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Some Equalities on q-Gamma and q-Digamma Functions

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Abstract. In this paper, we give some equalities on q-gamma and q-digamma functions for negative integer values of x by aid of using the concepts of neutrix and neutrix limit.

1. Introduction and Preliminaries

Q-calculus; the q-analogue of the classical calculus; is a quite popular subject today. The deformed calculus has found a lot of applications in mathematics, statistics and physics. These are many q-functions of the ordinary special functions such as q-beta, q-gamma and q-zeta functions. In a rude way, the definition of a q-analogue $\mathcal{M}_q$ of a mathematical object $\mathcal{M}$ is such that the limit of $\mathcal{M}_q$ as q tends to 1 is $\mathcal{M}$. In this paper we aim to give some equalities on q-gamma and q-digamma functions by using their q-integral representations and the neutrix calculus developed by van der Corput.

Definition 1.1. *(Neutrix)Let N' be a nonempty set and let $\mathcal{N}$ be a commutative, additive group of functions mapping N' into a commutative, additive group N''. The group $\mathcal{N}$ is called neutrix if the function which is identically equal to zero is the only constant function occurring $\mathcal{N}$. The function which belongs to $\mathcal{N}$ is called "negligible function" in $\mathcal{N}$.*
Let $N^{'}$ be a domain lying in a topological space with a limit point b not belonging to $N^{'}$ and $\mathcal{N}$ be a commutative additive group of functions defined on N' with the following property:

$$"f \in \mathcal{N}\ , \lim_{\varepsilon \to b} f(\varepsilon) = c\,(constant)\ for\ \varepsilon \in N^{'}\ then\ c = 0".$$

Then this group $\mathcal{N}$ is a neutrix.

Definition 1.2. *(Neutrix limit) Let f be a real valued function defined on N' and suppose that it is possible to find a constant c such that $f(x) - c$ is negligible in $\mathcal{N}$. Then c is called the neutrix limit of $f(x)$ as x tends to y and denoted by*

$$\mathrm{N}{-}\lim_{x \to y} f(x) = c. \tag{1}$$

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The reader may find the general definition of the neutrix and neutrix limit in [3].
In this work, we let N be the neutrix having domain the open interval $N' = (0, (1-q)^{-1})$ and range N'' as the real numbers with the negligible functions being finite linear sums of the functions

$$\epsilon^{\lambda} \ln^{r-1} \epsilon, \ln^{r} \epsilon, [\epsilon]^{\lambda}, \quad \lambda < 0, r = 1, 2, ...$$

and all being functions $f(\epsilon)$ which converge to zero in the usual sense as ϵ tends to zero.
Let q be a positive number $0 < q < 1$. For any complex number x, the basic number $[x]$ and the q-factorial $[n]!$ are defined by

$$[x] = \frac{1-q^{x}}{1-q}, \quad [n]! = [n][n-1]\dots[2][1], \quad n \in \mathbb{N}.$$

Let f be a function defined on a subject of real or complex plane. The q-analogue of the derivative of $f(x)$, called its q-derivative is given by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \text{ if } x \neq 0 \text{ and } (D_q f)(0) = f'(0)$$

provided $f'(0)$ exists.
The q-Jackson integral is defined for a function f to be

$$\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} q^n f(aq^n)$$

provided the sum converges absolutely and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

The q-integrating by parts is given for suitable functions f and g by

$$\int_a^b g(qx) d_q f(x) = f(b)g(b) - f(a)g(a) - \int_a^b f(x) d_q g(x). \tag{2}$$

One of the q-analogues of the exponential function e^x is defined as

$$E_q^x = \sum_{i=0}^{\infty} q^{\binom{i}{2}} \frac{x^i}{[i]!} = (-(1-q)x; q)^{\infty}.$$

Note that the q-derivative of E_q^x is E_q^{qx}. More information about q-calculus can be found in [1, 2].
The q-analogue of gamma function $\Gamma(x)$ is defined in [7, 8] by the q-integral representation

