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Approximation Properties of Bivariate Szász Durrmeyer Operators via Dunkl Analogue

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Abstract. In the present article, we construct a new sequence of bivariate Szász-Durrmeyer operators based on Dunkl analogue. We investigate the order of approximation with the aid of modulus of continuity in terms of well known Peetre's K-functional, weighted approximation results, Voronovskaja type theorems and Lipschitz maximal functions. Further, we also discuss here the approximation properties of the operators in Bögel-spaces by use of mixed-modulus of continuity.

1. Introduction

In the late of nineteenth century 1885, Weierstrass, a German mathematician proposed a prominent and historical theorem termed as Weierstrass approximation theorem [1]. This theorem plays a vital role and motivated several mathematicians to work on approximation theory. But there was a major drawback of the proof of the theorem, that it was very tedious and lengthy. In the year of 1912, Bernstein [2] proposed the polynomials with the aid of binomial distribution, which give the simplest and easiest proof of Weierstrass approximation theorem as follows:

$$B_n(f;x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f\left(\frac{\nu}{n}\right),\ n \in \mathbb{N}, \tag{1}$$

where $p_{n,\nu}(x) = \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}$. He proved that $B_n(f;x) \rightrightarrows f$ for each $f \in C[0,1]$ where $\rightrightarrows$ stands for uniform convergence. Szász [3] generalized the operators defined by (1) on unbounded interval, i.e., on $[0,\infty]$ as

$$S_n(f;x) = e^{-nx} \sum_{\nu=0}^{\infty} \frac{(nx)^{\nu}}{\nu!} f\left(\frac{\nu}{n}\right),\ n \in \mathbb{N}. \tag{2}$$

Many generalizations were discussed for (2), by the mathematician Acar [see [22], [23]], Acar et al. [see [24], [25]], Mohiuddin et al. ([15], [16]) to achieve the convergence properties by these sequences on positive

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semi axis. Operators (1) and (2) are restricted for continuous functions only. An integral modification of Bernstein operators (1) on an interval [0,1] was proposed by Durrmeyer [4] to study the approximation properties for Lebsgue integrable functions given by

$$D_n(f;x) = n\sum_{\nu=0}^{n} p_{n,\nu}(x)\int_0^1 p_{n,\nu}(t)f(t)dt. \tag{3}$$

Lately, several mathematician proposed Szász-type operators via various types of generating functions, i.e. Szász types operators via Charlier polynomials [6], Szász types operators via Sheffer polynomials [7]. Many generalizations have been studied in this directions [see [5], [11]-[33]]. The Szász types operators on the basis of Dunkl analogue was proposed by Sucu [8]. Using the generalization of exponential function is given by [9] as

$$e_\mu(x) = \sum_{\nu=0}^{\infty}\frac{x^\nu}{\lambda_\nu(\nu)} \tag{4}$$

where the coefficients $\gamma_\mu(\nu)$ are defined as follows:
For $\nu \in \mathbb{N}_0 = \{0\}\bigcup\mathbb{N}$ and $\mu > -1/2$

$$\gamma_\mu(2\nu) = \frac{2^{2\nu}\nu!\Gamma(\nu+\mu+1/2)}{\Gamma(\mu+1/2)},\ \gamma_\mu(2\nu+1) = \frac{2^{2\nu+1}\nu!\Gamma(\nu+\mu+3/2)}{\Gamma(\mu+1/2)}$$

and the recursive relation for γ_μ is defined as

$$\gamma_\mu(\nu+1) = (\nu+1+2\mu\theta_{\nu+1})\gamma_\mu(\nu),\ \nu\in\mathbb{N}_0, \tag{5}$$

with θ_ν is defined to be 0 if $\nu \in 2\mathbb{N}$ and 1 if $\nu \in 2\mathbb{N}+1$. For $f \in C[0,\infty)$, Sucu [8] defined Szász type operators using (4) given by

$$S_n^*(f;x) = \frac{1}{e_\mu(nx)}\sum_{\nu=0}^{\infty}\frac{(nx)^\nu}{\gamma_\mu(\nu)}f\left(\frac{\nu+2\mu\theta_\nu}{n}\right), \tag{6}$$

where $\mu, x \geq 0$ and $n \in \mathbb{N}$. One can notice that for $\mu = 0$, the operators (6) reduce to the operators (2). Wafi et al. [10] introduced Durrmeyer type modification of the operators defined by (6) to approximate the Lebesgue integrable function as follows:
For every $f \in C_\beta[0,\infty) = \{f \in C[0,\infty) : f(t) = O(t^\beta),\text{ as } t \to \infty\}$, we define

$$T_n^*(f;x) = \frac{1}{e_\mu(x)}\sum_{\nu=0}^{\infty}\frac{(nx)^\nu}{\gamma_\mu(\nu)}\frac{1}{B(\nu+2\mu\theta_\nu+1,n)}\int\limits_0^\infty \frac{t^{\nu+2\mu\theta_\nu}}{(1+t)^{\nu+2\mu\theta_\nu+n+1}}f(t)dt, \tag{7}$$

where $\beta > n$ and $B(\nu+2\mu\theta_\nu+1,n)$ is known as beta function which is defined as

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ m,\ n>0. \tag{8}$$

The motive of this article is to introduce bivariate Szász-Durremeyer type operators via Dunkl analogue. In order to get approximation results for these bivariate sequences, we yield results using modulus and mixed-modulus of continuity in Voronovskaja type theorem , K-functional and Lipschitz maximal functions, global approximation results. In addition to this, we also study the application of the GBS type operators with the help of mixed-modulus of continuity and yield results for Bögel continuous functions.

2. Construction of bivariate Szász-Durrmeyer-Operators $H^*_{n_1,n_2}(.;.)$ and their Basic Estimates

let $\mathcal{I}^2 = \{(y_1, y_2) : 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1\}$ and $C\left(\mathcal{I}^2\right)$ is the class of all continuous functions on $\mathcal{I}^2$ equipped with the norm $\|g\|_{C(\mathcal{I}^2)} = \sup_{(y_1,y_2)\in\mathcal{I}^2} |g(y_1, y_2)|$. Then for all $h \in C\left(\mathcal{I}^2\right)$ and $n_1, n_2 \in \mathbb{N}$, we construct a new sequences of bivariate Szász-Durrmeyer type operators via Dunkl analogue as follow :

$$H^*_{n_1,n_2}(h; y_1, y_2) = \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty} Q^*_1(n_1, y_1)Q^*_2(n_2, y_2)\int_0^{\infty}\int_0^{\infty} P^*_1(n_1, y_1)P^*_2(n_2, y_2)f(t_1, t_2)dt_1dt_2 \tag{9}$$

where $\mathcal{Q}^*_i(n_i, y_i) = \frac{1}{e_\mu(n_i,y_i)}\frac{(n_iy_i)^{\mu_i}}{\lambda_\mu(\vartheta_i)}\frac{1}{B(\nu_i+2\theta_{\nu_i}+1,n_i)}$ and $\mathcal{P}^*_i(n_i, y_i) = \frac{(t_i)^{\nu+2\vartheta_i\theta_{\nu_i}}}{(1+t_i)^{\vartheta_i+2\vartheta\theta_{\nu_i}+n_i+1}}$ $\quad i = 1, 2$

Lemma 2.1. *[10] Let $e_i(t) = t^i$, $i = 0, 1, 2$ be the test functions. Then, for the operators T^*_n given by* (5), *we have*

$$\begin{aligned} T^*_n(e_0; x) &= 1,\\ T^*_n(e_1; x) &= \frac{nx + 1}{n-1},\ n > 1,\\ T^*_n(e_2; x) &= \frac{1}{(n-2)(n-1)}\left\{n^2x^2 + \left(4 + 2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)nx + 2\right\},\ n > 2. \end{aligned}$$

Lemma 2.2. *[10] Let $\psi^i_x(t) = (t - x)^i$, $i = 0, 1, 2$ be the central moments. Then, for the operators defined by* (5), *we have*

$$\begin{aligned} T^*_n(\psi^0_x; x) &= 1,\\ T^*_n(\psi^1_x; x) &= \frac{x + 1}{n-1},\ n > 1,\\ T^*_n(\psi^2_x; x) &= \frac{1}{(n-2)(n-1)}\left\{(n + 2)x^2 + \left(1 + \mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)2nx + 2\right\},\ n > 2. \end{aligned}$$

