Filomat 30:2 (2016), 293–303
DOI 10.2298/FIL1602293A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Independent Transversal Dominating Sets in Graphs: Complexity and Structural Properties

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Abstract. A dominating set of a graph G which intersects every independent set of maximum cardinality in G is called an independent transversal dominating set. The minimum cardinality of an independent transversal dominating set is called the independent transversal domination number of G and is denoted by $\gamma_{it}(G)$. In this paper we study some complexity issues on some independent transversal domination related problems. On the other side, we prove that for every integers a, b, c with $a \leq b \leq a + c$, there exists a graph G such that G has domination number a, minimum degree c and independent transversal domination number b. We also give some other properties of independent transversal dominating sets in graphs.

1. Introduction

A transversal of a collection of sets is a set of distinct representatives of the elements in the collection. Transversals in graphs have received a high attention throughout the last thirty or more years and it is possible to find transversals regarding several types of vertex sets in graphs. Some of them, but maybe not every of the most remarkable ones, are related to the chromatic number and the independence number of a graph. For instance, [6] was addressed to the following problem. Given a partition of the vertex set of a graph satisfying a bound (lower or upper) on the quantity of elements in each set of the partition, is there a transversal of the partition that is an independent set or a dominating set? Several results on this problem were presented in [6], like possible applications to fault-tolerant data storage or some complexity aspects regarding the associated decision problems. In [2], the concept "partition domination number" was defined as the largest integer k such that given any partition of the vertex set of the graph having at most k elements in every set of the partition, there is transversal of the partition being a dominating set. Some complexity results regarding the associated decision problems and some bounds or exact values for some specific families of graphs were presented in [2]. Nevertheless, these are not the only examples of such a transversal-type results in the literature (for instance, see [1], for the case of strong partition independence or strong chromatic number to just mention at least two of them). Some other examples (and again not the only ones) are [3, 13], connecting transversals with the chromatic number. According to the amount of

2010 *Mathematics Subject Classification*. 05C69; 05C75.
Keywords. Independent transversal domination; Domination; Independence; Transversal.
Received: 29 June 2014; Accepted: 21 October 2015
Communicated by Francesco Belardo
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literature about this topic in every of its related variants, we restrict our references principally to those ones which are only citing papers that we really refer to in a non-superficial way.

More recently, some other authors have developed some works in this topic, in which the condition: "for every partition of the vertex set satisfying a property P there is a transversal satisfying property Q", has been relaxed to a collection of sets which do not necessarily satisfy the condition of being a partition of the vertex set. A recent work in this new style of transversal-type concepts has been presented in [10]: the independence domination transversal number. According to its novelty, this parameter remains relative unknown and just a few results on it are published. It is our goal to contribute with the topic of transversals in graphs throughout studying some other more properties of the independent transversal dominating sets in graphs.

All graphs considered in this paper are finite, undirected, loopless, and without multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. For a vertex x of G, $N(x)$ denotes the set of all neighbors of x in G and the degree of x is $\deg(x) = |N(x)|$. The minimum and maximum degrees of a vertex of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

We denote by *isol*(G) the set of *isolated vertices* of a graph G, and by *End*(G) the set of *end-vertices* (vertices of degree one) of G. An edge incident with an end-vertex is called a *pendant edge*. A vertex adjacent to an end-vertex is called a *stem*, and *Stem*(G) denotes the set of stems of G. A graph with a single vertex is called a *trivial graph* and a graph without edges is an *empty graph*. Also, a graph of order n with all the possible edges is the *complete graph* and is denoted by K_n.

A set S of vertices is *independent* if no two vertices from S are adjacent. An independent set of maximum cardinality is a *maximum independent set* of G. The *independence number* of G, denoted as $\beta_0(G)$, is the cardinality of a maximum independent set of G. An independent set of cardinality $\beta_0(G)$ is called a *$\beta_0(G)$-set*.

A *matching* of G is a set of pairwise non-incident edges of G. The maximum cardinality $\mu(G)$ of a matching in G is the *matching number* and a matching of cardinality $\mu(G)$ is a *maximum matching* or a $\mu(G)$-set.

A subset D of $V(G)$ is a *dominating set* in G if every vertex of $V(G) - D$ has at least one neighbor in D. The *domination number* of G, denoted by $\gamma(G)$, is the smallest size of any dominating set in G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. If a vertex v of a graph G belongs to some $\gamma(G)$-set, then v is called a *$\gamma(G)$-good vertex*.

A dominating set of G which intersects every independent set of maximum cardinality in G is called an *independent transversal dominating set*. The minimum cardinality of an independent transversal dominating set is called the *independent transversal domination number* of G and is denoted by $\gamma_{it}(G)$. An independent transversal dominating set of cardinality $\gamma_{it}(G)$ is called a *$\gamma_{it}(G)$-set*. The paper is organized as follows. In Section 2 we give a background of some known results, some of which are necessary to present our results. In Section 3 we study some topics regarding some complexity issues on independent transversal domination problems in graphs. In Section 4 we give some results regarding the realizability of the independent transversal domination number in connection with the domination number and the minimum degree of graphs. Finally, in Section 5 we give some properties of independent transversal dominating sets in graphs.

2. Some Known Results

The concept of independent transversal dominating set has been recently described in [10] and, according to that fact, just a few results are known in this moment. Some of them are stated at next according to its usefulness for our purposes. It is natural to think that $\gamma_{it}(G)$ and $\gamma(G)$ are related and, in this sense, the following "sandwich-type" bounds were presented in [10].

