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Lacunary Ward Continuity in 2-normed Spaces

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Abstract. In this paper, we introduce lacunary statistical ward continuity in a 2-normed space. A function f defined on a subset E of a 2-normed space X is lacunary statistically ward continuous if it preserves lacunary statistically quasi-Cauchy sequences of points in E where a sequence (x_k) of points in X is lacunary statistically quasi-Cauchy if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : ||x_{k+1} - x_k, z|| \geq \varepsilon\}| = 0$$

for every positive real number ε and $z \in X$, and (k_r) is an increasing sequence of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$, $I_r = (k_{r-1}, k_r]$. We investigate not only lacunary statistical ward continuity, but also some other kinds of continuities in 2-normed spaces.

1. Introduction

In 1928, Menger ([22]) introduced a concept of a generalized metric, and later on, Vulich ([31]) gave a notion of a higher dimensional normed linear space which had been neglected by many analysists until it was developed by Gähler in the mid of 1960's ([15], [16], and [17]). Recently, Mashadi [20], and many others ([8, 21, 23]) have studied this concept and obtained various results.

The concept of lacunary statistical convergence of a sequence of real numbers was introduced by Fridy and Orhan in [12, 13], and further investigated by several authors in [24], [27], [28], [29].

The idea in the definition of sequential continuity enabled some authors to introduce, and investigate certain kinds of continuities in [1, 3–5, 9, 30]. Lacunary statistical ward continuity, or S_θ ward continuity of a real function was introduced by Cakalli, Aras and Sonmez in [6].

The aim of this paper is to study the concept of lacunary statistical ward continuity in 2-normed spaces, and prove some interesting theorems.

2. Preliminaries

In this paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of all positive integers, and the set of all real numbers, respectively. Now we recall the definition of a 2-normed space. Let X be a real linear space with $dim X > 1$

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and $\|.,.\|: X\times X \to \mathbb{R}$ a function. Then $(X, \|.,.\|)$ is called a linear 2-normed space if (i) $\|x,y\| = 0 \Leftrightarrow x$ and y are linearly dependent, (ii) $\|x,y\| = \|y,x\|$, (iii) $\|\alpha x, y\| = |\alpha|\,\|x,y\|$, (iv) $\|x, y+z\| \le \|x,y\| + \|x,z\|$ for $\alpha \in \mathbb{R}$ and $x, y, z \in X$. The function $\|.,.\|$ is called a 2-norm on X. Throughout the paper X will denote a 2-normed space. Observe that in any 2-normed space $(X, \|.,.\|)$ we have $\|.,.\|$ is nonnegative, $\|x-z, x-y\| = \|x-z, y-z\|$, and $\|x, y+\alpha x\| = \|x,y\|$ for all $x, y \in X$, $\alpha \in \mathbb{R}$. A classical example is the 2-normed space $X = \mathbb{R}^2$ with the 2-norm $\|.,.\|$ defined by $\|a,b\| = |a_1b_2 - a_2b_1|$ where $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$. This is the area of the parallelogram determined by the vectors **a** and **b**. A sequence (x_n) of points in X is said to converge to $L \in X$ in the 2-norm X if $lim_{n\to\infty} \|x_n - L, z\| = 0$ for every $z \in X$. This is denoted by $lim_{n\to\infty}\|x_n, z\| = \|L, z\|$. A sequence (x_n) of points in X is said to be a Cauchy sequence with respect to the 2-norm X if $lim_{n,m\to\infty} \|x_n - x_m, z\| = 0$ for every $z \in X$. A sequence of functions (f_n) is said to be uniformly convergent to a function f on a subset E of X if for each $\epsilon > 0$, an integer N can be found such that $\|f_n(x) - f(x), z\| < \epsilon$ for $n \ge N$ and for all $x, z \in X$ ([14]). A lacunary sequence $\theta = (k_r)$ is an increasing sequence of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, the ratio k_r/k_{r-1} will be abbreviated by q_r, $q_1 = 0$ for convention, and we assume that $\lim inf_r\, q_r > 1$. A sequence (x_k) of points in X is called lacunary statistically convergent, or S_θ-convergent, to an element L of X if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : \|x_k - L, z\| \ge \varepsilon\}| = 0,$$

for every positive real number ε and $z \in X$ ([12], [2]), it is denoted by $S_\theta - lim_{k\to\infty}\|x_k, z\| = \|L, z\|$ for every $z \in X$.