Lemma 2.3. *Let $e_{i,j} = {y_1}^i{y_2}^j$. Then, for the operator $H^*_{n_1,n_2}(.;.)$, we have*

$$\begin{aligned} H^*_{n_1,n_2}(e_{0,0}; y_1, y_2) &= 1,\\ H^*_{n_1,n_2}(e_{1,0}; y_1, y_2) &= \frac{n_1y_1}{n_1-1},\quad n_1 > 1\\ H^*_{n_1,n_2}(e_{0,1}; y_1, y_2) &= \frac{n_2y_2}{n_2-1},\quad n_2 > 1\\ H^*_{n_1,n_2}(e_{1,1}; y_1, y_2) &= \frac{n_1y_1}{(n_1-1)}\frac{n_2y_2}{(n_2-2)},\quad n_1, n_2 > 1\\ H^*_{n_1,n_2}(e_{2,0}; y_1, y_2) &= \frac{1}{(n_1-2)(n_1-1)}\left\{n_1^2y_1^2 + \left(4 + 2\mu\frac{e_\mu(-n_1y_1)}{e_\mu(n_1y_1)}\right)n_1y_1 + 2\right\}, n_1 > 2,\\ H^*_{n_1,n_2}(e_{0,2}; y_1, y_2) &= \frac{1}{(n_2-2)(n_2-1)}\left\{n_2^2y_2^2 + \left(4 + 2\mu\frac{e_\mu(-n_2y_2)}{e_\mu(n_2y_2)}\right)n_2y_2 + 2\right\}, n_2 > 2. \end{aligned}$$

Proof. In the light of lemma (2.1) and linearly property, we have

$$\begin{aligned} H^*_{n_1,n_2}(e_{0,0}; y_1, y_2) &= H^*_{n_1,n_2}(e_0; y_1, y_2)H^*_{n_1,n_2}(e_0; y_1, y_2),\\ H^*_{n_1,n_2}(e_{1,0}; y_1, y_2) &= H^*_{n_1,n_2}(e_1; y_1, y_2)H^*_{n_1,n_2}(e_0; y_1, y_2),\\ H^*_{n_1,n_2}(e_{0,1}; y_1, y_2) &= H^*_{n_1,n_2}(e_0; y_1, y_2)H^*_{n_1,n_2}(e_1; y_1, y_2),\\ H^*_{n_1,n_2}(e_{1,1}; y_1, y_2) &= H^*_{n_1,n_2}(e_1; y_1, y_2)H^*_{n_1,n_2}(e_1; y_1, y_2),\\ H^*_{n_1,n_2}(e_{2,0}; y_1, y_2) &= H^*_{n_1,n_2}(e_2; y_1, y_2)H^*_{n_1,n_2}(e_0; y_1, y_2),\\ H^*_{n_1,n_2}(e_{0,2}; y_1, y_2) &= H^*_{n_1,n_2}(e_0; y_1, y_2)H^*_{n_1,n_2}(e_2; y_1, y_2), \end{aligned}$$

which proves Lemma (2.3). □

Lemma 2.4. *Let* $\Psi_{i,j}^{y_1,y_2}(t,s) = \eta_{i,j}(t,s) = (t-y_1)^i(s-y_2)^j, \quad i,j \in \{0,1,2\}$ *be the central moments. Then from the operators* $H^*_{n_1,n_2}(.;.)$ *defined by* (9) *satisfies the following identities*

$$\begin{aligned}
H^*_{n_1,n_2}(\eta_{0,0};y_1,y_2) &= 1\\
H^*_{n_1,n_2}(\eta_{1,0};y_1,y_2) &= \frac{y_1+1}{n_1-1},\ n_1>1,\\
H^*_{n_1,n_2}(\eta_{0,1};y_1,y_2) &= \frac{y_2+1}{n_2-1},\ n_2>1,\\
H^*_{n_1,n_2}(\eta_{1,1};y_1,y_2) &= \frac{y_1+1}{n_1-1}\frac{y_2+1}{n_2-1},\ n_1,n_2>1,\\
H^*_{n_1,n_2}(\eta_{2,0};y_1,y_2) &= \frac{1}{(n_1-2)(n_1-1)}\left\{(n_1+2)y_1^2+\left(1+\mu\frac{e_\mu(-n_1y_1)}{e_\mu(n_1y_1)}\right)2n_1y_1+2\right\},\ n_1>2.\\
H^*_{n_1,n_2}(\eta_{0,2};y_1,y_2) &= \frac{1}{(n_2-2)(n_2-1)}\left\{(n_2+2)y_2^2+\left(1+\mu\frac{e_\mu(-n_2y_2)}{e_\mu(n_2y_2)}\right)2n_2y_2+2\right\},\ n_2>2.
\end{aligned}$$

Proof. Using Lemma (2.2) and linearly property, we have

$$\begin{aligned}
H^*_{n_1,n_2}(\eta_{0,0};y_1,y_2) &= H^*_{n_1,n_2}(\eta_0;y_1,y_2)H^*_{n_1,n_2}(\eta_0;y_1,y_2),\\
H^*_{n_1,n_2}(\eta_{1,0};y_1,y_2) &= H^*_{n_1,n_2}(\eta_1;y_1,y_2)H^*_{n_1,n_2}(\eta_0;y_1,y_2),\\
H^*_{n_1,n_2}(\eta_{0,1};y_1,y_2) &= H^*_{n_1,n_2}(\eta_0;y_1,y_2)H^*_{n_1,n_2}(\eta_1;y_1,y_2),\\
H^*_{n_1,n_2}(\eta_{1,1};y_1,y_2) &= H^*_{n_1,n_2}(\eta_1;y_1,y_2)H^*_{n_1,n_2}(\eta_1;y_1,y_2),\\
H^*_{n_1,n_2}(\eta_{2,0};y_1,y_2) &= H^*_{n_1,n_2}(\eta_2;y_1,y_2)H^*_{n_1,n_2}(\eta_0;y_1,y_2),\\
H^*_{n_1,n_2}(\eta_{0,2};y_1,y_2) &= H^*_{n_1,n_2}(\eta_0;y_1,y_2)H^*_{n_1,n_2}(\eta_2;y_1,y_2),
\end{aligned}$$

which proves Lemma (2.4). □

Lemma 2.5. *For all* $y_1,y_2 \in \mathcal{I}^2$ *and sufficiently large* $n_1,n_2 \in \mathbb{N}$ *the operators* $H^*_{n_1,n_2}(.\,;\,.)$ *satisfy following*

$$\begin{aligned}
&(1)\quad H^*_{n_1,n_2}(\Psi^{2,0}_{y_1,y_2};y_1,y_2) = o\left(\frac{1}{n_1}\right)(y_1+1)^2 \le C_1(y_1+1)^2 \text{ as } n_1,n_2\to\infty;\\
&(2)\quad H^*_{n_1,n_2}(\Psi^{0,2}_{y_1,y_2};y_1,y_2) = o\left(\frac{1}{n_2}\right)(y_2+1)^2 \le C_2(y_2+1)^2 \text{ as } n_1,n_2\to\infty;\\
&(3)\quad H^*_{n_1,n_2}(\Psi^{4,0}_{y_1,y_2};y_1,y_2) = o\left(\frac{1}{n_1^2}\right)(y_1+1)^4 \le C_3(y_1+1)^4 \text{ as } n_1,n_2\to\infty;\\
&(4)\quad H^*_{n_1,n_2}(\Psi^{0,4}_{y_1,y_2};y_1,y_2) = o\left(\frac{1}{n_2^2}\right)(y_2+1)^4 \le C_4(y_2+1)^4 \text{ as } n_1,n_2\to\infty.
\end{aligned}$$

3. Some approximation results in weighted space and their degree of convergence

Let φ be weight function such that $\varphi(y_1,y_2) = 1+y_1^2+y_2^2$ and satisfying $B_\varphi\left(\mathcal{I}^2\right) = \{g :\mid g(y_1,y_2)\mid\le C_g\varphi(y_1,y_2), \quad C_g>0\}$, where $B_\varphi\left(\mathcal{I}^2\right)$ is the set of all bounded function on $\mathcal{I}^2 = [0,1]\times[0,1]$. Suppose $C^{(m)}\left(\mathcal{I}^2\right)$ be the m-times continuously differentiable functions defined on $\mathcal{I}^2 = \{(y_1,y_2)\in\mathcal{I}^2 : y_1,y_2\in[0,1]\}$. The equipped norm on B_φ defined by $\parallel g \parallel_\varphi = \sup_{y_1,y_2\in\mathcal{I}^2}\frac{|g(y_1,y_2)|}{\varphi(y_1,y_2)}$. Moreover we have classified here some classes of function as follows:

$$C^m_\varphi\left(\mathcal{I}^2\right) = \{g : g\in C_\varphi\left(\mathcal{I}^2\right); \quad \text{such that } \lim_{(y_1,y_2)\to\infty}\frac{g(y_1,y_2)}{\varphi(y_1,y_2)} = k_g < \infty\};$$

$$C^0_\varphi\left(\mathcal{I}^2\right) = \{f : f \in C^m_\varphi\left(\mathcal{I}^2\right); \quad \text{such that } \lim_{(y_1,y_2)\to\infty} \frac{g(y_1,y_2)}{\varphi(y_1,y_2)} = 0\}.$$

$$C_\varphi\left(\mathcal{I}^2\right) = \{g : g \in B_\varphi \cap C_\varphi\left(\mathcal{I}^2\right)\}.$$

