Filomat 30:11 (2016), 2925–2934
DOI 10.2298/FIL1611925D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Fixed Point Results of Single-Valued Mapping for c-Distance in Tvs-Cone Metric Spaces

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Abstract. The purpose of this paper is to prove some fixed point theorems in a tvs-cone metric space using c-distances. The presented theorems extend, generalize and improve the corresponding results of Fadail et al. [10], Dordevic et al. [8] and the results cited therein under the continuity condition for maps. In the last section we give an examples in support of our theorems.

1. Introduction

It is well known that the classic contraction mapping principle of Banach is a fundamental result in fixed point theory. Several authors have obtained various extensions and generalizations of Banach's theorem by considering contractive mappings on many different metric spaces. In 2007, Huang and Zhang [16] first introduced the concept of cone metric spaces. Cone metric spaces is a generalized version of metric spaces, where each pair of points is assigned to a member of a real Banach space over the cone. They also established and proved the existence of fixed point theorems which is an extension of the Banach's contraction mapping principle in to cone metric spaces. Later, many authors have generalized and studied fixed point theorems in cone metric spaces (see[1], [2], [4], [17]).

In 2010, Du [9] introduced the concept of tvs-cone metric spaces which is improved version of cone metric spaces [16]. Afterward many authors (see [3, 8, 21, 25]) have generalized and proved the fixed point results in tvs-cone metric spaces (in both the cases, that is either the underlying cone of an ordered tvs is solid and normal or with the underlying cone which are not normal); see also [17] for a survey of fixed point results in these spaces.

Recently, Wang and Guo [29] introduced the concept of c-distance in a cone metric spaces (also see [6]) and proved some fixed point theorems in ordered cone metric spaces. This is cone metric version of w-distance of Kada et al.[19]. Then several authors have proved fixed point theorems for c-distance in cone metric spaces (see[10],[11],[12],[13],[14],[15],[26],[27]).

In [10], Fadail et al. proved the following theorems for c-distance in cone metric spaces-

Theorem 1. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Let $f : X \to X$ be a mapping and suppose that there exists mapping $k : X \to [0, 1)$ such that the following hold:*

2010 *Mathematics Subject Classification*. 47H09, 47H10
Keywords. tvs-cone metric spaces, c-distance, w-distance
Received: 25 June 2014; Accepted: 27 August 2016
Communicated by Dragan S. Djordjević
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(a) $k(fx) \leq k(x)$ *for all* $x \in X$,

(b) $q(fx, fy) \leq k(x)q(x, y)$ *for all* $x, y \in X$.

Then f *has a fixed point* $x^* \in X$ *and for any* $x \in X$, *iterative sequence* $\{f^n x\}$ *converges to the fixed point. If* $v = fv$, *then* $q(v, v) = \theta$. *The fixed point is unique.*

Theorem 2. *Let* (X, d) *be a complete cone metric space and* q *is a c-distance on* X. *Let* $f : X \to X$ *be a continuous mapping and suppose that there exists mapping* $k, l, r : X \to [0, 1)$ *such that the following hold:*

(a) $k(fx) \leq k(x), l(fx) \leq l(x), r(fx) \leq r(x)$ *for all* $x \in X$,

(b) $(k + l + r)(x) < 1$ *for all* $x \in X$,

(c) $q(fx, fy) \leq k(x)q(x, y) + l(x)q(x, fx) + r(x)q(y, fy)$ *for all* $x, y \in X$.

Then f *has a fixed point* $x^* \in X$ *and for any* $x \in X$, *iterative sequence* $\{f^n x\}$ *converges to the fixed point. If* $v = fv$, *then* $q(v, v) = \theta$. *The fixed point is unique.*

Theorem 3. *Let* (X, d) *be a complete cone metric space and* q *is a c-distance on* X. *Let* $f : X \to X$ *be a mapping and suppose that there exists mapping* $k, l, r : X \to [0, 1)$ *such that the following hold:*

(a) $k(fx) \leq k(x), l(fx) \leq l(x), r(fx) \leq r(x)$ *for all* $x \in X$,

(b) $(2k + l + r)(x) < 1$ *for all* $x \in X$,

(c) $(1 - r(x))q(fx, fy) \leq k(x)q(x, fy) + l(x)q(x, fx)$ *for all* $x, y \in X$.