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1} E_q^{-qt} d_q t$$

and its derivatives are defined by

$$\Gamma_q^{(r)} = \int_0^{\frac{1}{1-q}} t^{x-1} \ln^r t E_q^{-qt} d_q t, \quad r = 0, 1, 2, \dots. \tag{3}$$

Using the regularization technique, it has been shown in [6] that for $x > -n, n = 1, 2, \dots, x \neq 0, -1, -2, \dots$, the q-gamma function is defined by the neutrix limit as

$$\Gamma_q(x) = \mathrm{N}-\lim_{\epsilon \to 0} \int_{\epsilon}^{1/1-q} t^{x-1} E_q^{-qt} d_q t$$

and in [9] the authors give an equation for the function $\Gamma_q(x)$ for negative integer values of x with using the Heaviside's function $H(x)$; which is equal to zero for $x < 0$ and to 1 for $x > 0$. That is

$$\begin{aligned}\Gamma_q(-n) &= \int_0^{1/1-q} t^{-n-1}\ln t\Big[E_q^{-qt} - \sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!}t^j - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]!}t^n H(1-t)\Big]d_qt \\ &+ (1-q)^{n+1}\sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{(q,q)_i(1-q^{i-n})}. \end{aligned} \tag{4}$$

2. Main Results

In this section, using neutrix calculus, we give some results on q-gamma function with its first derivative and then show that q-digamma function can be defined at negative integers. At first, we need the following lemmas.

Lemma 2.1. *We have*

$$\Gamma_q(0) = \frac{q-1}{lnq}\Gamma_q'(1). \tag{5}$$

Proof. By taking $g(qt) = E_q^{-qt}$, $d_q(f(t)) = t^{-1}$ and using q-integration by parts given in (2), then we obtain

$$\begin{aligned}\Gamma_q(0) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^{1/1-q} t^{-1}E_q^{-qt}d_qt \\ &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\left\{\frac{q-1}{\ln q}\ln(1-q)E_q^{-\frac{1}{1-q}} - \frac{q-1}{\ln q}\ln\epsilon E_q^{-q\epsilon} + \frac{q-1}{\ln q}\int_\epsilon^{1/1-q}\ln t E_q^{-qt}d_qt\right\}.\end{aligned}$$

Since $E_q^t = (1+(1-q)t)_q^\infty$, the first term on the right side is equal to zero and second one includes $\ln\epsilon$, which is negligible function, so the neutrix limit of the last term collides with ordinary one which is equal to the equation (3) at $x = 1$, then we get the desired result. □

Lemma 2.2. *For $n = 1, 2, \ldots,$*

$$\Gamma_q(-n) = \frac{1}{[-n]}\Gamma_q(-n+1) + \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n][n]!}. \tag{6}$$

Proof. On q-integrating by parts to equation (4) we have

$$\begin{aligned}\Gamma_q(-n) &= \frac{1}{[-n]}\left\{\int_1^{1/1-q} t^{-n}E_q^{-qt}d_qt + \int_0^1 t^{-n}\Big[E_q^{-t} - \sum_{j=0}^{n}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!}t^j\Big]d_qt\right\} \\ &- \frac{E_q^{-1}}{[-n]} + \frac{1}{[-n]}\Big[E_q^{-1} - \sum_{j=0}^{n}\frac{(-1)^j q^{\frac{j(j-1)}{2}}}{[j]!}\Big] + \sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]![-n+j]} \\ &= \frac{1}{[-n]}\left\{\int_1^{1/1-q} t^{-n}E_q^{-qt}d_qt + \int_0^1 t^{-n}\Big[E_q^{-t} - \sum_{j=0}^{n}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!}t^j\Big]d_qt\right. \\ &+ \left.\sum_{j=0}^{n-2}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[i]![-n+j+1]}\right\} - \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[-n][n]!} \\ &= \frac{1}{[-n]}\Gamma_q(-n+1) + \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n][n]!}\end{aligned}$$