Suppose $\omega_\varphi(g;\delta_1,\delta_2)$ is the weighted modulus of continuity for all $g \in C^0_\varphi\left(\mathcal{I}^2\right)$ and $\delta_1,\delta_2 > 0$, defined by

$$\omega_\varphi(g;\delta_1,\delta_2) = \sup_{(y_1,y_2)\in[0,1]}\ \sup_{0\leq|\theta_1|\leq\delta_1,\ 0\leq|\theta_2|\leq\delta_2} \frac{\mid g(y_1+\theta_1,y_2+\theta_2) - g(y_1,y_2)\mid}{\varphi(y_1,y_2)\varphi(\theta_1,\theta_2)}. \tag{10}$$

For any $\eta_1,\eta_2 > 0$ one has

$$\omega_\varphi(g;\eta_1\delta_1,\eta_2\delta_2) \leq 4(1+\eta_1)(1+\eta_2)(1+\delta_1^2)(1+\delta_2^2)\omega_\varphi(g;\delta_1,\delta_2),$$

$$\begin{aligned}\mid g(t,s) - g(y_1,y_2)\mid &\leq \varphi(y_1,y_2)\varphi\left(\mid t-y_1\mid,\mid s-y_2\mid\right)\omega_\varphi\left(g;\mid t-y_1\mid,\mid s-y_2\mid\right)\\ &\leq (1+y_1^2+y_2^2)(1+(t-y_1)^2)(1+(s-y_2)^2)\omega_\varphi\left(g;\mid t-y_1\mid,\mid s-y_2\mid\right).\end{aligned}$$

Theorem 3.1. *Let $g \in C^0_\varphi\left(\mathcal{I}^2\right)$, then for sufficiently large $n_1,n_2 \in \mathbb{N}$ operator $H^*_{n_1,n_2}$ satisfying the inequality*

$$\frac{\mid H^*_{n_1,n_2}(g;y_1,y_2) - g(y_1,y_2)\mid}{(1+y_1^2+y_2^2)} \leq \Psi_{y_1,y_2}\left(1+o\left(n_1^{-1}\right)\right)\left(1+o\left(n_2^{-1}\right)\right)\omega_\varphi\left(g;o\left(n_1^{-\frac{1}{2}}\right),\left(n_2^{-\frac{1}{2}}\right)\right),$$

where $\Psi_{y_1,y_2} = \left(1+(y_1+1)+C_1(y_1+1)^2+\sqrt{C_3}(y_1+1)^3\right)\left(1+(y_2+1)+C_2(y_2+1)^2+\sqrt{C_4}(y_2+1)^3\right)$ *and* $C_1,\ C_2,\ C_3,\ C_4 > 0.$

Proof. For all $\delta_{n_1},\delta_{n_2} > 0$ we have

$$\begin{aligned}\mid g(t,s) - g(y_1,y_2)\mid &\leq 4(1+y_1^2+y_2^2)\left(1+(t-y_1)^2\right)\left(1+(s-y_2)^2\right)\\ &\times \left(1+\frac{\mid t-y_1\mid}{\delta_{n_1}}\right)\left(1+\frac{\mid s-y_2\mid}{\delta_{n_2}}\right)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi\left(g;\delta_{n_1},\delta_{n_2}\right)\\ &= 4(1+y_1^2+y_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\\ &\times \left(1+\frac{\mid t-y_1\mid}{\delta_{n_1}}+(t-y_1)^2+\frac{1}{\delta_{n_1}}\mid t-y_1\mid(t-y_1)^2\right)\\ &\times \left(1+\frac{\mid s-y_2\mid}{\delta_{n_2}}+(s-y_2)^2+\frac{\mid s-y_2\mid}{\delta_{n_2}}(s-y_2)^2\right)\omega_\varphi\left(g;\delta_{n_1},\delta_{n_2}\right).\end{aligned}$$

There for apply operator $H^*_{n_1,n_2}$ and then use Cauchy-Schwarz inequality,

$$
\begin{aligned}
\mid H^*_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid &\leq H^*_{n_1,n_2}(\mid g(.,.)-g(y_1,y_2)\mid;y_1,y_2)\,4(1+y_1^2+y_2^2)\\
&\times H^*_{n_1,n_2}\left(1+\frac{\mid t-y_1\mid}{\delta_{n_1}}+(t-y_1)^2+\frac{1}{\delta_{n_1}}\mid t-y_1\mid(t-y_1)^2;y_1,y_2\right)\\
&\times H^*_{n_1,n_2}\left(1+\frac{\mid s-y_2\mid}{\delta_{n_2}}+(s-y_2)^2+\frac{\mid s-y_2\mid}{\delta_{n_2}}(s-y_2)^2;y_1,y_2\right)\\
&\times (1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi(g;\delta_{n_1},\delta_{n_2})\\
&= 4(1+y_1^2+y_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi(g;\delta_{n_1},\delta_{n_2})\\
&\times \left(1+\frac{1}{\delta_{n_1}}H^*_{n_1,n_2}(\mid t-y_1\mid;y_1,y_2)+H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)\right.\\
&+ \left.\frac{1}{\delta_{n_1}}H^*_{n_1,n_2}(\mid t-y_1\mid(t-y_1)^2;y_1,y_2\right)\\
&\times \left(1+\frac{1}{\delta_{n_2}}H^*_{n_1,n_2}(\mid s-y_2\mid;y_1,y_2)+H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)\right.\\
&+ \left.\frac{1}{\delta_{n_2}}H^*_{n_1,n_2}(\mid s-y_2\mid(s-y_2)^2;y_1,y_2\right);
\end{aligned}
$$

$$
\begin{aligned}
\mid H^*_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid &\leq 4(1+y_1^2+y_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi(g;\delta_{n_1},\delta_{n_2})\\
&\times \left[1+\frac{1}{\delta_{n_1}}\sqrt{H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)}+H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)\right.\\
&+ \left.\frac{1}{\delta_{n_1}}\sqrt{H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)}\sqrt{H^*_{n_1,n_2}((t-y_1)^4;y_1,y_2)}\right]\\
&\times \left[1+\frac{1}{\delta_{n_2}}\sqrt{H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)}+H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)\right.\\
&+ \left.\frac{1}{\delta_{n_2}}\sqrt{H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)}\sqrt{H^*_{n_1,n_2}((s-y_2)^4;y_1,y_2)}\right].
\end{aligned}
$$

In view of Lemma (2.5) and choose $\delta_{n_1}=o\left(n_1^{-\frac{1}{2}}\right)$ and $\delta_{n_2}=o\left(n_2^{-\frac{1}{2}}\right)$, then

$$
\begin{aligned}
\mid H^*_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid &\leq 4(1+y_1^2+y_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi(g;\delta_{n_1},\delta_{n_2})\\
&\times \left[1+\frac{1}{\delta_{n_1}}\sqrt{o\left(\frac{1}{n_1}\right)(y_1+1)^2}+o\left(\frac{1}{n_1}\right)(y_1+1)^2\right.\\
&+ \left.\frac{1}{\delta_{n_1}}\sqrt{o\left(\frac{1}{n_1}\right)(y_1+1)^2}\sqrt{o\left(\frac{1}{n_1}\right)(y_1+1)^4}\right]\\
&\times \left[1+\frac{1}{\delta_{n_2}}\sqrt{o\left(\frac{1}{n_2}\right)(y_2+1)^2}+o\left(\frac{1}{n_2}\right)(y_2+1)^2\right.\\
&+ \left.\frac{1}{\delta_{n_2}}\sqrt{o\left(\frac{1}{n_2}\right)(y_2+1)^2}\sqrt{o\left(\frac{1}{n_2}\right)(y_2+1)^4}\right]\\
&\leq 4(1+y_1^2+y_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi(g;\delta_{n_1},\delta_{n_2})\\
&\times \left[1+(y_1+1)+C_1(y_1+1)^2+\sqrt{C_2}(y_1+1)^3\right]\left[1+(y_2+1)\right.\\
&+ \left.C_3(y_2+1)^2+\sqrt{C_4}(y_2+1)^3\right].
\end{aligned}
$$