Then f *has a fixed point* $x^* \in X$ *and for any* $x \in X$, *iterative sequence* $\{f^n x\}$ *converges to the fixed point. If* $v = fv$, *then* $q(v, v) = \theta$. *The fixed point is unique.*

The purpose of this paper is to extend and generalize some fixed point results on c-distance in tvs-cone metric spaces (with the underlying cone which is not normal).

2. Preliminaries

The following definitions and results will be needed in the sequel. Throughout this paper we assume $\mathbb{R}$ as a set of real numbers and $\mathbb{N}$ as a set of natural numbers.

Let E be a tvs with the zero vector θ. A nonempty and closed subset P of E is called a cone if $P + P \subseteq P$ and $\lambda P \subseteq P$ for $\lambda \geq 0$. A cone P is said to be proper if $P \bigcap (-P) = \theta$. For a given cone $P \subseteq E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$; $x \prec y$ will stand for $x \prec y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is said to be solid if it has a nonempty interior. The pair (E, P) is an ordered topological vector space.

Ordered topological vector space (E, P) is order convex if it has a base of neighborhoods of θ consisting of order-convex subsets. In this case, the cone P is said to be normal. If E is normed space, this condition means that the unit ball is order-convex which is equivalent to the condition that there is a number K such that $x, y \in E$ and $0 \leq x \leq y$ implies that $\|x\| \leq K\|y\|$.

Definition 1 ([5, 9, 20]). *Let* X *be a nonempty set and* (E, P) *an ordered tvs. A vector-valued function* $d : X \times X \to E$ *is said to be a tvs-cone metric, if the following conditions hold:*

(C_1) $\theta \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$;

(C_2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(C_3) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

Then the pair (X, d) *is called a tvs-cone metric space.*

Definition 2 ([5, 9, 20]). *Let (X, d) be a tvs-cone metric space, $x \in X$ and let $\{x_n\}$ be a sequence in X. Then*

(i) *$\{x_n\}$ tvs-cone converges to x whenever for every $c \in E$ with $\theta \ll c$ there is a natural number n_0 such that $d(x_n, x) \ll c$, for all $n \geq n_0$. We denote this by $lim_{n\to\infty} x_n = x$;*

(ii) *$\{x_n\}$ is a tvs-cone Cauchy sequence whenever for every $c \in E$ with $\theta \ll c$ there is a natural number n_0 such that $d(x_n, x_m) \ll c$, for all $n, m \geq n_0$;*

(iii) *(X, d) is tvs-cone complete if every tvs-cone Cauchy sequence in X is tvs- cone convergent.*

Let (X, d) be a tvs-cone metric space. The following properties are often used, particularly in the case when the underlying cone is non-normal:

(P_1) *If $u, v, w \in E$, $u \leq v$ and $v \ll w$ then $u \ll w$.*

(P_2) *If $u \in E$ and $\theta \leq u \ll c$ for each $c \in intP$ then $u = \theta$.*

(P_3) *If $u_n, v_n, u, v \in E$, $\theta \leq u_n \leq v_n$ for each $n \in \mathbb{N}$, and $u_n \to u$, $v_n \to v$ $(n \to \infty)$, then $\theta \leq u \leq v$.*

(P_4) *If $x_n, x \in X$, $u_n \in E$, $d(x_n, x) \leq u_n$ and $u_n \to \theta$ $(n \to \infty)$, then $x_n \to x$ $(n \to \infty)$.*

(P_5) *If $u \leq \lambda u$, where $u \in P$ and $0 \leq \lambda < 1$, then $u = \theta$.*

(P_6) *If $c \gg \theta$ and $u_n \in E$, $u_n \to \theta$ $(n \to \infty)$, then there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq n_0$.*

Next, we give the notion of c-distance on a cone metric space (X, d) of Wang and Guo in [29] which is a generalization of w-distance of Kada et al. [19] and some properties of Cho et al. [6] converted it to the setting of cone metric spaces.