3. Results

First we note that lacunary statistical limit is unique.

Proposition 3.1. *If a sequence is lacunary statistically convergent to L_1 and L_2 in X, then $L_1 = L_2$.*

Proof. Although the proof follows from the fact that the set of semi-norms $\{p_z : z \in X\}$, where $p_z(x) = \|x, z\|$ for every $x \in X$ and for each $z \in X$ separates points, we give a direct proof for completeness. Now suppose that a sequence (x_k) of points in X has two different lacunary statistical limits, L_1 and L_2, say. Write $\alpha_k = L_1 - L_2$ for every $k \in \mathbb{N}$. Take any $z \in X$, then write $\varepsilon_0 = \frac{\|L_1 - L_2, z\|}{2}$. So for all $r \in \mathbb{N}$ we have

$$\{k \in I_r : \|\alpha_k, z\| \ge \varepsilon_0\} \subset \left\{k \in I_r : \|L_1 - x_k, z\| \ge \frac{\varepsilon_0}{2}\right\} \cup \left\{k \in I_r : \|x_k - L_2, z\| \ge \frac{\varepsilon_0}{2}\right\}.$$

Now it follows from this that for all $z \in X$, $r \in \mathbb{N}$

$$|\{k \in I_r : \|\alpha_k, z\| \ge \varepsilon_0\}| \le \left|\left\{k \in I_r : \|L_1 - x_k, z\| \ge \frac{\varepsilon_0}{2}\right\}\right| + \left|\left\{k \in I_r : \|x_k - L_2, z\| \ge \frac{\varepsilon_0}{2}\right\}\right|.$$

Lacunary statistical convergence of (x_k) to L_1 implies that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|L_1 - x_k, z\| \ge \frac{\varepsilon_0}{2}\right\}\right| = 0,$$

and lacunary statistical convergence of (x_k) to L_2 implies that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|L_2 - x_k, z\| \ge \frac{\varepsilon_0}{2}\right\}\right| = 0$$

for all $z \in X$. Thus for all $z \in X$

$$1 = \lim_{r\to\infty} \tfrac{1}{h_r} |\{k \in I_r : \|\alpha_k, z\| \ge \varepsilon_0\}|$$

$$\le \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|L_1 - x_k, z\| \ge \frac{\varepsilon_0}{2}\right\}\right| + \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|x_k - L_2, z\| \ge \frac{\varepsilon_0}{2}\right\}\right| = 0 + 0 = 0.$$

It follows from this contradiction that $L_1 = L_2$. □

Definition 3.2. *A subset E of X is called S_θ-sequentially compact if any sequence of points in E has an S_θ-convergent subsequence with an S_θ-limit in E.*

We note that the union of two S_θ-sequentially compact subsets of X is S_θ-sequentially compact, the sum of two S_θ-sequentially compact subsets of X is S_θ-sequentially compact, the intersection of any family of S_θ-sequentially compact subsets is S_θ-sequentially compact, any compact subset of X is S_θ-sequentially compact, and any finite subset of X is S_θ-sequentially compact.

Definition 3.3. *A function f defined on a subset E of X is said to be S_θ-sequentially continuous on E if it preserves S_θ-convergent sequences, i.e. $(f(x_k))$ is an S_θ-convergent sequence whenever (x_k) is an S_θ-convergent sequence.*

We see that if (x_k) is an S_θ-convergent sequence with $S_\theta - lim_{k\to\infty}\|x_k, z\| = \|x_0, z\|$ for every $z \in X$, then $(f(x_k))$ is an S_θ-convergent sequence with $S_\theta - lim_{k\to\infty}\|f(x_k), z\| = \|f(x_0), z\|$ for every $z \in X$. We note that the sum of two S_θ-sequentially continuous function at a point x_0 of X is S_θ-sequentially continuous at x_0, and the composition of two S_θ-sequentially continuous function at a point x_0 of X is S_θ-sequentially continuous at x_0. In the classical case, that is in the single normed case it is known that uniform limit of sequentially continuous functions is sequentially continuous, now we see that it is also true that not only uniform limit of sequentially continuous functions is sequentially continuous, but also uniform limit of S_θ-sequentially continuous functions is S_θ-sequentially continuous in 2-normed spaces. Now we give the latter in the following.