Theorem 2.1 (Hamid [10]). *For any graph G of minimum degree $\delta(G)$, $\gamma(G) \leq \gamma_{it}(G) \leq \gamma(G) + \delta(G)$.*

Notice that, if a graph G has an end-vertex, then the result above leads to that $\gamma_{it}(G)$ is either $\gamma(G)$ or $\gamma(G) + 1$, which raises some interesting problems regarding to know whether $\gamma_{it}(G)$ equals $\gamma(G)$ or $\gamma(G) + 1$, for instance, if G is a tree. Nevertheless, not only with $\gamma(G)$ is related $\gamma_{it}(G)$ as we can see at next, where is given some connection with the vertex cover number. We recall that a set S is a *vertex cover* if every edge of

G is incident to a vertex of S and the minimum cardinality of a vertex cover is the *vertex cover number* and is denoted by $\alpha_0(G)$.

Theorem 2.2 (Hamid [10]). *Let a graph G without isolated vertices. Then $\gamma_{it}(G) \leq \alpha_0(G) + 1$. If the equality holds, then $\gamma(G) = \alpha_0(G)$.*

Specific families of graphs were also studied in [10]. Particularly, for bipartite graphs and trees were presented the following results.

Theorem 2.3 (Hamid [10]). *Let G be a bipartite graph with bipartition (X, Y) such that $|X| \leq |Y|$ and $\gamma(G) = |X|$. Then $\gamma_{it}(G) = \gamma(G) + 1$ if and only if every vertex in X is adjacent to at least two end-vertices.*

Theorem 2.4 (Hamid [10]). *If T is a tree, then $\gamma_{it}(T)$ is either $\gamma(T)$ or $\gamma(T) + 1$.*

3. Complexity of Independent Transversal Domination Problems

In this section we consider several issues regarding the complexity of independent transversal domination problems in graphs. For more information on complexity classes and related topics we suggest [8]. We begin with the following decision problem.

INDEPENDENT TRANSVERSAL DOMINATION PROBLEM
INSTANCE: A non-trivial graph G and a positive integer r
PROBLEM: Deciding whether $\gamma_{it}(G)$ is less than r

The complexity of the INDEPENDENT TRANSVERSAL DOMINATION PROBLEM (ITD-PROBLEM for short) is clearly related to the existence of a polynomial time verification algorithm which checks that a given set of vertices of a graph G is indeed an independent transversal dominating set. In this sense, we need to consider the following problem.

INDEPENDENT TRANSVERSAL DOMINATING SET PROBLEM
INSTANCE: A non-trivial graph G and a subset S of vertices of G
PROBLEM: Deciding whether S is an independent transversal dominating set in G

At next we prove that the INDEPENDENT TRANSVERSAL DOMINATING SET PROBLEM (ITDS-PROBLEM for short) is a Co-NP-complete problem. It is further not so difficult to prove that a polynomial algorithm exists for the ITDS-PROBLEM if and only if P=NP. To do so, we consider the complement of the ITDS-PROBLEM.

NOT INDEPENDENT TRANSVERSAL DOMINATING SET PROBLEM
INSTANCE: A non-trivial graph G and a subset S of vertices of G
PROBLEM: Deciding whether S is not an independent transversal dominating set in G

To prove that the ITDS-PROBLEM is a Co-NP-complete problem, we need to prove that the NOT INDEPENDENT TRANSVERSAL DOMINATING SET PROBLEM (NOT-ITDS PROBLEM for short) is NP-complete, which we do at next.

Claim 3.1. *Let G be a non-trivial graph. The set $S \subset V(G)$ is not an independent transversal dominating set in G if and only if at least one of the following conditions is satisfied.*

- *S is not a dominating set, or*
- *S does not intersect a $\beta_0(G)$-set.*

Notice that checking a given set of a graph G is not a dominating set, or that it does not intersect a $\beta_0(G)$-set can be done in polynomial time. Therefore, the ITDS-PROBLEM is clearly in NP. Next we show that it is also an NP-complete problem.

The characterization below (Lemma 3.3) together with the NP-completeness [11] of the problem of deciding whether the independence number of a graph G is greater than a positive integer r are crucial to prove that the NOT-ITDS PROBLEM is NP-complete.

INDEPENDENCE PROBLEM
INSTANCE: A non-trivial graph G and a positive integer r
PROBLEM: Deciding whether the independence number of G is greater than r

Theorem 3.2. [11] *INDEPENDENCE PROBLEM is NP-complete.*

Given a set of vertices X of a graph G, the graph obtained from G, by deleting the vertices of X together with all the edges incident with at least one vertex of X is denoted by $G - X$.

Lemma 3.3. *Let G be a non-trivial graph and let $S \subset V(G)$ be a dominating set. Then S is not an independent transversal dominating set in G if and only if $\beta_0(G - S) = \beta_0(G)$.*

Proof. First we notice that for any set X of vertices of G, it follows that $\beta_0(G - X) \leq \beta_0(G)$. Now, if $S \subset V(G)$ is not an independent transversal dominating set in G, then there exists an independent set $Y \subset V(G) - S$ such that $|Y| = \beta_0(G)$. Thus, Y is an independent set in $G - S$ and, as a consequence, we have that $\beta_0(G - S) \geq |Y| = \beta_0(G)$. Therefore, we obtain that $\beta_0(G - S) = \beta_0(G)$.