Definition 3 ([6]). *Let (X, d) be a tvs-cone metric space. A function $q : X \times X \to E$ is called a c-distance in X if the following conditions hold:*

(q_1) *$\theta \leq q(x, y)$ for all $x, y \in X$,*

(q_2) *$q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$,*

(q_3) *If a sequence $\{y_n\} \in X$ converges to a point $y \in X$, and for some $x \in X$ and $u = u_x \in P$, $q(x, y_n) \leq u$ holds for each $n \in \mathbb{N}$, then $q(x, y) \leq u$,*

(q_4) *for each $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \leq e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ implies $d(x, y) \ll c$.*

Example 1 ([8]). *Let (X, d) be a tvs-cone metric space such that the metric $d(., .)$ is a continuous function in second variable. Then $q(x, y) = d(x, y)$ is a c-distance. Indeed, only property (q_3) is non-trivial and it follows from $q(x, y_n) = d(x, y_n) \leq u$, passing to the limit when $n \to \infty$ and using continuity of d.*

Example 2 ([8]). *Let (X, d) be a tvs-cone metric space and let $u \in X$ be fixed. Then, $q(x, y) = d(u, y)$ defines a c-distance on X. Indeed, (q_1) and (q_3) are clear, (q_2) follows from $q(x, z) = d(u, z) \leq d(u, y) + d(u, z) = q(x, y) + q(y, z)$. Finally, ($q_4$) is obtained by taking $e = \frac{c}{2}$.*

Definition 4 ([8]). *A sequence $\{u_n\}$ in P is a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$. It is easy to show that if $\{u_n\}$ and $\{v_n\}$ are c-sequences in E and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence.*

The following lemma is a tvs-cone metric version of lemmas from [6, 19].

Lemma 1 ([8]). *Let (X, d) be a tvs-cone metric space and let q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be a sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are c-sequences in P. Then the following hold:*

(1) *If* $q(x_n, y) \preceq u_n$ *and* $q(x_n, z) \preceq v_n$ *for* $n \in \mathbb{N}$, *then* $y = z$. *In particular, if* $q(x, y) = \theta$ *and* $q(x, z) = \theta$, *then* $y = z$.

(2) *If* $q(x_n, y_n) \preceq u_n$ *and* $q(x_n, z) \preceq v_n$ *for* $n \in \mathbb{N}$, *then* $\{y_n\}$ *converges to* z.

(3) *If* $q(x_n, x_m) \preceq u_n$ *for* $m > n > n_0$, *then* $\{x_n\}$ *is a Cauchy sequence in* X.

(4) *If* $q(y, x_n) \preceq u_n$ *for* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is a Cauchy sequence in* X.

Remark 1 ([29]). (1) *If* $q(x, y) = q(y, x)$ *does not necessarily for all* $x, y \in X$;

(2) *If* $q(x, y) = \theta$ *is not necessarily equivalent to* $x = y$ *for all* $x, y \in X$.

Now we are ready to state and prove our main results.

3. Main Result

Theorem 4. *Let* (X, d) *be a complete tvs-cone metric space, and* q *is a c-distance on* X. *Let* $f : X \to X$ *be a continuous mapping and suppose that there exists mapping* $k, l : X \to [0, 1)$ *such that the following conditions hold:*

(*a*) $k(fx) \leq k(x)$, $l(fx) \leq l(x)$, *for all* $x \in X$;

(*b*) $(k + 2l)(x) < 1$ *for all* $x \in X$;

(*c*) (*i*) $q(fx, fy) \preceq k(x)q(x, y) + l(x)[q(fx, y) + q(x, fy)]$,
(*ii*) $q(fy, fx) \preceq k(y)q(y, x) + l(y)[q(y, fx) + q(fy, x)]$ *for all* $x, y \in X$.

Then the map f *has a unique fixed point* $x^* \in X$. *If* $v = fv$, *then* $q(v, v) = \theta$.

Proof. Let $x_0 \in X$ be arbitrary and form the sequence with $x_n = fx_{n-1} = f^n x_0$. In order to prove that it is a Cauchy sequence, put $x = x_n$ and $y = x_{n-1}$ in [(c)(i)] to get

$$\begin{aligned} q(fx_n, fx_{n-1}) &= q(x_{n+1}, x_n) \\ &\preceq k(x_n)q(x_n, x_{n-1}) + l(x_n)[q(fx_n, x_{n-1}) + q(x_n, fx_{n-1})] \\ &= k(fx_{n-1})q(x_n, x_{n-1}) + l(fx_{n-1})[q(x_{n+1}, x_{n-1})] \\ &\preceq k(x_{n-1})q(x_n, x_{n-1}) + l(x_{n-1})[q(x_{n+1}, x_n) + q(x_n, x_{n-1})] \end{aligned}$$

continuing in this manner, we can get,

$$q(x_{n+1}, x_n) \preceq k(x_0)q(x_n, x_{n-1}) + l(x_0)[q(x_{n+1}, x_n) + q(x_n, x_{n-1})], \tag{1}$$

