.3. *Let (X,d) be a complete metric space and let $\mathcal{R}$ be a binary relation over X. Let $T : X \to X$ be a comparable mapping satisfying*

$$d(Tx,Ty) \leq \alpha(d(x,y))\frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)} + \beta(d(x,y))M(x,y),$$

for each $x\,\mathcal{S}\,y, x \neq y$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\},$$

$\alpha,\beta : [0,\infty) \to [0,1)$ are mappings such that α is continuous and $\alpha(t) + \limsup_{s\to t^+}\beta(s) < 1$, for each $t \geq 0$. Assume X has the property:

$$\text{if } x_n\,\mathcal{S}\,x_{n+1} \text{ for each } n \geq 0 \text{ and } x_n \to x \text{ then } x_n\,\mathcal{S}\,x.$$

If there exists $x_0 \in X$ such that $x_0\,\mathcal{S}\,Tx_0$, then T has a fixed point.

Now we illustrate our main results by the following examples.

Example 2.4. *Let (X,d) be a metric space, where $X = \{1,2,3,4\}$, $d(1,2) = d(1,3) = 1$, $d(1,4) = \frac{7}{4}$, $d(2,3) = d(2,4) = d(3,4) = 2$. Let $T : X \to X$ be given by $T1 = 1, T2 = 4, T3 = 4$ and $T4 = 1$. Let $\mathcal{R} = X \times X$. Since X is a finite set then every Cauchy sequence in (X,d) is eventually constant and so is convergent. Then (X,d) is a complete metric space. Moreover, since (X,d) is a discrete topological space then $T : X \to X$ is a comparable continuous mapping. Now, we show that we cannot invoke the above mentioned Theorems of Jaggi and of Dass and Gupta to prove the existence of a fixed point for T. Let $x = 1$ and $y = 2$. Then, we have*

$$d(Tx,Ty) = \frac{7}{4},\ d(x,Tx) = 0,\ d(y,Ty) = 2,\ d(x,y) = 1,$$

and so the contractive conditions of Theorems 1.1 and Theorem 1.2 are not satisfied because

$$d(Tx,Ty)=\frac{7}{4}\le \alpha\frac{d(x,Tx)d(y,Ty)}{d(x,y)}+\beta d(x,y)=\beta$$

and

$$d(Tx,Ty)=\frac{7}{4}\le \alpha\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+\beta d(x,y)=\alpha+\beta,$$

imply that $\alpha+\beta\ge\frac{7}{4}$.
Now let $\alpha=\frac{3}{7}$ *and* $\beta=\frac{1}{2}$. *It is easy to see that for each* $x,y\in X$, *we have*

$$d(Tx,Ty)\le \alpha\frac{d(y,Ty)(1+d(x,Tx))(1+d(y,Tx))}{1+d(x,y)}+\beta M(x,y),$$

and so the contractive condition (2.1) of Theorem 2.1 is satisfied by T, where $\lambda(u_1,u_2,u_3,u_4,u_5)=\frac{u_3(1+u_2)(1+u_5)}{1+u_1}$ *for each* $u_i\ge 0$. *Then by Theorem 2.1, T has a fixed point.*

Example 2.5. *Let* (X,d) *be a metric space, where* $X=\{1,2,3\}$, $d(1,2)=d(1,3)=1,\ \ d(2,3)=2$. *Let* $T:X\to X$ *be given by* $T1=1,T2=3$ *and* $T3=1$. *Since X is a finite set then every Cauchy sequence in* (X,d) *is eventually constant and so is convergent. Then* (X,d) *is a complete metric space. Moreover, since* (X,d) *is a discrete topological space then* $T:X\to X$ *is a continuous mapping. We first show that T does not satisfy the contractive conditions of Theorems 1.1 and 1.2. On the contrary, assume that there exist* $\alpha,\beta\ge 0$ *with* $\alpha+\beta<1$ *such that (1.1) and (1.2) are satisfied for each* $x,y\in X$. *Let* $x=1$ *and* $y=2$. *Then*

$$d(Tx,Ty)=1,\ \ d(x,Tx)=0,\ \ d(y,Ty)=2,\ \ d(x,y)=1.$$

Then from (1.1) and (1.2), we have

$$d(Tx,Ty)=1\le \alpha\frac{d(x,Tx)d(y,Ty)}{d(x,y)}+\beta d(x,y)=\beta,$$

and

$$d(Tx,Ty)=1\le \alpha\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+\beta d(x,y)=\alpha+\beta,$$

which give $\alpha+\beta\ge 1$, *a contradiction.*
Now, it is easy to see that the inequality

$$d(Tx,Ty)\le \alpha\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+\beta M(x,y),$$

holds for each $x,y\in X$, *where* $\alpha=0$ *and* $\beta=\frac{1}{2}$. *Then by Corollary 2.3, T has a fixed point.*

Example 2.6. *Let* $X=[0,1]$ *be endowed with the Euclidean metric* $d(x,y)=|x-y|$ *for each* $x,y\in X$. *Let* $\mathcal{R}=X\times X$. *Let* $T:[0,1]\to[0,1]$ *be given by* $Tx=\frac{x}{2}$ *for* $x\in[0,1)$ *and* $T(1)=1$. *Now we show that the mapping T does not satisfy neither (1.1) nor (1.2). To show the claim, let* $x=0$ *and let* $y=1$. *Then (1.1) and (1.2) reduce to* $1\le\beta$, *which is impossible.*
Now let $\alpha(t)=\frac{1}{3}$ *and* $\beta(t)=\frac{1}{2}$ *for each* $t\ge 0$ *and let*

$$\lambda(u_1,u_2,u_3,u_4,u_5)=3u_5\ \text{ for each }\ (u_1,u_2,u_3,u_4,u_5)\in\mathbb{R}^5_+.$$

It is easy to see that

$$|Tx-Ty|\le\frac{1}{3}N(x,y)+\frac{1}{2}M(x,y),$$

for each $x,y\in X$. *Then all of the hypothesis of Theorem 2.1 is satisfied and so T has a fixed point (note that T has two fixed points).*

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