as desired. □

Note that by using the equations (5) and (6) and mathematical induction, q-gamma function satisfies the equation

$$\Gamma_q(-n) = \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]!}\Big(\varphi_q(n) + \Gamma_q(0)\Big) \tag{7}$$

where (for example, see [10] and references therein)

$$\varphi_q(n) = \sum_{j=1}^{n} \frac{1}{[j]}$$

for $n = 0, -1, -2, \ldots$ and this result tends to the equation

$$\Gamma(-n) = (-1)^n\Big(\varphi(n) + \Gamma(0)\Big)$$

where

$$\varphi(n) = \sum_{j=0}^{n} \frac{1}{j}$$

shown in [4] and [11] as $q \to 1$.

Theorem 2.3. *Let H denotes Heaviside's function. Then for $n \in \mathbb{N}$,*

$$\begin{aligned} \Gamma_q(-n) &= \int_0^{1/1-q} t^{-n-1} \ln t \Big[E_q^{-qt} - \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!} t^j - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]!} t^n H(1-t)\Big] d_q t \\ &+ \ln q^{-1}(1-q)^{-1} \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]![-n+j]} (1-q)^{n-j} + \frac{\ln q^{-1}}{q-1} \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]![-n+j]^2} (1-q)^{n-j}. \end{aligned} \tag{8}$$

Proof. By definitions, we have

$$\begin{aligned} \int_\epsilon^{\frac{1}{1-q}} t^{-n-1} \ln t E_q^{-qt} d_q t &= \int_\epsilon^{1/1-q} t^{-n-1} \ln t \Big[E_q^{-qt} - \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!} t^j - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]!} t^n H(1-t)\Big] d_q t \\ &+ \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!} \int_\epsilon^{\frac{1}{1-q}} t^{-n+j-1} \ln t d_q t + \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]!} \int_\epsilon^1 t^{-1} \ln t d_q t. \end{aligned}$$

Now calculating the last two integrals on the right side of the equation and then taking the neutrix limit of the both sides of the equation we get

$$\begin{aligned} \underset{\epsilon \to 0}{\mathrm{N-lim}} \int_\epsilon^{\frac{1}{1-q}} t^{-n-1} \ln t E_q^{-qt} d_q t &= \int_\epsilon^{1/1-q} t^{-n-1} \ln t \Big[E_q^{-qt} - \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!} t^j - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]!} t^n H(1-t)\Big] d_q t \\ &+ \sum_{j=0}^{n-1} \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!} \left\{ \frac{(1-q)^{n-j} \ln(q^{-1}(1-q)^{-1})}{[-n+j]} + \frac{\ln q^{-1}(1-q)^{n-j}}{(q-1)[-n+j]^2} \right\} \end{aligned}$$

as desired. □

Theorem 2.4. *For $n = 1, 2, \ldots$*

$$\Gamma'_q(-n) = \frac{\ln q^{-1}}{(q-1)[-n]}\Gamma_q(-n) + \frac{\ln q^{-1}}{[-n]}\Gamma_q(-n+1) + \frac{1}{[-n]}\Gamma'_q(-n+1). \tag{9}$$