Similarly putting $y = x_{n-1}$ and $x = x_n$ in [(c)(ii)] to get,

$$\begin{aligned} q(fx_{n-1}, fx_n) &= q(x_n, x_{n+1}) \\ &\preceq k(x_{n-1})q(x_{n-1}, x_n) + l(x_{n-1})[q(x_{n-1}, fx_n) + q(fx_{n-1}, x_n)] \\ &= k(fx_{n-2})q(x_{n-1}, x_n) + l(fx_{n-2})[q(x_{n-1}, x_{n+1})] \\ &\preceq k(x_{n-2})q(x_{n-1}, x_n) + l(x_{n-2})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \end{aligned}$$

continuing in this manner, we can get,

$$q(x_n, x_{n+1}) \preceq k(x_0)q(x_{n-1}, x_n) + l(x_0)[q(x_{n-1}, x_n) + q(x_n, x_{n+1})], \tag{2}$$

denote $v_n = q(x_{n+1}, x_n) + q(x_n, x_{n+1})$. Adding equations (1) and (2), we get,

$$v_n \preceq (k(x_0) + l(x_0))v_{n-1} + l(x_0)v_n,$$

i.e.

$$v_n \preceq hv_{n-1} \quad \text{with} \quad 0 \leq h = \frac{k(x_0) + l(x_0)}{1 - l(x_0)} < 1,$$

since $(k+2l)(x) < 1$ for all $x \in X$. By induction

$$v_n \preceq h^n v_0 \quad \text{and} \quad q(x_n, x_{n+1}) \preceq v_n \preceq h^n(q(x_1, x_0) + q(x_0, x_1)).$$

Then it follows that

$$q(x_n, x_m) \preceq \frac{h^n}{1-h}(q(x_1, x_0) + q(x_0, x_1)) = u_n,$$

for $m > n$, where $\{u_n\}$ is a c-sequence. Thus, Lemma 1 (3) implies that $\{x_n\}$ is a Cauchy sequence in X and since X is complete, $x_n \to x^* \in X$ ($n \to \infty$). Since f is continuous then $x_{n+1} = fx_n \to fx^*$, and since the limit of a sequence in tvs-cone metric space is unique, we get that $fx^* = x^*$, i.e. x^* is a unique fixed point of f. Suppose that $fv = v$. Then [(c)(i)] implies that,

$$\begin{aligned} q(v,v) &= q(fv, fv) \\ &\preceq k(x_0)q(v,v) + l(x_0)[q(fv,v) + q(fv,v)] \\ &= (k+2l)(x_0)q(v,v). \end{aligned}$$

Since $(k+2l)(x_0) < 1$ and by property (P_5) shows that $q(v,v) = \theta$. Finally, suppose there is another fixed point y^* of f, then we have,

$$\begin{aligned} q(x^*, y^*) &= q(fx^*, fy^*) \\ &\preceq k(x^*)q(x^*, y^*) + l(x^*)[q(fx^*, y^*) + q(x^*, fy^*)] \\ &= k(x^*)q(x^*, y^*) + l(x^*)[q(x^*, y^*) + q(x^*, y^*)] \\ &= (k+2l)(x^*)q(x^*, y^*). \end{aligned}$$

Since $(k+2l)(x^*) < 1$ and by property (P_5) shows that $q(x^*, y^*) = \theta$ and also we have $q(x^*, x^*) = \theta$, hence by Lemma 1(1), $x^* = y^*$. Therefore the fixed point is unique. □

Corollary 1. *Let (X, d) be a complete tvs-cone metric space and q is c-distance on X. Let $f : X \to X$ be a continuous mapping and suppose that there exists mapping $k, l : X \to [0,1)$ such that the following conditions hold:*

(*a*) $k(fx) \preceq k(x), l(fx) \preceq l(x)$, *for all* $x \in X$;

(*b*) $(k+2l)(x) < 1$ *for all* $x \in X$;

(*c*) (*i*) $q(fx, fy) \preceq k(x)q(x,y) + l(x)[q(x, fx) + q(y, fy)]$

(*ii*) $q(fy, fx) \preceq k(y)q(y,x) + l(y)[q(fx, x) + q(fy, y)]$ *for all* $x, y \in X$.