Theorem 3.4. *Let f_k be a lacunary statistically sequentially continuous function defined on a subset E of X into X for each $k \in \mathbb{N}$, and (f_n) be uniformly convergent to a function f, and then f is lacunary statistically sequentially continuous.*

Proof. Let (f_k) be a uniformly convergent sequence with uniform limit f, and (x_k) be any S_θ-convergent sequence of points in E with $S_\theta - lim_{k\to\infty}\|x_k, z\| = \|x_0, z\|$ for every $z \in X$. Take any $\varepsilon > 0$. By uniform convergence of (f_k), there exists an $N \in \mathbb{N}$ such that $\|f(x) - f_k(x), z\| < \frac{\varepsilon}{3}$ for $k \geq N$ and every $x \in E$ and $z \in X$. Since f_N is lacunary statistically sequentially continuous on E we have

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \|f_N(x_0) - f_N(x_k), z\| \geq \frac{\epsilon}{3} \right\} \right| = 0.$$

On the other hand, we have

$$\{k \in I_r : \|f(x_0) - f(x_k), z\| \geq \epsilon\} \subset \left\{ k \in I_r : \|v_k, z\| \geq \frac{\epsilon}{3} \right\} \cup \left\{ k \in I_r : \|f_N(x_0) - f_N(x_k), z\| \geq \frac{\epsilon}{3} \right\}$$
$$\cup \left\{ k \in I_r : \|f_N(x_k) - f(x_k), z\| \geq \frac{\epsilon}{3} \right\}$$

where $v_k = f(x_0) - f_N(x_0)$ for every $k \in \mathbb{N}$. Thus it follows from this inclusion that

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \{k \in I_r : \|f(x_0) - f(x_n), z\| \geq \epsilon\} \right| \leq \lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \|v_k, z\| \geq \frac{\epsilon}{3} \right\} \right|$$
$$+ \lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \|f_N(x_0) - f_N(x_n), z\| \geq \frac{\epsilon}{3} \right\} \right| + \lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \|f_N(x_n) - f(x_n), z\| \geq \frac{\epsilon}{3} \right\} \right| = 0$$

for every $z \in X$. So f is lacunary statistically sequentially continuous on E, and the proof is completed. □

Theorem 3.5. *S_θ-sequentially continuous image of any S_θ-sequentially compact subset of X is S_θ-sequentially compact.*

Proof. Assume that f is an S_θ-sequentially continuous function on a subset E of X, and A is an S_θ-sequentially compact subset of E. Let $(f(x_n))$ be any sequence of points in $f(A)$ where $x_n \in A$ for each positive integer n. Since A is S_θ-sequentially compact, there is a subsequence $(\gamma_k) = (x_{n_k})$ of (x_n) with $S_\theta - \lim_{k\to\infty}\|\gamma_k, z\| = \|\ell, z\|$ for every $z \in E$. Write $(t_k) = (f(\gamma_k))$. As f is S_θ-sequentially continuous, $(f(\gamma_k))$ is S_θ-sequentially convergent which is a subsequence of the sequence $(f(x_n))$ with $S_\theta - \lim_{k\to\infty}\|t_k, z\| = \|\ell, z\|$ for $\forall z \in E$. This completes the proof of the theorem. □

The concept of a strongly lacunary quasi-Cauchy sequence in a 2-normed space was studied in [8]. Now we give the following definition of an S_θ-quasi-Cauchy sequence.

Definition 3.6. *A sequence (x_k) of points in X is called to be lacunary statistically quasi-Cauchy if $S_\theta - lim_{k\to\infty}\|\Delta x_k, z\| = 0$ for every $z \in X$ where $\Delta x_k = x_{k+1} - x_k$ for each $k \in \mathbb{N}$. The set of lacunary statistically quasi-Cauchy sequences is denoted by ΔS_θ.*

Definition 3.7. *A subset E of X is called S_θ-ward compact if any sequence of points in E has an S_θ-quasi-Cauchy subsequence.*

The union of two S_θ-ward compact subset of X is S_θ-ward compact. The intersection of any family of S_θ-ward compact subsets is S_θ-ward compact. Any finite subset of X is S_θ-ward compact.