Proof. On q-integrating by parts to equation (8) we have

$$\begin{aligned}
\Gamma'_q(-n) &= \frac{\ln q^{-1}}{[-n](q-1)}\int_0^{1/1-q} t^{-n-1}\Big[E_q^{-qt} - \sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!}t^j - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]!}t^n H(1-t)\Big]d_qt \\
&+ \frac{\ln q^{-1}}{[-n]}\int_0^{1/1-q} t^{-n}\Big[E_q^{-qt} - \sum_{j=0}^{n-2}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!}t^j - \frac{(-1)^{n-1} q^{\frac{n(n-1)}{2}}}{[n-1]!}t^{n-1} H(1-t)\Big]d_qt \\
&+ \frac{1}{[-n]}\int_0^{1/1-q} t^{-n}\ln t\Big[E_q^{-qt} - \sum_{j=0}^{n-2}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!}t^j - \frac{(-1)^{n-1} q^{\frac{n(n-1)}{2}}}{[n-1]!}t^{n-1} H(1-t)\Big]d_qt \\
&+ \ln q^{-1}(1-q)^{-1}\sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]![-n+j]}(1-q)^{n-j} + \frac{\ln q^{-1}}{q-1}\sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]![-n+j]^2}(1-q)^{n-j} \\
&- \frac{\ln q^{-1}}{[-n]}\sum_{j=0}^{n}\frac{(-1)^j q^{\frac{j(j-1)}{2}}}{[j]!}(1-q)^{n-j} - \frac{\ln(1-q)^{-1}}{[-n]}\sum_{j=0}^{n}\frac{(-1)^j q^{\frac{j(j-1)}{2}}}{[j]!}(1-q)^{n-j}.
\end{aligned}$$

The first three terms on the right side of the equation are the integral parts of the definitions of $\Gamma_q(-n)$, $\Gamma_q(-n+1)$ and $\Gamma'_q(-n+1)$ respectively, because of that, adding and extracting the missing series of these definitions one can see that sums of the remaining series are equal to zero and this completes the proof. □

Theorem 2.5. *For all real values of x,*

$$\Gamma'_q(x) = \operatorname*{N-lim}_{\epsilon\to 0}\Gamma'_q(x+\epsilon). \tag{10}$$

Proof. Since $\Gamma'_q(x)$ is a continuous function for $x \neq 0, -1, -2, \ldots$ its neutrix limit becomes normal limit as ϵ tends to zero and the result follows for $x \neq 0, -1, -2, \ldots$. Now we will consider $\Gamma'_q(x)$ at the point $x = -n$, $n = 1, 2, \ldots$. For $0 < \epsilon < 1$, we have from equation (8) that

$$\begin{aligned}
\Gamma'_q(-n+\epsilon) &= \int_0^{1/1-q} t^{-n+\epsilon-1}\ln t\Big[E_q^{-qt} - \sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!}t^j)\Big]d_qt \\
&+ \ln q^{-1}(1-q)^{-1}\sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]![-n+\epsilon+j]}(1-q)^{n-\epsilon-j} + \frac{\ln q^{-1}}{q-1}\sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]![-n+\epsilon+j]^2}(1-q)^{n-\epsilon-j} \\
&= \int_0^{1/1-q} t^{-n+\epsilon-1}\ln t\Big[E_q^{-qt} - \sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!}t^j - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]!}t^n H(1-t)\Big]d_qt \\
&+ \ln q^{-1}(1-q)^{-1}\sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]![-n+\epsilon+j]}(1-q)^{n-\epsilon-j} + \frac{\ln q^{-1}}{q-1}\sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]![-n+\epsilon+j]^2}(1-q)^{n-\epsilon-j} \\
&+ \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]!}\int_\epsilon^1 t^{\epsilon-1}\ln t d_qt.
\end{aligned}$$

Note that the neutrix limit is unique and its precisely the same as the ordinary one, if it exists. Then taking neutrix limit of both sides, we obtain

$$\begin{aligned}
\mathrm{N}\text{-}\lim_{\epsilon\to 0}\Gamma_q'(-n+\epsilon) &= \int_0^{1/1-q} t^{-n-1}\ln t\Big[E_q^{-qt}-\sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]!}t^j-\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]!}t^n H(1-t)\Big]d_qt \\
&+ \ln q^{-1}(1-q)^{-1}\sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]![-n+j]}(1-q)^{n-j}+\frac{\ln q^{-1}}{q-1}\sum_{j=0}^{n-1}\frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]![-n+j]^2}(1-q)^{n-j} \\
&= \Gamma_q(-n).
\end{aligned}$$