Then the map f has a unique fixed point x^* in X. If $v = fv$, then $q(v,v) = \theta$.

Theorem 5. *Let (X, d) be a complete tvs-cone metric space and q is c-distance on X. Let $f : X \to X$ be a continuous mapping and suppose that there exists mapping $k, l, r : X \to [0,1)$ such that the following conditions hold:*

(*a*) $k(fx) \preceq k(x), l(fx) \preceq l(x), r(fx) \preceq r(x)$, *for all* $x \in X$;

(*b*) $(k+2l+2r)(x) < 1$ *for all* $x \in X$;

(*c*) (*i*) $q(fx, fy) \preceq k(x)q(x,y) + l(x)[q(x, fy) + q(y, fx)] + r(x)[q(x, fx) + q(y, fy)]$,

(*ii*) $q(fy, fx) \preceq k(y)q(y,x) + l(y)[q(fy, x) + q(fx, y)] + r(y)[q(fx, x) + q(fy, y)]$ *for all* $x, y \in X$.

Then the map f has a unique fixed point x^* in X. If $v = fv$, then $q(v,v) = \theta$.

Proof. Let $x_0 \in X$ be arbitrary and form the sequence with $x_n = fx_{n-1} = f^n x_0$. In order to prove that it is a Cauchy sequence, put $x = x_n$ and $y = x_{n-1}$ in [(c)(i)] to get

$$\begin{aligned} q(fx_n, fx_{n-1}) &= q(x_{n+1}, x_n) \\ &\preceq k(x_n)q(x_n, x_{n-1}) + l(x_n)[q(x_n, fx_{n-1}) + q(x_{n-1}, fx_n)] + r(x_n)[q(x_n, fx_n) + q(x_{n-1}, fx_{n-1})] \\ &= k(fx_{n-1})q(x_n, x_{n-1}) + l(fx_{n-1})[q(x_n, x_n) + q(x_{n-1}, x_{n+1})] + r(fx_{n-1})[q(x_n, x_{n+1}) + q(x_{n-1}, x_n)] \\ &\preceq k(x_{n-1})q(x_n, x_{n-1}) + l(x_{n-1})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] + r(x_{n-1})[q(x_n, x_{n+1}) + q(x_{n-1}, x_n)], \end{aligned}$$

continuing in this manner we can get,

$$q(x_{n+1}, x_n) \preceq k(x_0)q(x_n, x_{n-1}) + l(x_0)[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] + r(x_0)[q(x_n, x_{n+1}) + q(x_{n-1}, x_n)]. \quad (3)$$

Similarly, putting $y = x_{n-1}$ and $x = x_n$ in [(c)(ii)] to get,

$$\begin{aligned} q(fx_{n-1}, fx_n) &= q(x_n, x_{n+1}) \\ &\preceq k(x_{n-1})q(x_{n-1}, x_n) + l(x_{n-1})[q(fx_{n-1}, x_n) + q(fx_n, x_{n-1})] + r(x_{n-1})[q(fx_n, x_n) + q(fx_{n-1}, x_{n-1})] \\ &= k(fx_{n-2})q(x_{n-1}, x_n) + l(fx_{n-2})[q(x_n, x_n) + q(x_{n+1}, x_{n-1})] + r(fx_{n-2})[q(x_{n+1}, x_n) + q(x_n, x_{n-1})] \\ &\preceq k(x_{n-2})q(x_{n-1}, x_n) + l(x_{n-2})[q(x_{n+1}, x_n) + q(x_n, x_{n-1})] + r(x_{n-2})[q(x_{n+1}, x_n) + q(x_n, x_{n-1})], \end{aligned}$$

continuing in this manner we can get,

$$q(x_n, x_{n+1}) \preceq k(x_0)q(x_{n-1}, x_n) + l(x_0)[q(x_{n+1}, x_n) + q(x_n, x_{n-1})] + r(x_0)[q(x_{n+1}, x_n) + q(x_n, x_{n-1})], \quad (4)$$

denote $v_n = q(x_{n+1}, x_n) + q(x_n, x_{n+1})$. Adding equations (3) and (4), we get,