On the other hand, we assume that $\beta_0(G - S) = \beta_0(G)$. Thus, there exists an $\beta_0(G - S)$-set Y' such that $|Y'| = \beta_0(G)$ and Y' remains being an independent set in G. Therefore, it is clear that S is not an independent transversal dominating set in G, since it does not intersect the $\beta_0(G)$-set Y'. □

From the Lemma above we deduce that the problem of deciding whether a dominating set $S \subset V(G)$ is not an independent transversal dominating set in G can be reduced to the INDEPENDENCE PROBLEM for the graph $G - S$. According to this and the fact that ITDS-PROBLEM is in NP, we obtain the following result.

Theorem 3.4. *NOT-ITDS PROBLEM is NP-complete.*

As a consequence of the result above we obtain that the ITDS PROBLEM (which is the complement of the NOT-ITDS PROBLEM) is a Co-NP-complete problem.

Corollary 3.5. *ITDS PROBLEM is Co-NP-complete.*

The Corollary above along with the next results, which is satisfied for any decision problem, conclude the complexity of the ITDS-PROBLEM.

Theorem 3.6. *ITDS PROBLEM is in P if and only if P=NP.*

Proof. First we notice that, if P = NP, then the NOT-ITDS PROBLEM, which is a NP problem, is also in the class P. So, its complement, the ITDS-PROBLEM, will also be in the class P.

On the contrary, if the ITDS PROBLEM is in the class P, then its complement, the NOT-ITDS PROBLEM, is also in P. Since the NOT-ITDS PROBLEM is an NP-Complete problem (by Theorem 3.4), we obtain that P = NP. □

At next we continue with the other issue regarding the complexity of independent transversal domination problems. That is, we prove that the ITD-PROBLEM (deciding whether the independent transversal domination number of G is less than a positive integer r) is NP-Hard and, moreover, it is NP-complete if and only if P=NP.

3.1. Complexity of the INDEPENDENT TRANSVERSAL DOMINATION PROBLEM

First we observe that, as a requirement for proving that the ITD-PROBLEM will be in the NP class, there must be a polynomial verifier which can check the existence of an independent transversal dominating set of cardinality less than or equal to a positive integer r, in such a case that the evidence of this existence has been given. That is, a set S of cardinality r or less. In this sense, a polynomial algorithm must verify that S is indeed an independent transversal dominating set in G. Nevertheless, by Theorem 3.6, this algorithm exists only in the case P = NP. Therefore, if P≠NP, it follows that the ITD-PROBLEM is not in NP.

To continue with our results we present the following problem regarding the domination number of a graph.

DOMINATION PROBLEM
INSTANCE: A non-trivial graph G and a positive integer r
PROBLEM: Deciding whether the domination number of G is less than r

Theorem 3.7. [8] *DOMINATION PROBLEM is NP-complete.*

Let G be a non-empty graph of order $n \geq 2$ with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. To analyze the complexity of the ITD-PROBLEM, we construct the following graph H_G. We consider three sets of vertices $A = \{a_1, a_2, ..., a_n\}$, $B = \{b_1, b_2, ..., b_n\}$ and $C = \{c_1, c_2, ..., c_n\}$, each of which of cardinality n. With the sets A and B we form a complete graph K_{2n} (we add all the possible edges between any two vertices of $A \cup B$). Then, to obtain the graph H_G, for every $i \in \{1, ..., n\}$, we add an edge between v_i and a_i and other edge between b_i and c_i. See Figure 1 for an example of the graph H_{P_4}.

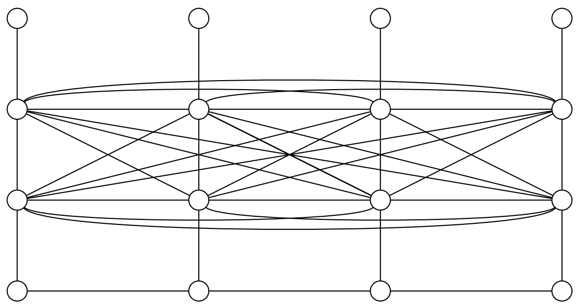

Figure 1: The graph H_{P_4}

Now we give some claims and properties on the graph H_G.

Lemma 3.8. *For any non-empty graph G of order n, $\gamma(H_G) = n + \gamma(G)$.*

Proof. Let $S \subset V(G)$ be a $\gamma(G)$-set. According to the construction of the graph H_G, it is straightforward to observe that $S \cup B$ is a dominating set in H_G. So, $\gamma(H_G) \leq |B| + |S| = n + \gamma(G)$. On the other hand, let X be a $\gamma(H_G)$-set. Since every vertex of C has degree one, for every $i \in \{1, ..., n\}$ it follows that $X \cap \{b_i, c_i\} \neq \emptyset$. Thus, $|X \cap (A \cup B)| \geq n$. Also, it is clear that $|X \cap (A \cup V(G))|$ is a dominating set in G. Thus, $|X \cap (A \cup V(G))| \geq \gamma(G)$. As a consequence, the proof is completed by the following.

$$\gamma(H_G) = |X| = |X \cap (A \cup B)| + |X \cap (A \cup V(G))| \geq n + \gamma(G).$$

□

Lemma 3.9. *For any non-empty graph G of order n, $\beta_0(H_G) = n + 1 + \beta_0(G)$.*

Proof. Let $S \subset V(G)$ be a $\beta_0(G)$-set. Since G is non-empty, there exists at least one vertex $v_j \notin S$. Now, according to the construction of the graph H_G, it is straightforward to observe that $S \cup \{a_j\} \cup C$ is an independent set in H_G (a_j is the vertex of A adjacent to v_j). So, $\beta_0(H_G) \geq |C| + 1 + |S| = n + 1 + \beta_0(G)$.