Which completes the proof. □

Lemma 3.2 ([43, 44]). *For the positive sequence of operators $\{L_{n_1,n_2}\}_{n_1,n_2\geq 1}$, which acting $C_\varphi \to B_\varphi$ defined earlier then there exists some positive K such that*

$$\| L_{n_1,n_2}(\varphi) \|_\varphi \leq K.$$

Theorem 3.3 ([43, 44]). *or the positive sequence of operators $\{L_{n_1,n_2}\}_{n_1,n_2\geq 1}$ acting $C_\varphi \to B_\varphi$ defined earlier satisfying the following conditions*

$$\begin{array}{rrcl} (1) & \lim\limits_{n_1,n_2\to\infty} \| L_{n_1,n_2}(1) - 1 \|_\varphi & = & 0; \\ (2) & \lim\limits_{n_1,n_2\to\infty} \| L_{n_1,n_2}(t) - y_1 \|_\varphi & = & 0; \\ (3) & \lim\limits_{n_1,n_2\to\infty} \| L_{n_1,n_2}(s) - y_2 \|_\varphi & = & 0; \\ (4) & \lim\limits_{n_1,n_2\to\infty} \| L_{n_1,n_2}((t^2+s^2)) - (y_1^2+y_2^2) \|_\varphi & = & 0. \end{array}$$

Then for all $g \in C_\varphi^0$,

$$\lim_{n_1,n_2\to\infty} \| L_{n_1,n_2} g - g \|_\varphi = 0$$

and there exists another function $f \in C_\varphi \setminus C_\varphi^0$, such that

$$\lim_{n_1,n_2\to\infty} \| L_{n_1,n_2} f - f \|_\varphi \geq 1.$$

Theorem 3.4. *If $g \in C_\varphi^0\left(\mathcal{I}^2\right)$, then we have*

$$\lim_{n_1,n_2\to\infty} \| H^*_{n_1,n_2}(g) - g \|_\varphi = 0.$$

Proof.

$$\begin{aligned} \| H^*_{n_1,n_2}(\varphi; y_1, y_2) \|_\varphi &= \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{| H^*_{n_1,n_2}(1+y_1^2+y_2^2; y_1, y_2) |}{1+y_1^2+y_2^2} \\ &= 1 + \sup_{(y_1,y_2)\in\mathcal{I}^2} \left[\frac{1}{1+y_1^2+y_2^2} \left| \left(1 + H^*_{n_1,n_2}(y_1^2; y_1, y_2) + H^*_{n_1,n_2}(y_2^2; y_1, y_2) \right) \right| \right] \end{aligned}$$

$$\begin{aligned} \| H^*_{n_1,n_2}(\varphi; y_1, y_2) \|_\varphi &= 1 + \left| \frac{n_1^2}{(n_1-2)(n_1-1)} \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{y_1^2}{1+y_1^2+y_2^2} \\ &+ \left| \left(4 + 2\mu \frac{e_\mu(-n_1y_1)}{e_\mu(n_1y_1)} \right) \frac{n_1}{(n_1-2)(n_1-1)} \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{y_1}{1+y_1^2+y_2^2} \\ &+ \left| \frac{2}{(n_1-2)(n_1-1)} \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{1}{1+y_1^2+y_2^2} \\ &+ \left| \frac{n_2^2}{(n_2-2)(n_2-1)} \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{y_2^2}{1+y_1^2+y_2^2} \\ &+ \left| \left(4 + 2\mu \frac{e_\mu(-n_2y_2)}{e_\mu(n_2y_2)} \right) \frac{n_2}{(n_2-2)(n_2-1)} \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{y_2}{1+y_1^2+y_2^2} \\ &+ \left| \frac{2}{(n_2-2)(n_2-1)} \right| \sup_{(y_1,y_2)\in\mathcal{I}^2} \frac{1}{1+y_1^2+y_2^2} \end{aligned}$$

$$
\begin{aligned}
\parallel H^*_{n_1,n_2}(\varphi; y_1, y_2) \parallel_\varphi &\leq 1 + \left|\frac{n_1^2}{(n_1-2)(n_1-1)}\right| + \left|\left(4 + 2\mu\frac{e_\mu(-n_1y_1)}{e_\mu(n_1y_1)}\right)\frac{n_1}{(n_1-2)(n_1-1)}\right| \\
&+ \left|\frac{2}{(n_1-2)(n_1-1)}\right| + \left|\frac{n_2^2}{(n_2-2)(n_2-1)}\right| \\
&+ \left|\left(4 + 2\mu\frac{e_\mu(-n_2y_2)}{e_\mu(n_2y_2)}\right)\frac{n_2}{(n_2-2)(n_2-1)}\right| + \left|\frac{2}{(n_2-2)(n_2-1)}\right|.
\end{aligned}
$$

Now for all $n_1, n_2 \in \mathbb{N} \setminus \{1, 2\}$, there exists a positive constant K such that

$$\parallel H^*_{n_1,n_2}(\varphi; y_1, y_2) \parallel_\varphi \leq K.$$

Therefore, in order to prove Theorem 3.4 it is sufficient from the Lemma 3.2 and Theorem 3.3. Thus we arrive at the prove of Theorem 3.4. □

For any $g \in C(\mathcal{I}^2)$ and $\delta > 0$ modulus of continuity of order second is given by

$$\omega\,(g; \delta_{n_1}, \delta_{n_2}) = \sup\{\mid g(t,s) - g(y_1, y_2) \mid : (t,s), (y_1, y_2) \in \mathcal{I}^2\}$$

with $\mid t - y_1 \mid \leq \delta_{n_1}$, $\mid s - y_2 \mid \leq \delta_{n_2}$ with the partial modulus of continuity defined as:

$$\omega_1\,(g; \delta) = \sup_{0 \leq y_2 \leq 1}\ \sup_{|x_1 - x_2| \leq \delta} \{\mid g(x_1, y_2) - g(x_2, y_2) \mid\},$$

$$\omega_2\,(g; \delta) = \sup_{0 \leq y_1 \leq 1}\ \sup_{|y_1 - y_2| \leq \delta} \{\mid g(y_1, y_1) - g(y_1, y_2) \mid\}.$$

Theorem 3.5. *For any $g \in C(\mathcal{I}^2)$ we have*

$$\mid H^*_{n_1,n_2}(g; y_1, y_2) - g(y_1, y_2) \mid \leq 2\Big(\omega_1(g; \delta_{y_1,n_1}) + \omega_2(g; \delta_{n_2,y_2})\Big).$$

Proof. In order to give the prove of Theorem 3.5, in general we use well-known Cauchy-Schwarz inequality. Thus we see that

$$
\begin{aligned}
\mid H^*_{n_1,n_2}(g; y_1, y_2) - g(y_1, y_2) \mid &\leq H^*_{n_1,n_2}\,(\mid g(t,s) - g(y_1, y_2) \mid; y_1, y_2) \\
&\leq H^*_{n_1,n_2}\,(\mid g(t,s) - g(y_1, s) \mid; y_1, y_2) \\
&+ H^*_{n_1,n_2}\,(\mid g(y_1, s) - g(y_1, y_2) \mid; y_1, y_2) \\
&\leq H^*_{n_1,n_2}\,(\omega_1(g; \mid t - y_1 \mid); y_1, y_2) + H^*_{n_1,n_2}\,(\omega_2(g; \mid s - y_2 \mid); y_1, y_2) \\
&\leq \omega_1(g; \delta_{n_1})\left(1 + \delta_{n_1}^{-1} H^*_{n_1,n_2}(\mid t - y_1 \mid; y_1, y_2)\right) \\
&+ \omega_2(g; \delta_{n_2})\left(1 + \delta_{n_2}^{-1} H^*_{n_1,n_2}(\mid s - y_2 \mid; y_1, y_2)\right) \\
&\leq \omega_1(g; \delta_{n_1})\left(1 + \frac{1}{\delta_{n_1}}\sqrt{H^*_{n_1,n_2}((t - y_1)^2; y_1, y_2)}\right) \\
&+ \omega_2(g; \delta_{n_2})\left(1 + \frac{1}{\delta_{n_2}}\sqrt{H^*_{n_1,n_2}((s - y_2)^2; y_1, y_2)}\right).
\end{aligned}
$$

If we choose $\delta_{n_1}^2 = \delta_{n_1,y_1}^2 = H^*_{n_1,n_2}((t - y_1)^2; y_1, y_2)$ and $\delta_{n_2}^2 = \delta_{n_2,y_2}^2 = H^*_{n_1,n_2}((s - y_2)^2; y_1, y_2)$, then we easily to reach our desired results. □