$$v_n \preceq (k(x_0) + l(x_0) + r(x_0))v_{n-1} + (l(x_0) + r(x_0))v_n,$$

i.e. $v_n \preceq hv_{n-1}$ with

$$0 \leq h = \frac{k(x_0) + l(x_0) + r(x_0)}{1 - l(x_0) - r(x_0)} < 1,$$

since $(k + 2l + 2r)(x) < 1$ for all $x \in X$. By induction $v_n \preceq h^n v_0$ and

$$q(x_n, x_{n+1}) \preceq v_n \preceq h^n(q(x_1, x_0) + q(x_0, x_1)).$$

Then it follows that,

$$q(x_n, x_m) \preceq \frac{h^n}{1-h}(q(x_1, x_0) + q(x_0, x_1)) = u_n,$$

for $m > n$, where $\{u_n\}$ is a c-sequence. Thus, Lemma 1(3) implies that $\{x_n\}$ is Cauchy sequence in X and since X is complete, $x_n \to x^* \in X$ $(n \to \infty)$. Since f is continuous, then $x_{n+1} = fx_n \to fx^*$, and since the limit of a sequence in tvs-cone metric space is unique, we get that $fx^* = x^*$, i.e. x^* is a unique fixed point of f. Suppose that $fv = v$. Then by [(c)(i)] we have,

$$\begin{aligned} q(v, v) &= q(fv, fv) \\ &\preceq k(x_0)q(v, v) + l(x_0)[q(v, fv) + q(v, fv)] + r(x_0)[q(v, fv) + q(v, fv)] \\ &= (k + 2l + 2r)(x_0)q(v, v), \end{aligned}$$

since $(k + 2l + 2r)(x_0) < 1$ and by property (P_5) implies that $q(v, v) = \theta$. Finally, for the uniqueness of the fixed point, suppose there is another fixed point y^* of f, then we have,

$$\begin{aligned} q(x^*, y^*) &= q(fx^*, fy^*) \\ &\preceq k(x^*)q(x^*, y^*) + l(x^*)[q(x^*, fy^*) + q(y^*, fx^*)] + r(x^*)[q(x^*, fx^*) + q(y^*, fy^*)] \\ &= k(x^*)q(x^*, y^*) + l(x^*)[q(x^*, y^*) + q(y^*, x^*)] + r(x^*)[q(x^*, x^*) + q(y^*, y^*)] \\ &= (k + 2l)(x^*)q(x^*, y^*), \end{aligned}$$

since $(k + 2l + 2r)(x^*) < 1$ and by property (P_5) shows that $q(x^*, y^*) = \theta$ and also we have $q(x^*, x^*) = \theta$ hence by Lemma 1(1), $x^* = y^*$. Therefore the fixed point is unique. □

Corollary 2. *Let (X, d) be a complete tvs-cone metric space and q is c-distance on X. Let $f : X \to X$ be a continuous mapping and suppose that there exists mapping $k, l, r : X \to [0, 1)$ such that the following conditions hold:*

(*a*) $k(fx) \le k(x), l(fx) \le l(x), r(fx) \le r(x)$ *for all* $x \in X$;

(*b*) $(k + 2l + 2r)(x) < 1$ *for all* $x \in X$;

(*c*) (*i*) $q(fx, fy) \le k(x)q(x, y) + l(x)[q(x, fx) + q(x, fy)] + r(x)[q(y, fx) + q(y, fy)]$

(*ii*) $q(fy, fx) \le k(y)q(y, x) + l(y)[q(fx, x) + q(fy, x)] + r(y)[q(fx, y) + q(fy, y)]$ *for all* $x, y \in X$.

Then the map f has a unique fixed point x^* in X. If $v = fv$, then $q(v, v) = \theta$.

Theorem 6. *Let (X, d) be a complete tvs-cone metric space and q is c-distance on X. Let $f : X \to X$ be a continuous mapping and suppose that there exists mapping $k, r, l, t : X \to [0, 1)$ such that the following conditions hold:*

(*a*) $k(fx) \le k(x), r(fx) \le r(x), l(fx) \le l(x)$ *and* $t(fx) \le t(x)$ *for all* $x \in X$;

(*b*) $(k + l + r + 2t)(x) < 1$*for all* $x \in X$;

(*c*) (*i*) $q(fx, fy) \le k(x)q(x, y) + r(x)q(fx, x) + l(x)q(fy, y) + t(x)[q(fx, y) + q(fy, x)]$

(*ii*) $q(fy, fx) \le k(y)q(y, x) + r(y)q(x, fx) + l(y)q(y, fy) + t(y)[q(y, fx) + q(x, fy)]$ *for all* $x, y \in X$.