On the other side, let X be a $\beta_0(H_G)$-set. Since the subgraph induced by $A \cup B$ is isomorphic to a complete graph K_{2n}, it follows that $|X \cap (A \cup B)| \leq 1$. Also, $X \cap V(G)$ is an independent set in G. Thus, we have the following.

$$\beta_0(H_G) = |X| = |X \cap C| + |X \cap (A \cup B)| + |X \cap V(G)| \leq n + 1 + \beta_0(G),$$

and, as a consequence, the result follows. □

According to the result above and its proof, it is clear that every $\beta_0(H_G)$-set contains all the vertices of the set C. That is stated in the following claim.

Claim 3.10. *Let G be a non-empty graph and let H_G be the graph constructed as in the procedure above. Then, every $\beta_0(H_G)$-set contains all the vertices of the set C.*

Lemma 3.11. *For any non-empty graph G of order n, $\gamma_{it}(H_G) = \gamma(H_G) = n + \gamma(G)$.*

Proof. Let S be a $\gamma(G)$-set and let $b_i \in B, c_i \in C$ (notice that b_i, c_i are adjacent). It is clear that $X = (C - \{c_i\}) \cup \{b_i\} \cup S$ is a dominating set in H_G and it has cardinality $n + \gamma(G)$. So, it is a $\gamma(H_G)$-set by Lemma 3.8. Also, by Claim 3.10 we have that X intersects every $\beta_0(H_G)$-set. Thus, $\gamma_{it}(H_G) \leq \gamma(H_G) = n + \gamma(G)$. The proof is complete by Theorem 2.1. □

Now, by using the above result we finally prove our main result in this section. We recall that our goal is to reduce the INDEPENDENT TRANSVERSAL DOMINATION PROBLEM to the DOMINATION PROBLEM.

Theorem 3.12. *INDEPENDENT TRANSVERSAL DOMINATION PROBLEM is an NP-hard problem. Moreover, it is NP-complete if and only if P=NP.*

Proof. We consider a non-trivial graph G and we construct a graph H_G by the procedure above. It is clear that such a construction can be done in polynomial time. By Lemma 3.11 we have that $\gamma_{it}(H_G) = \gamma(H_G) = n+\gamma(G)$. So, for any j, k with $j = n + k$, it follows that $\gamma(G) \leq k$ if and only if $\gamma_{it}(H_G) \leq j$. Therefore, we have reduced the DOMINATION PROBLEM to the ITD-PROBLEM and, as a consequence, ITD-PROBLEM is NP-hard.

Now, to complete the proof that the ITD-PROBLEM is NP-complete we need to show that the ITD-PROBLEM is in NP. Equivalently, there must be a polynomial verifier which can check that a set S of cardinality r or less is indeed an independent transversal dominating set in G. If P=NP, then by Theorem 3.6, this can be done in polynomial time. Therefore, the proof in this direction is complete.

We assume now that the ITD-PROBLEM is NP-complete. Thus, there is a polynomial verifier which checks that a set S of cardinality r or less is indeed an independent transversal dominating set in G, which means that the ITDS-PROBLEM is in the class P. So, by Theorem 3.6, we obtain that P=NP and we are done. □

4. Realizability Results for the Independent Transversal Domination Number

As mentioned at the beginning of Section 2, the bound $\gamma(G) \leq \gamma_{it}(G) \leq \gamma(G) + \delta(G)$ plays an important role while studying $\gamma_{it}(G)$. In this sense, it is natural to ask the following question, which was already presented as an open problem in [10]. Given three integers a, b, c with $a \leq b \leq a + c$, is there a graph G such that $\gamma(G) = a$, $\gamma_{it}(G) = b$ and $\delta(G) = c$? A positive answer to this question is known for $\delta(G) = 1$, by taking G as a tree. According to that fact, for any tree T, it follows that $\gamma(T) \leq \gamma_{it}(T) \leq \gamma(T) + 1$. Thus, it is of interest to characterize when $\gamma_{it}(T)$ equals $\gamma(T)$ or $\gamma(T) + 1$. This problem was also stated in [10], where a tree T was classified to be of class 1 or of class 2 according to the fact that $\gamma_{it}(T)$ is $\gamma(T)$ or $\gamma(T) + 1$, respectively. On the other hand, if G is not a tree and $\delta(G) = c > 1$, then it would be desirable to describe graphs such that

$\gamma(G) = a$ and $\gamma(G)_{it} = b$. Next we give an answer to such a problem. To this end, we need to introduce the following graph operation, which was defined first in [7].

Let G and H be two graphs of order n_1 and n_2, respectively. The *corona graph* $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G. Hereafter, we will denote by $V = \{u_1, u_2, ..., u_n\}$ the set of vertices of G, and by $H_i = (V_i, E_i)$ the i^{th} copy of H in $G \odot H$. For some domination related properties of corona graphs we suggest [9].