The case of $x=-n-\epsilon$ for $0<\epsilon<1$ can be proved similarly. □

For any $x>0$, we have

$$\Gamma_q(x+1)=[x]\Gamma_q(x). \tag{11}$$

It has been shown in [9] that equation (11) can be given for all real numbers by the neutrix limit such that

$$\Gamma_q(x+1)=\mathrm{N}\text{-}\lim_{\epsilon\to 0}[x+\epsilon]\Gamma_q(x+\epsilon). \tag{12}$$

Differentiating equation (11), we get

$$\Gamma_q'(x+1)=\frac{-q^x\ln q}{1-q}\Gamma_q(x)+[x]\Gamma_q'(x) \tag{13}$$

for $x\neq 0,-1,-2,\cdots$.
Now we give that, equation (13) can be extended for all real values of x.

Theorem 2.6. *For all x we have*

$$\Gamma_q'(x+1)=\mathrm{N}\text{-}\lim_{\epsilon\to 0}\frac{q^{x+\epsilon}\ln q}{q-1}\Gamma_q(x+\epsilon)+[x+\epsilon]\Gamma_q'(x+\epsilon).$$

Proof. The result can easily be obtained because of the continuity of $\Gamma_q'(x)$ for $x\neq 0,-1,-2,\ldots$. Equation (8) also satisfies for all real values of x. By rewriting this equation for $n=1,2,\ldots$ and $0<\epsilon<1$ as

$$\Gamma_q'(-n+\epsilon+1)=[-n]\Gamma_q'(-n+\epsilon)-\frac{\ln q^{-1}}{q-1}\Gamma_q(-n+\epsilon)-\ln q^{-1}\Gamma_q(-n+\epsilon+1)$$

then we get from taking the neutrix limit of both sides that

$$\mathrm{N}\text{-}\lim_{\epsilon\to 0}\Gamma_q'(-n+\epsilon+1)=\mathrm{N}\text{-}\lim_{\epsilon\to 0}[-n+\epsilon]\Gamma_q'(-n+\epsilon)-\frac{\ln q^{-1}}{q-1}\Gamma_q(-n+\epsilon)-\ln q^{-1}\Gamma_q(-n+\epsilon+1).$$

Hence with the previous theorem and equation (2) we have

$$\begin{aligned}
&= \mathrm{N}\text{-}\lim_{\epsilon\to 0}[-n+\epsilon]\Gamma_q'(-n+\epsilon)+\frac{\ln q}{q-1}\,\mathrm{N}\text{-}\lim_{\epsilon\to 0}\Gamma_q(-n+\epsilon)+\ln q\,\mathrm{N}\text{-}\lim_{\epsilon\to 0}[-n+\epsilon]\Gamma_q(-n+\epsilon) \\
&= \mathrm{N}\text{-}\lim_{\epsilon\to 0}[-n+\epsilon]\Gamma_q'(-n+\epsilon)+\frac{\ln q}{q-1}\,\mathrm{N}\text{-}\lim_{\epsilon\to 0}\Big[1+(q-1)[-n+\epsilon]\Big]\Gamma_q(-n+\epsilon) \\
&= \mathrm{N}\text{-}\lim_{\epsilon\to 0}[-n+\epsilon]\Gamma_q'(-n+\epsilon)+\frac{\ln q}{q-1}\,\mathrm{N}\text{-}\lim_{\epsilon\to 0}\Big[1+q^{-n+\epsilon}-1\Big]\Gamma_q(-n+\epsilon) \\
&= \mathrm{N}\text{-}\lim_{\epsilon\to 0}\frac{-q^{-n+\epsilon}\ln q}{1-q}\Gamma_q(-n+\epsilon)+[-n+\epsilon]\Gamma_q'(-n+\epsilon)
\end{aligned}$$