Here we find convergence in terms of the Lipschitz class for bivariate function. For $M > 0$ and $\rho_1, \rho_2 \in [0,1]$, Lipschitz maximal function space on $E \times E \subset \mathcal{I}^2$ defined by

$$\begin{aligned}\mathcal{L}_{\rho_1,\rho_2}(E\times E) &= \Big\{g : \sup(1+t)^{\rho_1}(1+s)^{\rho_2}\left(g_{\rho_1,\rho_2}(t,s) - g_{\rho_1,\rho_2}(y_1,y_2)\right)\\ &\leq M\frac{1}{(1+y_1)^{\rho_1}}\frac{1}{(1+y_2)^{\rho_2}}\Big\},\end{aligned}$$

where g is continuous and bounded on $\mathcal{I}^2$, and

$$g_{\rho_1,\rho_2}(t,s) - g_{\rho_1,\rho_2}(y_1,y_2) = \frac{\mid g(t,s) - g(y_1,y_2)\mid}{\mid t - y_1\mid^{\rho_1}\mid s - y_2\mid^{\rho_2}}; \quad (t,s),(y_1,y_2) \in \mathcal{I}^2. \tag{11}$$

Theorem 3.6. *Let $g \in \mathcal{L}_{\rho_1,\rho_2}(E\times E)$, then for any $\rho_1, \rho_2 \in [0,1]$, there exists $M > 0$ such that*

$$\begin{aligned}\mid H^*_{n_1,n_2}(g; y_1, y_2) - g(y_1,y_2)\mid &\leq M\Big\{\Big((d(y_1,E))^{\rho_1} + \left(\delta^2_{n_1,y_1}\right)^{\frac{\rho_1}{2}}\Big)\Big((d(y_2,E))^{\rho_2} + \left(\delta^2_{n_2,y_2}\right)^{\frac{\rho_2}{2}}\Big)\\ &+ (d(y_1,E))^{\rho_1}(d(y_2,E))^{\rho_2}\Big\},\end{aligned}$$

where δ_{n_1,y_1} and δ_{n_2,y_2} defined by Theorem 3.5.

Proof. Take $\mid y_1 - x_0 \mid = d(y_1,E)$ and $\mid y_2 - y_0 \mid = d(y_2,E)$. For any $(y_1,y_2) \in \mathcal{I}^2$, and $(x_0,y_0) \in E\times E$ we let $d(y_1,E) = \inf\{\mid y_1 - y_2\mid : y_2 \in E\}$. Thus we can write here

$$\mid g(t,s) - g(y_1,y_2)\mid \leq M\mid g(t,s) - g(x_0,y_0)\mid + \mid g(x_0,y_0) - g(y_1,y_2)\mid. \tag{12}$$

Apply $H^*_{n_1,n_2}$, we obtain

$$\begin{aligned}\mid H^*_{n_1,n_2}(g;y_1,y_2) - g(y_1,y_2)\mid &\leq H^*_{n_1,n_2}\left(\mid g(y_1,y_2) - g(x_0,y_0)\mid + \mid g(x_0,y_0) - g(y_1,y_2)\mid\right)\\ &\leq MH^*_{n_1,n_2}\left(\mid t - x_0\mid^{\rho_1}\mid s - y_0\mid^{\rho_2}; y_1, y_2\right)\\ &+ M\mid y_1 - x_0\mid^{\rho_1}\mid y_2 - y_0\mid^{\rho_2}.\end{aligned}$$

For all $A, B \geq 0$ and $\rho \in [0,1]$ we know inequality $(A+B)^\rho \leq A^\rho + B^\rho$, thus

$$\mid t - x_0\mid^{\rho_1} \leq \mid t - y_1\mid^{\rho_1} + \mid y_1 - x_0\mid^{\rho_1},$$

$$\mid s - y_0\mid^{\rho_1} \leq \mid s - y_2\mid^{\rho_2} + \mid y_2 - y_0\mid^{\rho_2}.$$

Therefore

$$\begin{aligned}\mid H^*_{n_1,n_2}(g;y_1,y_2) - g(y_1,y_2)\mid &\leq MH^*_{n_1,n_2}\left(\mid t - y_1\mid^{\rho_1}\mid s - y_2\mid^{\rho_2}; y_1,y_2\right)\\ &+ M\mid y_1 - x_0\mid^{\rho_1} H^*_{n_1,n_2}\left(\mid s - y_2\mid^{\rho_2}; y_1, y_2\right)\\ &+ M\mid y_2 - y_0\mid^{\rho_2} H^*_{n_1,n_2}\left(\mid t - y_1\mid^{\rho_1}; y_1,y_2\right)\\ &+ 2M\mid y_1 - x_0\mid^{\rho_1}\mid y_2 - y_0\mid^{\rho_2} H^*_{n_1,n_2}\left(\mu_{0,0}; y_1,y_2\right).\end{aligned}$$

On apply Hölder inequality on $H^*_{n_1,n_2}$, we get

$$\begin{aligned} H^*_{n_1,n_2}(\mid t-y_1\mid^{\rho_1}\mid s-y_2\mid^{\rho_2};y_1,y_2) &= \mathcal{U}^{\alpha_1}_{n_1,k}(\mid t-y_1\mid^{\rho_1};y_1,y_2)\mathcal{V}^{\alpha_2}_{n_2,l}(\mid s-y_2\mid^{\rho_2};y_1,y_2) \\ &\leq \left(H^*_{n_1,n_2}(\mid t-y_1\mid^2;y_1,y_2)\right)^{\frac{\rho_1}{2}}\left(H^*_{n_1,n_2}(\mu_{0,0};y_1,y_2)\right)^{\frac{2-\rho_1}{2}} \\ &\times \left(H^*_{n_1,n_2}(\mid s-y_2\mid^2;y_1,y_2)\right)^{\frac{\rho_2}{2}}\left(H^*_{n_1,n_2}(\mu_{0,0};y_1,y_2)\right)^{\frac{2-\rho_2}{2}}. \end{aligned}$$

Thus we can obtain

$$\begin{aligned} \mid H^*_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid &\leq M\left(\delta^2_{n_1,y_1}\right)^{\frac{\rho_1}{2}}\left(\delta^2_{n_2,y_2}\right)^{\frac{\rho_2}{2}}+2M\left(d(y_1,E)\right)^{\rho_1}\left(d(y_2,E)\right)^{\rho_2} \\ &+ M\left(d(y_1,E)\right)^{\rho_1}\left(\delta^2_{n_2,y_2}\right)^{\frac{\rho_2}{2}}+L\left(d(y_2,E)\right)^{\rho_2}\left(\delta^2_{n_1,y_1}\right)^{\frac{\rho_1}{2}}. \end{aligned}$$

We have complete the proof. □

Theorem 3.7. *If $g \in C'(\mathcal{I}^2)$, then for all $(y_1,y_2) \in \mathcal{I}^2$, operator $H^*_{n_1,n_2}$ satisfying*

$$\mid H^*_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid \leq \parallel g_{y_1}\parallel_{C(\mathcal{I}^2)}\left(\delta^2_{n_1,y_1}\right)^{\frac{1}{2}}+\parallel g_{y_2}\parallel_{C(\mathcal{I}^2)}\left(\delta^2_{n_2,y_2}\right)^{\frac{1}{2}},$$

where δ_{n_1,y_1} and δ_{n_2,y_2} are defined by Theorem 3.5.

Proof. Let $g \in C'(\mathcal{I}^2)$, and for any fixed $(y_1,y_2) \in \mathcal{I}^2$ we have

$$g(t,s)-g(y_1,y_2) = \int_{y_1}^{t} g_{\varrho}(\varrho,s)\mathrm{d}\varrho+\int_{y_2}^{s} g_{\mu}(y_1,\mu)\mathrm{d}\mu.$$

On apply $H^*_{n_1,n_2}$

$$H^*_{n_1,n_2}(g(t,s);y_1,y_2)-g(y_1,y_2) = H^*_{n_1,n_2}\left(\int_{y_1}^{t} g_{\varrho}(\varrho,s)\mathrm{d}\varrho;y_1,y_2\right)+H^*_{n_1,n_2}\left(\int_{y_2}^{s} g_{\mu}(y_1,\mu)\mathrm{d}\mu;y_1,y_2\right). \tag{13}$$

From the sup-norm on $\mathcal{I}^2$ we can see that

$$\left|\int_{y_1}^{t} g_{\varrho}(\varrho,s)\mathrm{d}\varrho\right| \leq \int_{y_1}^{t}\mid g_{\varrho}(\varrho,s)\mathrm{d}\varrho\mid \leq \parallel g_{y_1}\parallel_{C(\mathcal{I}^2)}\mid t-y_1\mid \tag{14}$$

$$\left|\int_{y_2}^{s} g_{\mu}(y_1,\mu)\mathrm{d}\mu\right| \leq \int_{y_2}^{s}\mid g_{\mu}(y_1,\mu)\mathrm{d}\mu\mid \leq \parallel g_{y_2}\parallel_{C(\mathcal{I}^2)}\mid s-y_2\mid. \tag{15}$$