On the other hand, given a graph G, we will say that a $d_\beta(G)$ represents the largest number of pairwise disjoint $\beta_0(G)$-sets. From now on, we say that S is a $d_\beta(H)$-set of G if any two different vertices of S belong to two disjoint $\beta_0(G)$-sets. Since every graph has at least one $\beta_0(G)$-set, it follows that $d_\beta(G) \geq 1$ for any graph G. For instance, it is straightforward to observe that for any cycle C_n, $d_\beta(C_n) = 2$. Also, for any regular complete multipartite graph $K_{n,n,...,n}$ having k partite sets, it follows $d_\beta(G) = k$. This parameter is useful for our purposes on the realizability of the independence transversal domination, as we show at next. But we need before the following basic results.

Lemma 4.1. *For any graphs G and H, $\gamma(G \odot H) = n$.*

Lemma 4.2. *For any graphs G and H, $\beta_0(G \odot H) = n\beta_0(H)$. Moreover, if $\beta_0(H) \geq 2$, then every $\beta_0(G \odot H)$-set contains only vertices belonging to the copies H_i of the graph H in $G \odot H$ and, it is given by the union of n $\beta_0(H_i)$-sets.*

Next we study the independent transversal domination number of corona graphs.

Theorem 4.3. *Let G be a graph of order $n \geq 2$. Then, for any graph H such that $\beta_0(H) \geq 2$,*

$$n - 1 + d_\beta(H) \leq \gamma_{it}(G \odot H) \leq n + d_\beta(H).$$

Moreover, if there is a $d_\beta(H)$-set which is a dominating set in H, then $\gamma_{it}(G \odot H) = n - 1 + d_\beta(H)$.

Proof. Let X be a $\gamma_{it}(G \odot H)$-set. Since $\beta_0(H) \geq 2$, by Lemma 4.2 we have that every $\beta_0(G \odot H)$-set contains only vertices belonging to the copies H_i of the graph H in $G \odot H$ and, it is given by the union of n $\beta_0(H_i)$-sets. Thus, as X intersects every $\beta_0(G \odot H)$-set, there exists $j \in \{1, ..., n\}$, such that $X \cap V(H_j) \neq \emptyset$. Moreover, if there are k pairwise disjoint maximum independent sets in H_j, then $|X \cap V(H_j)| \geq k$. On the other hand, to dominate each set $V(H_i)$ with $i \in \{1, ..., n\}$ and $i \neq j$, we need at least one vertex. Thus, $|X \cap (V(H_i) \cup \{u_i\})| \geq 1$ for every $i \in \{1, ..., n\}$ and $i \neq j$ (notice that $\{u_1, ..., u_n\}$ is the vertex set of G). Therefore, we have that

$$\begin{aligned} |X| &= \sum_{i=1}^{n} |X \cap (V(H_i) \cup \{u_i\})| \\ &= \sum_{i=1, i\neq j}^{n} |X \cap (V(H_i) \cup \{u_i\})| + |X \cap V(H_j)| \\ &\geq (n-1) + k \\ &\geq n - 1 + d_\beta(H), \end{aligned}$$

and the proof of the lower bound is complete.

On the other hand, we consider a set Y in $G \odot H$, given in the following way. For the copy H_1 of H in $G \odot H$, we take a $d_\beta(H_1)$-set A (notice that A could be not a dominating set in H). Now, we make $Y = V(G) \cup A$. Now, it is straightforward to observe that Y is a dominating set in $G \odot H$ and, that it intersects every maximum independent set in $G \odot H$. Thus, $\gamma_{it}(G \odot H) \leq |Y| = n + d_\beta(H)$ and the upper bound is proved. Finally, it is clear that, if A is a dominating set in H, then $Y = (V(G) - \{u_1\}) \cup A$ is an independent transversal dominating set and we have the equality $\gamma_{it}(G \odot H) = n - 1 + d_\beta(H)$. □

By using the result above as a tool, we now deal with the problem of the existence of graphs G such that $\gamma(G) = a$, $\gamma(G)_{it} = b$ and $\delta(G) = c$ for every integers a, b, c with $a \leq b \leq a + c$. Since the case $\delta(G) = 1$ is already studied, we focus in the possibilities $\delta(G) = c > 1$.

Theorem 4.4. *Let a, b, c be three positive integers, such that $c \geq 2$ and $a \leq b \leq a + c$. Then, there exists a graph G of minimum degree $\delta(G) = c$, such that $\gamma(G) = a$ and $\gamma_{it}(G) = b$.*

Proof. We consider first the case $a \leq b = a + c$. If $a = 1$, then the complete graph K_b satisfies that $\delta(K_b) = b - 1 = b - a = c$, $\gamma(K_b) = 1 = a$, and $\gamma_{it}(K_b) = b$. Hence, we assume now that $a \geq 2$ and we consider the following cases.

Case 1: $c \leq a$. Let $H_1 \cong K_a$ and $H_2 \cong K_{c+1}$. Let G be the graph obtained from H_1 and H_2 by taking one copy of H_1 and a copies of H_2 and then, joining by an edge the i^{th}-vertex of H_1 with exactly one vertex of the i^{th}-copy of H_2. Then, it is clear that $\delta(G) = c$ and that, all the vertices of degree $c + 1$ form the only $\gamma(G)$-set. So $\gamma(G) = a$. Also, $\beta_0(G) = \gamma(G) + 1 = a + 1$ and each $\beta_0(G)$-set contains exactly one vertex of H_1 and exactly one vertex of each copy of H_2. As a consequence, $\gamma_{it}(G) = a + c = b$.