as desired. □

The $\psi_q(x)$ function is defined by

$$\psi_q(x) = \left(\ln \Gamma_q(x)\right)' = \frac{\Gamma_q'(x)}{\Gamma_q(x)}$$

for $n \neq 0, -1, -2, \ldots$. By the previous results, it gives the idea that we define the q-digamma function $\psi_q(-n)$ by

$$\psi_q(-n) = \mathrm{N}\!-\!\lim_{\epsilon\to 0} \frac{\Gamma_q'(-n+\epsilon)}{\Gamma_q(-n+\epsilon)}$$

for $n = 0, -1, -2, \ldots$, provided the neutrix limit exists. Now we will show the existence of the function $\psi_q(-n)$ for $n = 0, 1, 2, \ldots$.

Theorem 2.7. *For $n = 0, 1, 2, \ldots$*

$$\psi_q(-n) = \psi_q(1) + \frac{\ln q}{q-1}\varphi_q(n).$$

Proof. If we take $0 < |\epsilon| < 1$ and use equation (11) and (13), we get

$$\begin{aligned}\frac{\Gamma_q'(\epsilon)}{\Gamma_q(\epsilon)} &= \frac{\Gamma_q'(\epsilon+1) + \frac{q^{\epsilon}\ln q}{1-q}\Gamma_q(\epsilon)}{[\epsilon]\Gamma_q(\epsilon)} \\ &= \frac{\Gamma_q'(\epsilon+1)}{\Gamma_q(\epsilon+1)} + \frac{q^{\epsilon}\ln q}{(1-q)[\epsilon]}.\end{aligned}$$

Now taking neutrix limit of both sides, it follows that

$$\begin{aligned}\mathrm{N}\!-\!\lim_{\epsilon\to 0}\frac{\Gamma_q'(\epsilon)}{\Gamma_q(\epsilon)} &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\frac{\Gamma_q'(\epsilon+1)}{\Gamma_q(\epsilon+1)} + \frac{q^{\epsilon}\ln q}{(1-q)[\epsilon]} \\ &= \frac{\Gamma_q'(1)}{\Gamma_q(1)} = \psi_q(1).\end{aligned}$$

providing that $\psi_q(0)$ exists and$\psi_q(0) = \psi_q(1)$. For the case of $n = 1, 2, \ldots$, assuming the existence of $\psi_q(-n+1)$, we have

$$\frac{\Gamma_q'(-n+\epsilon)}{\Gamma_q(-n+\epsilon)} = \frac{\Gamma_q'(-n+\epsilon+1) + \frac{q^{-n+\epsilon}\ln q}{1-q}\Gamma_q(-n+\epsilon)}{[-n+\epsilon]\Gamma_q(-n+\epsilon)} = \frac{\Gamma_q'(-n+\epsilon+1)}{\Gamma_q(-n+\epsilon+1)} + \frac{q^{-n+\epsilon}\ln q}{(1-q)[-n+\epsilon]}.$$

Then we get

$$\begin{aligned}\mathrm{N}\!-\!\lim_{\epsilon\to 0}\frac{\Gamma_q'(-n+\epsilon)}{\Gamma_q(-n+\epsilon)} &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\frac{\Gamma_q'(-n+\epsilon+1)}{\Gamma_q(-n+\epsilon+1)} + \frac{q^{-n+\epsilon}\ln q}{(q-1)[-n+\epsilon]} \\ \psi_q(-n) &= \psi_q(-n+1) + \frac{\ln q}{(1-q)[n]}.\end{aligned}$$

By induction, it follows that

$$\psi_q(-n) = \psi_q(1) + \frac{\ln q}{q-1}\varphi_q(n)$$

for $n = 1, 2, \ldots$. Hence the proof is completed. □

Note that all results that we obtain in this paper, tends to the results in [4] and [5] as $q \to 1$.

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