In the view of (13), (14) and (15) we can obtain

$$\begin{aligned} \mid H^*_{n_1,n_2}(g(y_1,y_2);y_1,y_2)-g(y_1,y_2)\mid &\leq H^*_{n_1,n_2}\left(\left|\int_{y_1}^{t} g_{\varrho}(\varrho,s)\mathrm{d}\varrho\right|;y_1,y_2\right) \\ &+ H^*_{n_1,n_2}\left(\left|\int_{y_2}^{s} g_{\mu}(y_1,\mu)\mathrm{d}\mu\right|;y_1,y_2\right) \\ &\leq \parallel g_{y_1}\parallel_{C(\mathcal{I}^2)} H^*_{n_1,n_2}(\mid t-y_1\mid;y_1,y_2) \\ &+ \parallel g_{y_2}\parallel_{C(\mathcal{I}^2)} H^*_{n_1,n_2}(\mid s-y_2\mid;y_1,y_2) \\ &\leq \parallel g_{y_1}\parallel_{C(\mathcal{I}^2)}\left(H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)H^*_{n_1,n_2}(1;y_1,y_2)\right)^{\frac{1}{2}} \\ &+ \parallel g_{y_2}\parallel_{C(\mathcal{I}^2)}\left(H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)H^*_{n_1,n_2}(1;y_1,y_2)\right)^{\frac{1}{2}} \\ &= \parallel g_{y_1}\parallel_{C(\mathcal{I}^2)}\left(\delta^2_{n_1,y_1}\right)^{\frac{1}{2}}+\parallel g_{y_2}\parallel_{C(\mathcal{I}^2)}\left(\delta^2_{n_2,y_2}\right)^{\frac{1}{2}}. \end{aligned}$$

□

Theorem 3.8. *For any $f \in C(\mathcal{I}^2)$, if we define an auxiliary operator such that*

$$R^{\alpha_1,\alpha_2}_{n_1,n_2}(f;y_1,y_2) \quad = \quad H^*_{n_1,n_2}(g;y_1,y_2) + f(y_1,y_2) - f\Big(\mathcal{U}^{\alpha_1}_{n_1,k}(\mu_{1,0};y_1,y_2), \mathcal{V}^{\alpha_2}_{n_2,l}(\mu_{0,1};y_1,y_2)\Big).$$

where, from Lemma (2.4), $\mathcal{U}^{\alpha_1}_{n_1,k}(\mu_{1,0};y_1,y_2) = \frac{y_1+1}{n_1-1} \quad n_1 > 1$ *and*
$\mathcal{V}^{\alpha_2}_{n_2,l}(\mu_{0,1};y_1,y_2) = \frac{y_2+1}{n_2-1} \quad n_2 > 1.$

Then for all $g \in C'(\mathcal{I}^2)$, $R^{\alpha_1,\alpha_2}_{n_1,n_2}$ satisfying

$$\begin{aligned} R^{\alpha_1,\alpha_2}_{n_1,n_2}(g;t,s) - g(y_1,y_2) \quad &\leq \quad \left\{\delta^2_{n_1,y_1} + \delta^2_{n_2,y_2} + \left(\frac{y_1+1}{n_1-1} - y_1\right)^2\right. \\ &+ \quad \left.\left(\frac{y_2+1}{n_2-1} - y_2\right)^2\right\} \| g \|_{C^2(\mathcal{I}^2)} . \end{aligned}$$

Proof. In the light of operator $R^{\alpha_1,\alpha_2}_{n_1,n_2}(f;y_1,y_2)$ and Lemma 2.2, we obtain $R^{\alpha_1,\alpha_2}_{n_1,n_2}(1;y_1,y_2) = 1$, $R^{\alpha_1,\alpha_2}_{n_1,n_2}(t - y_1;y_1,y_2) = 0$ and $R^{\alpha_1,\alpha_2}_{n_1,n_2}(s-y_2;y_1,y_2) = 0$. For any $g \in C'(\mathcal{I}^2)$ the Taylor series give us:

$$\begin{aligned} g(t,s) - g(y_1,y_2) \quad &= \quad \frac{\partial g(y_1,y_2)}{\partial y_1}(t-y_1) + \int_{y_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda \\ &+ \quad \frac{\partial g(y_1,y_2)}{\partial y_2}(s-y_2) + \int_{y_2}^{s}(s-\psi)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi. \end{aligned}$$

On apply $R^{\alpha_1,\alpha_2}_{n_1,n_2}$, we see that

$$R^{\alpha_1,\alpha_2}_{n_1,n_2}\,(g(t,s);y_1,y_2) - R^{\alpha_1,\alpha_2}_{n_1,n_2}\,(g(y_1,y_2)$$

$$\begin{aligned} &= \quad R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_{y_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda;y_1,y_2\right) + R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_{y_2}^{s}(s-\psi)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi;y_1,y_2\right) \\ &= \quad H^*_{n_1,n_2}\left(\int_{y_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda;y_1,y_2\right) + H^*_{n_1,n_2}\left(\int_{y_2}^{s}(s-\psi)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi;y_1,y_2\right) \\ &- \quad \int_{y_1}^{\frac{y_1+1}{n_1-1}}\left(\frac{y_1+1}{n_1-1} - \lambda\right)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda \\ &- \quad \int_{y_2}^{\frac{y_2+1}{n_2-1}}\left(\frac{y_2+1}{n_2-1} - \psi\right)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi. \end{aligned}$$

From hypothesis we easily obtain

$$\left|\int_{y_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda\right| \leq \int_{y_1}^{t}\left|(t-\lambda)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda\right| \leq \| g \|_{C^2(\mathcal{I}^2)}\,(t-y_1)^2,$$

$$\left|\int_{y_2}^{s}(s-\psi)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi\right| \leq \int_{y_2}^{s}\left|(s-\psi)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi\right| \leq \| g \|_{C^2(\mathcal{I}^2)}\,(s-y_2)^2,$$

$$\begin{aligned}
&\Big|\int_{y_1}^{\frac{y_1+1}{n_1-1}}\left(\frac{y_1+1}{n_1-1}-\lambda\right)\frac{\partial^2 g(\lambda,y_2)}{\partial\lambda^2}\mathrm{d}\lambda\Big| \\
\leq\ & \parallel g\parallel_{C^2(\mathcal{I}^2)}\left(\frac{y_1+1}{n_1-1}-y_1\right)^2
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_{y_2}^{\frac{y_2+1}{n_2-1}}\left(\frac{y_2+1}{n_2-1}-\psi\right)\frac{\partial^2 g(y_1,\psi)}{\partial\psi^2}\mathrm{d}\psi\Big| \\
\leq\ & \parallel g\parallel_{C^2(\mathcal{I}^2)}\left(\frac{y_2+1}{n_2-1}-y_2\right)^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mid R_{n_1,n_2}^{\alpha_1,\alpha_2}(g;t,s)-g(y_1,y_2)\mid\ \leq\ & \Big\{H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)+H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2) \\
&+\left(\frac{y_1+1}{n_1-1}-y_1\right)^2 \\
&+\left(\frac{y_2+1}{(n_2-1)}-y_2\right)^2\Big\}\parallel g\parallel_{C^2(\mathcal{I}^2)}.
\end{aligned}$$

We complete the proof of desired Theorem 3.8.

□

4. Some approximation results in Bögel space

Take any function $g:\mathcal{I}_1\times\mathcal{I}_2\to\mathbb{R}$ for a real compact intervals of $\mathcal{I}_1\times\mathcal{I}_2$. For all $(t,s),\ (y_1,y_2)\in\mathcal{I}_1\times\mathcal{I}_2$ suppose $\Delta^*_{(t,s)}g(y_1,y_2)$ denotes the bivariate mixed difference operators defined as follows:

$$\Delta^*_{(t,s)}g(y_1,y_2)=g(t,s)-g(t,y_2)-g(y_1,s)+g(y_1,y_2).$$

If at any point $(y_1,y_2)\in\mathcal{I}_1\times\mathcal{I}_2$ the function $g:\mathcal{I}_1\times\mathcal{I}_2\to\mathbb{R}$ defined on $\mathcal{I}_1\times\mathcal{I}_2$, then $\lim_{(t,s)\to(y_1,y_2)}\Delta^*_{(t,s)}g(y_1,y_2)=0$.