Case 2 $c > a$. Let $H_1 = K_c$ and $H_2 = K_d$, where $d > c$. Let G be the graph obtained from H_1 and H_2 by taking one copy of H_1 and a copies of H_2 by doing the following actions.

- Join by an edge the i^{th}-vertex of H_1 with exactly one vertex of the i^{th}-copy of H_2, $i = 1, 2, .., a - 1$.
- Join by an edge exactly one vertex of the last copy of H_2 with each one of the remaining vertices of H_1.

In this sense, it is obtained that $\delta(G) = c$ and all the vertices of the copies of H_2 that have a neighbor in H_1 form the only $\gamma(G)$-set. So $\gamma(G) = a$. Also, $\beta_0(G) = \gamma(G) + 1 = a + 1$ and each $\beta_0(G)$-set contains exactly one vertex of H_1 and exactly one vertex of each copy of H_2. Thus, $\gamma_{it}(G) = a + c = b$.

From now on, we consider the case $a \leq b < a + c$. If $a = 1$, then let G be the graph obtained from two complete graphs K_b and $K_{b(c-b+1)+1}$ by doing the following actions.

- Identify one vertex v_1 of K_b with one vertex u_1 of $K_{b(c-b+1)+1}$.
- Join by an edge each vertex of K_b different from v_1 with the same $c-b+1$ vertices of $K_{b(c-b+1)+1}$ different from u_1.

Thus, we have that the minimum degree of G is attained in a vertex of K_b different from v_1 and so, $\delta(G) = b - 1 + c - b + 1 = c$. Also, $\gamma(G) = 1$ and every $\beta_0(G)$-set is formed by one vertex of K_b different from v_1 and other one from $K_{b(c-b+1)+1}$ different from u_1. Thus, $\gamma_{it}(G) = b$.

Finally we analyze the case $a \leq b < a + c$ with $a \geq 2$. To this end, we begin with a graph G' of order a without isolated vertices. Then, we need a graph H such that $\delta(H) = c - 1$ and $d_\beta(H) = b - a + 1$ having a $d_\beta(H)$-set which is a dominating set in H. Thus, by taking G as the corona graph $G' \odot H$ we have the following. The graph G has minimum degree $\delta(G) = \delta(H) + 1 = c$, since every other vertex u of G' has degree, in G, equal to $\delta_{G'}(u) + |V(H)| \geq \delta_{G'}(u) + \delta(H) + 1 = \delta_{G'}(u) + c > c$. Also, from Lemma 4.1 we have that $\gamma(G) = a$ and, from Theorem 4.3 we obtain that $\gamma_{it}(G) = a - 1 + d_\beta(H) = a - 1 + b - a + 1 = b$.

Now, we will describe a graph H satisfying that $\delta(H) = c-1$, $d_\beta(H) = b-a+1$ and having a $d_\beta(H)$-set which is a dominating set in H. We consider a complete multipartite graph $K_{t_1,t_2,...,t_r}$ with partite sets $U_1, U_2, ..., U_r$ ($|U_i| = t_i$ and $r > b - a + 1$) such that there exists exactly $b - a + 1$ sets U_j having the same cardinality R and $R = |U_j| = t_j = \max\{t_i : 1 \leq i \leq r\}$. Also, there is one set U_l such that $t_l = 1$. On the other hand, we consider a complete graph K_c. Then, to obtain the graph H, we join by an edge the vertex of U_l with at most $c - 1$ vertices of the complete graph K_c (Figure 2 shows an example of such a graph for $a = 3$, $b = 5$ and $c = 4$).

Notice that the graph H has minimum degree $c - 1$ achieved in one of the vertices of the complete graph K_c which is not adjacent to the vertex of the set U_l. Also, it is not difficult to check that $\beta_0(H) = R + 1$ and, that every maximum independent set is formed by one of the partite sets U_j of cardinality R together with one vertex of the complete graph K_c. Since there are exactly $b - a + 1$ of such a partite sets U_j and also, $c > b - a$ (by the premise of the theorem), which is equivalent to $c \geq b - a + 1$, we have that there are $b - a + 1$ pairwise disjoint maximum independent sets in H. Therefore, $d_\beta(H) = b - a + 1$ and, as every $d_\beta(H)$-set is a dominating set in H, the proof is complete. □

As a consequence of the results above, it is also possible to observe that there are graphs G such that the difference $\gamma_{it}(G) - \gamma(G)$ can be arbitrarily large.

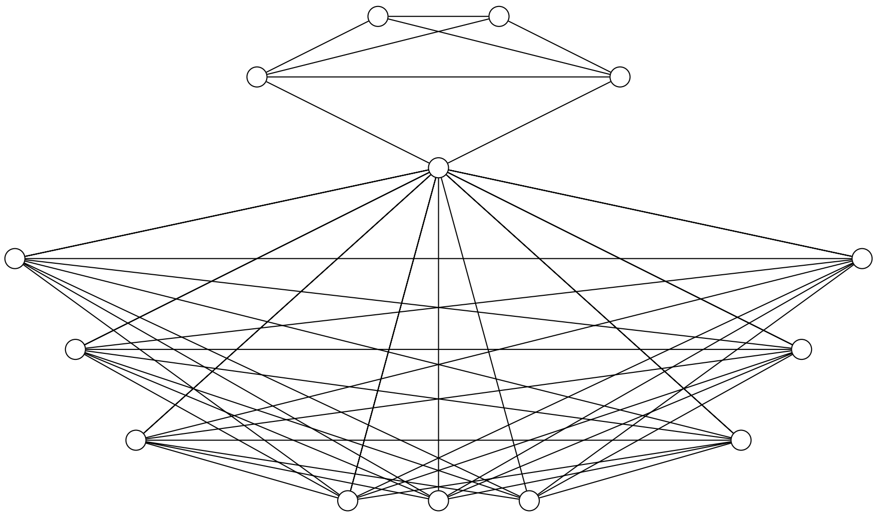

Figure 2: The graph H for $a = 3$, $b = 5$ and $c = 4$. Notice that $\delta(G) = c - 1 = 3$, $\beta(H) = 4$ and $d_\beta(H) = b - a + 1 = 3$