Now we state the definition of lacunary statistical ward continuity in a 2-normed space in the following:

Definition 3.8. *A real valued function f defined on a subset E of X is called lacunary statistically ward continuous, or S_θ-ward continuous on E if it preserves lacunary statistically quasi-Cauchy sequences of points in E, i.e. $(f(x_k))$ is a lacunary statistically quasi-Cauchy sequence whenever (x_k) is a lacunary statistically quasi-Cauchy sequence of points in E.*

The sum of two lacunary statistically ward continuous functions is lacunary statistically ward continuous, and the composition of lacunary statistically ward continuous functions is lacunary statistically ward continuous.

Theorem 3.9. *If a real valued function is lacunary statistically ward continuous on a subset E of X, then it is lacunary statistically sequentially continuous on E.*

Proof. Suppose that f is a lacunary statistically ward continuous function on a subset E of X. Let (x_n) be a lacunary statistically quasi-Cauchy sequence of points in E. Then the sequence

$$(x_1, x_0, x_2, x_0, x_3, x_0, ..., x_{n-1}, x_0, x_n, x_0, ...)$$

is a lacunary statistically quasi-Cauchy sequence. Since f is lacunary statistically ward continuous, the sequence

$$(y_n) = (f(x_1), f(x_0), f(x_2), f(x_0), ..., f(x_n), f(x_0), ...)$$

is a lacunary statistically quasi-Cauchy sequence. Therefore $S_\theta - \lim_{n\to\infty}\|\Delta y_n, z\| = 0$ for every $z \in X$. Hence $S_\theta - \lim_{n\to\infty}\|f(x_n) - f(x_0), z\| = 0$ for $\forall z \in X$. It follows that the sequence $(f(x_n))$ is lacunary statistically convergent to $f(x_0)$. This completes the proof of the theorem. □

Now we prove the following theorem.

Theorem 3.10. *If a real valued function f is uniformly continuous on a subset E of X, then $(f(x_n))$ is lacunary statistically quasi-Cauchy whenever (x_n) is a quasi-Cauchy sequence of points in E.*

Proof. Let f be uniformly continuous on E. Take any quasi-Cauchy sequence (x_n) of points in E. Let ε be any positive real number. Since f is uniformly continuous, there exists a $\delta > 0$ such that $\|f(x) - f(y), w\| < \varepsilon$ for any $w \in X$ whenever $\|x - y, z\| < \delta$ for any $x, y \in E$ and $z \in X$. As (x_n) is a quasi-Cauchy sequence, for this δ there exists an $n_0 \in \mathbb{N}$ such that $\|\Delta x_n, z\| < \delta$ for $n \geq n_0$ for $z \in X$. Therefore $\|\Delta f(x_n), z\| < \varepsilon$ for $n \geq n_0$, so the number of indices k for which $\|f(x_{n+1}) - f(x_n), z\| \geq \varepsilon$ is less than n_0. Hence

$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : \|f(x_{n+1}) - f(x_n), z\| \geq \varepsilon\}| \leq \lim_{r\to\infty} \frac{n_0}{h_r} = 0$.

This completes the proof of the theorem. □

Theorem 3.11. *Uniform limit of lacunary statistically ward continuous function is lacunary statistically ward continuous.*

Proof. Let (f_k) be a uniformly convergent sequence with uniform limit f. To prove that f is lacunary statistically ward continuous on E, take any lacunary statistically quasi-Cauchy sequence (x_n) of points in E. Let ε be any positive real number. Since (f_n) is uniformly convergent to f, there exists a positive integer N such that $\|f_n(x) - f(x), z\| < \frac{\varepsilon}{3}$ whenever $n \geq N$ for all $x, z \in E$. Since f_N is lacunary statistically ward continuous on E we have

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|f_N(x_{k+1}) - f_N(x_k), z\| \geq \frac{\epsilon}{3}\right\}\right| = 0.$$

On the other hand we have

$$\{k \in I_r : \|f(x_{k+1}) - f(x_k), z\| \geq \epsilon\} \subset \left\{k \in I_r : \|f(x_{k+1}) - f_N(x_{k+1}), z\| \geq \frac{\epsilon}{3}\right\}$$
$$\cup \left\{k \in I_r : \|f_N(x_{k+1}) - f_N(x_k), z\| \geq \frac{\epsilon}{3}\right\} \cup \left\{k \in I_r : \|f_N(x_k) - f(x_k), z\| \geq \frac{\epsilon}{3}\right\}.$$