If set of all the space of all Bögel-continuous(B-continuous) denoted by $C_B(\mathcal{I}_1\times\mathcal{I}_2)$ on $(y_1,y_2)\in\mathcal{I}_1\times\mathcal{I}_2$ and be defined such that $C_B(\mathcal{I}_1\times\mathcal{I}_2)=\{g,$ such that $g:\mathcal{I}_1\times\mathcal{I}_2\to\mathbb{R}$ is $g,\ B-$bounded on $\mathcal{I}_1\times\mathcal{I}_2\}$. Next, the Bögel-differentiable function on $(y_1,y_2)\in\mathcal{I}_1\times\mathcal{I}_2$ be $g:\mathcal{I}_1\times\mathcal{I}_2\to\mathbb{R}$ and limit exists finite defined by

$$\lim_{(t,s)\to(y_1,y_2),\ t\neq y_1,\ s\neq y_2}\frac{1}{(t-y_1)(s-y_2)}\left(\Delta^*_{(t,s)}\right)=D_Bg(y_1,y_2)<\infty.$$

Let the classes of all Bögel-differentiable function denoted by $D_\varphi g(y_1,y_2)$ and be $D_\varphi(\mathcal{I}_1\times\mathcal{I}_2)=\{g,$ such that $g:\mathcal{I}_1\times\mathcal{I}_2\to\mathbb{R}$ is $g,$ B$-$differentiable on $\mathcal{I}_1\times\mathcal{I}_2\}$. Suppose the function g is B-bounded on D and be $g:\mathcal{I}_1\times\mathcal{I}_2\to\mathbb{R}$, then for all $(t,s),\ (y_1,y_2)\in\mathcal{I}_1\times\mathcal{I}_2$ there exists positive constant M such that $\mid\Delta^*_{(t,s)}g(y_1,y_2)\mid\leq M$. The classes of all B-continuous function is called a B-bounded on $\mathcal{I}_1\times\mathcal{I}_2$, whene $\mathcal{I}_1\times\mathcal{I}_2$ is compact subset. Let $B_\varphi(\mathcal{I}_1\times\mathcal{I}_2)$ denote the classes of all B-bounded function defined on $\mathcal{I}_1\times\mathcal{I}_2$ which equipped the norm on B as $\parallel g\parallel_B=\sup_{(t,s),(y_1,y_2)\in\mathcal{I}_1\times\mathcal{I}_2}\mid\Delta^*_{(t,s)}g(y_1,y_2)\mid$. As we know to approximate the degree for a set of all B-continuous function on positive linear operators, it is essential to use the properties of mixed-modulus of continuity. So we let for all (t,s), $(y_1,y_2)\in\mathcal{I}_1\times\mathcal{I}_2$ and $g\in B_\varphi(\mathcal{I}_{\alpha_n})$, the mixed-modulus of continuity of function g bt $\omega_B:[0,\infty)\times[0,\infty)\to\mathbb{R}$ and be defined such as:

$$\omega_B(g;\delta_1,\delta_2)=\sup\{\Delta^*_{(t,s)}g(y_1,y_2):\mid t-y_1\mid\leq\delta_1,\ \mid s-y_2\mid\leq\delta_2\}.$$

For any $\mathcal{I}^2=[0,1]\times[0,1]$, we suppose the classes of all B-continuous function defined on $\mathcal{I}^2$ denoted by $C_\varphi(\mathcal{I}^2)$. Moreover, let set of all ordinary continuous function defined on $\mathcal{I}^2$ be $C(\mathcal{I}^2)$. For further details on space of Bögel functions related to this paper we propose the article [40, 41].

Let $(y_1,y_2)\in\mathcal{I}^2$ and $n_1,n_2\in\mathbb{N}$ then for all $g\in C(\mathcal{I}^2)$ we define the GBS type operators for the positive linear operators $H^*_{n_1,n_2}$. Thus we suppose

$$K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);y_1,y_2)=H^*_{n_1,n_2}\Big(g(t,y_2)+g(y_1,s)-g(t,s);y_1,y_2\Big). \tag{16}$$

More precisely, the generalized GBS operator for bivariate function is defined as follows:

$$K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);y_1,y_2)=\sum_{k,l=0}^{\infty}\mathcal{Q}^*_1(n_1,y_1)\mathcal{Q}^*_2(n_2,y_2)\int_0^\infty\int_0^\infty\mathcal{P}^*_1(n_1,y_1)P^*_2(n_2,y_2) \tag{17}$$

where $P_{y_1,y_2}(t,s)=\Big(g(t,y_2)+g(y_1,s)-g(t,s)\Big)$.

Theorem 4.1. *For all $g\in C_\varphi(\mathcal{I}^2)$, it follows that*

$$\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);y_1,y_2)-g(y_1,y_2)\mid\leq 4\omega_B\left(g;\delta_{n_1,y_1},\delta_{n_2,y_2}\right),$$

where δ_{n_1,y_1} and δ_{n_2,y_2} are defined by Theorem 3.5.

Proof. Let $(t,s),(y_1,y_2)\in\mathcal{I}^2$. For all $n_1,n_2\in\mathbb{N}$ and $\delta_{n_1},\delta_{n_2}>0$, it follows that

$$\begin{aligned}\mid\Delta^*_{(y_1,y_2)}g(t,s)\mid &\leq \omega_B\left(g;\mid t-y_1\mid\mid s-y_2\mid\right)\\ &\leq \left(1+\frac{t-y_1}{\delta_{n_1}}\right)\left(1+\frac{s-y_2}{\delta_{n_2}}\right)\omega_B\left(g;\delta_{n_1},\delta_{n_2}\right).\end{aligned}$$

From (16) and well-known Cauchy-Schwarz inequality, we easily conclude that

$$\begin{aligned}\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);y_1,y_2)-g(y_1,y_2)\mid &\leq H^*_{n_1,n_2}\left(\mid\Delta^*_{(y_1,y_2)}g(t,s)\mid;y_1,y_2\right)\\ &\leq \left(H^*_{n_1,n_2}(\phi_{0,0};y_1,y_2)+\frac{1}{\delta_{n_1}}\left(H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)\right)^{\frac{1}{2}}\right.\\ &+ \frac{1}{\delta_{n_2}}\left(H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)\right)^{\frac{1}{2}}\\ &+ \frac{1}{\delta_{n_1}}\left(H^*_{n_1,n_2}((t-y_1)^2;y_1,y_2)\right)^{\frac{1}{2}}\\ &\times \left.\frac{1}{\delta_{n_2}}\left(H^*_{n_1,n_2}((s-y_2)^2;y_1,y_2)\right)^{\frac{1}{2}}\right)\omega_B\left(g;\delta_{n_1},\delta_{n_2}\right).\end{aligned}$$

In the view of Theorem 3.5 we easily get our results.

□

If we let $x = (t, s),\ y = (y_1, y_2) \in \mathcal{I}^2$, then the Lipschitz function in terms of B-continuous functions defined by

$$Lip_M^{\xi} = \left\{ g \in C(\mathcal{I}^2) : \mid \Delta^*_{(y_1,y_2)} g(x, y) \mid \leq M \parallel x - y \parallel^{\xi}, \right\}$$

where M is a positive constant, $0 < \xi \leq 1$, and Euclidean norm $\parallel x - y \parallel = \sqrt{(t - y_1)^2 + (s - y_2)^2}$.

Theorem 4.2. *For all $g \in Lip_M^{\xi}$ operator $K^{\alpha_1,\alpha_2}_{n_1,n_2}$ satisfying*

$$\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x, y); y_1, y_2) - g(y_1, y_2) \mid \leq M\{\delta^2_{n_1,y_1} + \delta^2_{n_2,y_2}\}^{\frac{\xi}{2}},$$

where δ_{n_1,y_1} and δ_{n_2,y_2} are defined by Theorem 3.5.

Proof. We easily see that

$$\begin{aligned}
K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x, y); y_1, y_2) &= H^*_{n_1,n_2}(g(y_1, y) + g(x, y_2) - g(x, s); y_1, y_2) \\
&= H^*_{n_1,n_2}\left(g(y_1, y_2) - \Delta^*_{(y_1,y_2)} g(x, s); y_1, y_2\right) \\
&= g(y_1, y_2) - H^*_{n_1,n_2}\left(\Delta^*_{(y_1,y_2)} g(x, s); y_1, y_2\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x, y); y_1, y_2) - g(y_1, y_2) \mid &\leq H^*_{n_1,n_2}\left(\mid \Delta^*_{(y_1,y_2)} g(x, y) \mid; y_1, y_2\right) \\
&\leq M H^*_{n_1,n_2}\left(\parallel x - y \parallel^{\xi}; y_1, y_2\right) \\
&\leq M\{H^*_{n_1,n_2}\left(\parallel x - y \parallel^2; y_1, y_2\right)\}^{\frac{\xi}{2}} \\
&\leq M\{H^*_{n_1,n_2}\left((t - y_1)^2; y_1, y_2\right) + H^*_{n_1,n_2}\left((s - y_2)^2; y_1, y_2\right)\}^{\frac{\xi}{2}}.
\end{aligned}$$

□

Theorem 4.3. *If $g \in D_{\varphi}(\mathcal{I}^2)$ and $D_B g \in B(\mathcal{I}^2)$, then*

$$\begin{aligned}
\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g; y_1, y_2) - g(y_1, y_2) \mid \leq\ & C\Big\{3 \parallel D_B g \parallel_{\infty} + \varpi_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2})\Big\}(y_1 + 1)(y_2 + 1) \\
+\ & \Big\{1 + \sqrt{C_2}(y_1 + 1) + \sqrt{C_1}(y_2 + 1)\Big\} \\
\times\ & \varpi_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2})(y_1 + 1)(y_2 + 1),
\end{aligned}$$

where $\delta_{n_1},\ \delta_{n_2}$ defined by Theorem 3.5 and C is any positive constant.