5. Properties of Independent Transversal Dominating Sets

Now, according to the complexity of the ITD-PROBLEM, it is also desirable to study the independent transversal domination number of several families of graphs or to study several properties of the independent transversal dominating sets of graphs. This could be done for instance, throughout giving sharp bounds for the independent transversal domination number in terms of other parameters of the graph or characterizing classes of graphs achieving some specific values of the studied parameter. To this end we need to introduce some terminology and notation. We denote by $\Omega(G)$ the set of all maximum independent sets in G, that is $\Omega(G) = \{S : S \text{ is a maximum independent set of } G\}$. According to this, we also say that $core(G) = \bigcap_{S\in\Omega(G)} S$ and $\xi(G) = |core(G)|$. Clearly, any isolated vertex of a graph G is contained in $core(G)$.

Observation 5.1. *Let G be a graph and let $v \in core(G)$.*

(i) *For any $\gamma(G)$-set D, $\{v\} \cup D$ is an independent transversal dominating set of G. In particular, $\gamma_{it}(G) \leq \gamma(G) + 1$.*
(ii) *If v is a $\gamma(G)$-good vertex, then $\gamma(G) = \gamma_{it}(G)$.*

Corollary 5.2. *Let G be a graph with a unique $\beta_0(G)$-set. Then $\gamma_{it}(G) \leq \gamma(G) + 1$. The equality holds if and only if no $\gamma(G)$-good vertex belongs to the $\beta_0(G)$-set.*

The Observation 5.1 (i) together with the known result below lead to a result in which we obtain the same bound as in the corollary above.

Lemma 5.3 (Boros, Golumbic and Levit [4]). *If G is a connected graph with $\beta_0(G) > \mu(G)$, then $\xi(G) \geq 1 + \beta_0(G) - \mu(G)$.*

Proposition 5.4. *If G is a connected graph with $\beta_0(G) > \mu(G)$, then $\gamma_{it}(G) \leq \gamma(G) + 1$.*

Theorem 5.5. *For any graph G without isolated vertices the following conditions are equivalent.*

(i) *$\gamma_{it}(G) = \alpha_0(G) + 1$.*
(ii) *Each $\gamma(G)$-set is a minimum vertex cover of G.*

Proof. (i) ⇒ (ii): By Theorem 2.2, $\gamma(G) = \alpha_0(G)$. Suppose there is a $\gamma(G)$-set U which is not a vertex cover of G. But then the set $V(G) - U$ is not independent and $|V(G) - U| = \beta_0(G)$. Hence $\gamma_{it}(G) \leq |U| = \gamma(G) = \alpha_0(G)$, a contradiction.

(ii) ⇒ (i): Consider any $\gamma_{it}(G)$-set D. Since D is dominating, $|D| \geq \gamma(G) = \alpha_0(G)$. If the equality holds, then D is a $\gamma(G)$-set, which implies that D is a minimum vertex cover of G. But, then $V(G) - D$ is a $\beta_0(G)$-set. □

Observation 5.6. *Let G be any graph. If each $\gamma(G)$-set is contained in some minimum vertex cover, then $\gamma_{it}(G) \geq \gamma(G) + 1$.*

Proposition 5.7. *Let G be a connected graph of order n and let D be a $\gamma(G)$-set. If $\beta_0(G) \geq (n - \gamma(G) + 1)/2$ and $V(G) - D$ contains exactly k $\beta_0(G)$-sets, then $\gamma_{it}(G) \leq \gamma(G) + k$. Moreover, if $k \in \{2, 3\}$, then $\gamma_{it}(G) \leq \gamma(G) + 1$.*

Proof. The inequality $\gamma_{it}(G) \leq \gamma(G)+k$ is obvious. Now, let $I_1, .., I_k$ be all $\beta_0(G)$-sets each of which is contained in $V(G) - D$. If I_1 and I_2 are vertex-disjoint, then $n \geq |D| + |I_1| + |I_2| = \gamma(G) + 2\beta_0(G) \geq n + 1$, a contradiction. Hence, $I_1 \cap I_2$ is not empty and, if $k = 2$ and $v \in I_1 \cap I_2$, then $D \cup \{v\}$ is an independent transversal dominating set of cardinality $\gamma(G) + 1$.