So it follows from this inclusion that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : \|f(x_{k+1}) - f(x_k), z\| \geq \epsilon\}\right| \leq \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|f(x_{k+1}) - f_N(x_{k+1}), z\| \geq \frac{\epsilon}{3}\right\}\right|$$
$$+ \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|f_N(x_{k+1}) - f_N(x_k), z\| \geq \frac{\epsilon}{3}\right\}\right| + \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|f_N(x_k) - f(x_k), z\| \geq \frac{\epsilon}{3}\right\}\right| = 0$$

for every $z \in X$. So f is lacunary statistically ward continuous on E, and the proof is completed. □

Theorem 3.12. *Let f_k be a function defined on a subset E of X into X that transforms convergent sequences to lacunary statistically quasi-Cauchy sequences for each $k \in \mathbb{N}$, and (f_k) be uniformly convergent to a function f, then f transforms convergent sequences to lacunary statistically quasi-Cauchy sequences.*

Proof. Let (f_k) be a uniformly convergent sequence with uniform limit f, and (x_k) be a convergent sequence of points in E with $lim\ \|x_k, z\| = \|x_0, z\|$ for every $z \in X$. Take any $\varepsilon > 0$. By uniform convergence of (f_k), there exists an $N \in \mathbb{N}$ such that $\|f(x) - f_k(x), z\| < \frac{\varepsilon}{3}$ for $k \geq N$ and every $x \in E$ and $z \in X$. Since f_N transforms convergent sequences to lacunary statistically quasi-Cauchy sequences on E we have

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|f_N(x_{k+1}) - f_N(x_k), z\| \geq \frac{\epsilon}{3}\right\}\right| = 0$$

for each $z \in X$. On the other hand, we have

$$\{k \in I_r : \|f(x_{k+1}) - f(x_k), z\| \geq \epsilon\} \subset \left\{k \in I_r : \|f(x_{k+1}) - f_N(x_k), z\| \geq \frac{\epsilon}{3}\right\}$$
$$\cup \left\{k \in I_r : \|f_N(x_k) - f_N(x_{k+1}), z\| \geq \frac{\epsilon}{3}\right\} \cup \left\{k \in I_r : \|f_N(x_{k+1}) - f(x_k), z\| \geq \frac{\epsilon}{3}\right\}.$$

So it follows from this inclusion that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : \|f(x_{k+1}) - f(x_k), z\| \geq \epsilon\}\right| \leq \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|f(x_{k+1}) - f_N(x_{k+1}), z\| \geq \frac{\epsilon}{3}\right\}\right|$$
$$+ \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|f_N(x_{k+1}) - f_N(x_k), z\| \geq \frac{\epsilon}{3}\right\}\right| + \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : \|f_N(x_k) - f(x_k), z\| \geq \frac{\epsilon}{3}\right\}\right| = 0$$

for every $z \in X$. Thus f transforms convergent sequences to lacunary statistically quasi-Cauchy sequences, so the proof of the theorem is completed. □

We note that any lacunary statistically ward continuous function transforms not only convergent sequences, but also slowly oscillating sequences to lacunary statistically quasi-Cauchy sequences.

4. Conclusion

We have introduced not only lacunary statistical ward continuity, but also some other kinds of continuities and proved interesting theorems. The results are more comprehensive than existing related ones in the literature, and there are some results obtained in this research that have not been appeared in the classical real number system as well. We note that lacunary statistical quasi-Cauchyness is equivalent to the notion of a lacunary statistical convergence in a complete non-Archimedean 2-normed space, and so the set of lacunary statistically ward continuous functions is the same as the set of lacunary statistically sequentially continuous functions in a complete non-Archimedean 2-normed space (see [25], and [10] for the related concepts in a non-Archimedean 2-normed space). As a further study, our suggestion is to investigate lacunary statistically quasi-Cauchy sequences of fuzzy points, lacunary statistical ward continuity of the fuzzy functions in a 2-normed fuzzy space. However due to the change in the setting, the definitions and methods of proofs will not always be analogous to those of the present work (see [7, 18, 24]). For another further study, we suggest to investigate lacunary statistically quasi-Cauchy sequences of double sequences in a 2-normed space, and lacunary statistical ward double continuity to find out whether lacunary statistical ward double continuity coincides with lacunary statistical ward (single) continuity or not (see [26] for the definitions and related concepts in the double case).

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