Proof. Suppose $\rho \in (y_1, t),\ \xi \in (y_2, s)$ and

$$\Delta^*_{(y_1,y_2)} g(t, s) = (t - y_1)(s - y_2) D_B g(\rho, \xi),$$

$$D_B g(\rho, \xi) = \Delta^*_{(y_1,y_2)} D_B g(\rho, \xi) + D_B g(\rho, y) + D_B g(x, \xi) - D_B g(y_1, y_2).$$

For all $D_B g \in B(\mathcal{I}^2)$, it follows that

$$\begin{aligned}
\mid K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Delta_{(y_1,y_2)}^{*} g(t,s); y_1, y_2\right) \mid &= \mid K_{n_1,n_2}^{\alpha_1,\alpha_2}\left((t-y_1)(s-y_2) D_B g(\rho,\xi); y_1, y_2\right) \mid \\
&\leq K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\mid t-y_1 \mid\mid s-y_2 \mid\mid \Delta_{(y_1,y_2)}^{*} D_B g(\rho,\xi) \mid; y_1, y_2\right) \\
&+ K_{n_1,n_2}^{\alpha_1,\alpha_2}\Big(\mid t-y_1 \mid\mid s-y_2 \mid (\mid D_B g(\rho,y_2) \mid \\
&+ \mid D_B g(y_1,\xi) \mid + \mid D_B g(y_1,y_2) \mid); y_1, y_2\Big) \\
&\leq K_{n_1,n_2}^{\alpha_1,\alpha_2}\Big(\mid t-y_1 \mid\mid s-y_2 \mid \\
&\times\ \varpi_{mixed}\left(D_B g; \mid \rho-y_1 \mid, \mid \xi-y_2 \mid\right); y_1, y_2\Big) \\
&+ 3 \parallel D_B g \parallel_\infty K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\mid t-y_1 \mid\mid s-y_2 \mid; y_1, y_2\right).
\end{aligned}$$

Here ϖ_{mixed}, is mixed-modulus of continuity and it follows that

$$\begin{aligned}
&\varpi_{mixed}\left(D_B g; \mid \rho-y_1 \mid, \mid \xi-y_2 \mid\right) \\
&\quad\leq \varpi_{mixed}\left(D_B g; \mid t-y_1 \mid, \mid s-y_2 \mid\right) \\
&\quad\leq \left(1+\delta_{n_1}^{-1} \mid t-y_1 \mid\right)\left(1+\delta_{n_2}^{-1} \mid s-y_2 \mid\right) \varpi_{mixed}\left(D_B g; \mid \delta_{n_1}, \delta_{n_2}\right).
\end{aligned}$$

Therefore, it is obvious that

$$\begin{aligned}
\mid K_{n_1,n_2}^{*}(g; y_1, y_2) - g(y_1,y_2) \mid &= \mid \Delta_{(y_1,y_2)}^{*} g(t,s); y_1, y_2 \mid \\
&\leq 3 \parallel D_B g \parallel_\infty \left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left((t-y_1)^2 (s-y_2)^2; y_1, y_2\right)\right)^{\frac{1}{2}} \\
&+ \Big(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\mid t-y_1 \mid\mid s-y_2 \mid; y_1, y_2\right) \\
&+ \delta_{n_1}^{-1} K_{n_1,n_2}^{\alpha_1,\alpha_2}\left((t-y_1)^2 \mid s-y_2 \mid; y_1, y_2\right)\Big) \\
&+ \delta_{n_2}^{-1} K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\mid t-y_1 \mid (s-y_2)^2; y_1, y_2\right) \\
&+ \delta_{n_1}^{-1}\delta_{n_2}^{-1} K_{n_1,n_2}^{\alpha_1,\alpha_2}\left((t-y_1)^2 (s-y_2)^2; y_1, y_2\right) \varpi_{mixed}\left(D_B g; \delta_{n_1}, \delta_{n_2}\right);
\end{aligned}$$

$$\begin{aligned}
\mid K_{n_1,n_2}^{*}(g; y_1, y_2) - g(y_1,y_2) \mid &= \mid \Delta_{(y_1,y_2)}^{*} g(t,s); y_1, y_2 \mid \\
&\leq 3 \parallel D_B g \parallel_\infty \left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{y_1,y_2}^{2,2}; y_1, y_2\right)\right)^{\frac{1}{2}} \\
&+ \Bigg\{\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{y_1,y_2}^{2,2}; y_1, y_2\right)\right)^{\frac{1}{2}} \\
&+ \delta_{n_1}^{-1}\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{y_1,y_2}^{4,2}; y_1, y_2\right)\right)^{\frac{1}{2}} + \delta_{n_2}^{-1}\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{y_1,y_2}^{2,4}; y_1, y_2\right)\right)^{\frac{1}{2}} \\
&+ \delta_{n_1}^{-1}\delta_{n_2}^{-1} K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{y_1,y_2}^{2,2}; y_1, y_2\right)\Bigg\} \varpi_{mixed}\left(D_B g; \delta_{n_1}, \delta_{n_2}\right).
\end{aligned}$$

Which follows that

$$
\begin{aligned}
\mid K^{*}_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid &= 3\parallel D_B g\parallel_\infty\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{y_1,y_2};y_1,y_2\right)K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}}\\
&+ \left\{\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{y_1,y_2};y_1,y_2\right)K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}}\right.\\
&+ \delta_{n_1}^{-1}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{4,0}_{y_1,y_2};y_1,y_2\right)K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}}\\
&+ \delta_{n_2}^{-1}\left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{y_1,y_2};y_1,y_2\right)K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,4}_{y_1,y_2};y_1,y_2\right)\right)^{\frac{1}{2}}\\
&+ \left.\delta_{n_1}^{-1}\delta_{n_2}^{-1}K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{y_1,y_2};y_1,y_2\right)K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{y_1,y_2};y_1,y_2\right)\right\}\\
&\times\ \omega_{mixed}\left(D_B g;\delta_{n_1},\delta_{n_2}\right).
\end{aligned}
$$

From Lemma (2.5), we demonstrate

$$
\begin{aligned}
\mid K^{*}_{n_1,n_2}(g;y_1,y_2)-g(y_1,y_2)\mid &\leq 3\parallel D_B g\parallel_\infty\left(\sqrt{C_1C_2}(y_1+1)(y_2+1)\right)\\
&+ \left\{\left(\sqrt{C_1C_2}(y_1+1)(y_2+1)\right)\right.\\
&+ \delta_{n_1}^{-1}\left(\sqrt{C_2}\sqrt{O\left(\frac{1}{n_1}\right)}(y_1+1)^2(y_2+1)\right)\\
&+ \delta_{n_2}^{-1}\left(\sqrt{C_1}\sqrt{O\left(\frac{1}{n_2}\right)}(y_2+1)^2(y_1+1)\right)\\
&+ \left.\delta_{n_1}^{-1}\delta_{n_2}^{-1}\left(\sqrt{O\left(\frac{1}{n_1}\right)}\sqrt{O\left(\frac{1}{n_2}\right)}(y_1+1)(y_2+1)\right)\right\}\\
&\times\ \omega_{mixed}\left(D_B g;\delta_{n_1},\delta_{n_2}\right).
\end{aligned}
$$

Which complete the proof of Theorem 4.3. □

5. Conclusion and Remarks

These types of generalization, that is, Bivariate Szász operators is a new generalization. In this, manuscript our investigation is to generalize the Szász Durrmeyer operators based on Dunkl analogue [46] by introducing the bivariate functions. We study the bivariate properties of Szász Durrmeyer operators with the help of modulus of continuity, mixed-modulus of continuity and then find the approximation results in Peetre's K-functional, Voronovskaja type theorem and Lipschitz maximal functions for these bivariate operators. Next, we also construct the GBS type operator of these generalized operators and study approximation in Bögel continuous functions by use of mixed-modulus of continuity.

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