Let us consider the case where $k = 3$. As above the sets $S_1 = I_1 \cap I_2$, $S_2 = I_1 \cap I_3$ and $S_3 = I_2 \cap I_3$ all are nonempty. Furthermore, clearly $S_1 \cup S_2 \cup S_3$ is an independent set of G. Suppose $I_1 \cap I_2 \cap I_3$ is empty. Thus, $n \geq |D| + (|I_1| + |I_2| + |I_3|) - (|S_1| + |S_2| + |S_3|) \geq \gamma(G) + 3\beta_0(G) - \beta_0(G) \geq n + 1$, a contradiction. Hence, $I_1 \cap I_2 \cap I_3$ is not empty and, if $u \in I_1 \cap I_2 \cap I_3$, then $D \cup \{u\}$ is an independent transversal dominating set of cardinality $\gamma(G) + 1$. □

Our next theorem requires the use of the following known result from [12].

Lemma 5.8 (Levit and Mandrescu [12]). *Let G be a graph with $\beta_0(G) > |V(G)|/2$. If $|isol(G)| \neq 1$, then $\xi(G) \geq 2$.*

Theorem 5.9. *Let G be a graph with $\beta_0(G) > |V(G)|/2$. Then, $\gamma_{it}(G) \leq \gamma(G) + 1$.*

Proof. If v is an isolated vertex of G then $v \in core(G)$ and the result follows by Observation 5.1. If G has no isolated vertices, then by Lemma 5.8, $core(G)$ is not empty. Hence, again Observation 5.1 leads to $\gamma_{it}(G) \leq \gamma(G) + 1$. □

The following conjecture is due to Hamid, and appears in [10].

Conjecture 5.10. *If G is a connected bipartite graph, then $\gamma_{it}(G)$ is either $\gamma(G)$ or $\gamma(G) + 1$.*

Next corollary shows that Conjecture 5.10 is true for at least all bipartite graphs of odd order.

Proposition 5.11. *Let G be a bipartite graph with bipartition (X, Y) such that $|X| \neq |Y|$. Then, $\gamma_{it}(G) \leq \gamma(G) + 1$. In particular, this is true when G has odd order.*

Proof. Since $|X| \neq |Y|$, it follows $\beta_0(G) > |V(G)|/2$. Thus, the result is deduced directly from Theorem 5.9. □

5.1. Trees

By Theorem 2.3, the next corollary immediately follows.

Corollary 5.12. *Let T be a tree with bipartition (X, Y) such that $1 \leq |X| \leq |Y|$ and $\gamma(T) = |X|$. Then, T is in class 1 if and only if there is a vertex in X which is adjacent to at most one end-vertex.*

Proposition 5.13. *Let T be a tree of order at least three. Then T is in class 1 if one of the following holds:*

(i) *$V(T) = End(T) \cup Stem(T)$ and there is a path x_1, y_1, y_2, x_2 where y_1 and y_2 are stems and x_i is the only end-vertex which is adjacent to y_i, $i = 1, 2$.*

(ii) *$\langle V(T) - (End(T) \cup Stem(T))\rangle \cong \overline{K}_r$ and there is a path x, y, z such that y is a stem, z is not a stem, and x is the only end-vertex which is adjacent to y.*

Proof. (i) First note that $Stem(T)$ is a $\gamma(T)$-set and $V(T) - Stem(T)$ is a $\beta_0(T)$-set. On the other hand $D = (Stem(T) - \{y_1, y_2\}) \cup \{x_1, x_2\}$ is also a $\gamma(T)$-set, while $V(T) - D$ is not independent.

(ii) Clearly $Stem(T)$ is a $\gamma(T)$-set and $V(T) - Stem(T)$ is a $\beta_0(T)$-set. Now, $D = (Stem(T) \cup \{x\}) - \{y\}$ is a $\gamma(T)$-set while $V(G) - D$ is not independent. □

5.2. Independent Transversal Dominating Sets Versus Cliques

A *clique* K of a graph G is a subset of its vertices such that every two vertices in the subset are connected by an edge. Equivalently, a clique induces a complete subgraph having at least two vertices which is maximal under inclusion. (According to this definition, isolated vertices are maximal complete subgraphs but not cliques)

A vertex set meeting all cliques will be called a *clique-transversal*. The *clique-transversal number*, $\tau_C(G)$ is defined as the minimum cardinality of a clique-transversal in G. The concept of clique transversal has been already mentioned by Payan [14] in 1979 and the first NP-hardness results for clique transversals are due to Erdős, Gallai and Tuza [5].

Theorem 5.14 (Erdős, Gallai, Tuza [5]). *Let k and n be natural numbers, $n \geq k+2$. If G is a graph on n vertices in which every clique has more than k vertices, then $\tau_C(G) \leq n - \sqrt{kn}$, unless $k = 1$, $n = 5$, and G is the cycle of length* 5.

Now, by using the above known theorem we give a result for the independent transversal domination number of graphs.

Proposition 5.15. *Let k and n be natural numbers, $n \geq k+2$. If G is a non complete graph on n vertices in which every maximal independent set has more than k vertices, then $\gamma_{it}(G) \leq n - \sqrt{kn} + \gamma(G)$.*

Proof. First we note that a set I is a maximal independent set of G if and only if I is a clique of $\overline{G}$. Hence, any clique-transversal set of $\overline{G}$ intersects every maximal independent set in G. Now, if F is a clique-transversal set of largest cardinality in $\overline{G}$ and D is a γ-set of G, then clearly $D \cup F$ intersects every independent set of maximum cardinality in G. If $G \not\cong C_5$, then by using Theorem 5.15, we have that $n - \sqrt{kn} + \gamma(G) \geq |F| + |D| \geq |F \cup D| \geq \gamma_{it}(G)$. Finally, if $G \cong C_5$, then the result is obvious. □

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