Then the map f has a unique fixed point x^* in X. If $v = fv$, then $q(v, v) = \theta$.

Proof. Let $x_0 \in X$ be arbitrary and form the sequence with $x_n = fx_{n-1} = f^n x_0$. In order to prove that it is a Cauchy sequence, put $x = x_n$ and $y = x_{n-1}$ in [(c)(i)] to get,

$$\begin{aligned}
q(fx_n, fx_{n-1}) &= q(x_{n+1}, x_n) \\
&\le k(x_n)q(x_n, x_{n-1}) + r(x_n)q(fx_n, x_n) + l(x_n)q(fx_{n-1}, x_{n-1}) + t(x_n)[q(fx_n, x_{n-1}) + q(fx_{n-1}, x_n)] \\
&= k(fx_{n-1})q(x_n, x_{n-1}) + r(fx_{n-1})q(x_{n+1}, x_n) + l(fx_{n-1})q(x_n, x_{n-1}) \\
&\quad + t(fx_{n-1})[q(x_{n+1}, x_{n-1}) + q(x_n, x_n)] \\
&\le k(x_{n-1})q(x_n, x_{n-1}) + r(x_{n-1})q(x_{n+1}, x_n) + l(x_{n-1})q(x_n, x_{n-1}) + t(x_{n-1})[q(x_{n+1}, x_n) + q(x_n, x_{n-1})];
\end{aligned}$$

continuing in this manner we can get,

$$q(x_{n+1}, x_n) \le k(x_0)q(x_n, x_{n-1}) + r(x_0)q(x_{n+1}, x_n) + l(x_0)q(x_n, x_{n-1}) + t(x_0)[q(x_{n+1}, x_n) + q(x_n, x_{n-1})]. \tag{5}$$

Similarly, putting $y = x_{n-1}$ and $x = x_n$ in [(c)(ii)], to get

$$\begin{aligned}
q(fx_{n-1}, fx_n) &= q(x_n, x_{n+1}) \\
&\le k(x_{n-1})q(x_{n-1}, x_n) + r(x_{n-1})q(x_n, fx_n) + l(x_{n-1})q(x_{n-1}, fx_{n-1}) \\
&\quad + t(x_{n-1})[q(x_{n-1}, fx_n) + q(x_n, fx_{n-1})] \\
&= k(fx_{n-2})q(x_{n-1}, x_n) + r(fx_{n-2})q(x_n, x_{n+1}) + l(fx_{n-2})q(x_{n-1}, x_n) \\
&\quad + t(fx_{n-2})[q(x_{n-1}, x_{n+1}) + q(x_n, x_n)] \\
&\le k(x_{n-2})q(x_{n-1}, x_n) + r(x_{n-2})q(x_n, x_{n+1}) + l(x_{n-2})q(x_{n-1}, x_n) + t(x_{n-2})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})],
\end{aligned}$$

continuing in this manner we can get,

$$q(x_n, x_{n+1}) \le k(x_0)q(x_{n-1}, x_n) + r(x_0)q(x_n, x_{n+1}) + l(x_0)q(x_{n-1}, x_n) + t(x_0)[q(x_{n-1}, x_n) + q(x_n, x_{n+1})], \tag{6}$$

denote $v_n = q(x_{n+1}, x_n) + q(x_n, x_{n+1})$. Adding equations (5) and (6), we get,

$$v_n \le (k(x_0) + l(x_0) + t(x_0))v_{n-1} + (r(x_0) + t(x_0))v_n,$$

i.e.

$$v_n \preceq hv_{n-1} \quad \text{with} \quad 0 \leq h = \frac{k(x_0)+l(x_0)+t(x_0)}{1-r(x_0)-t(x_0)} < 1,$$

since $(k+r+l+2t)(x) < 1$ for all $x \in X$. By induction $v_n \preceq h^n v_0$ and

$$q(x_n, x_{n+1}) \preceq v_n \preceq h^n(q(x_1,x_0)+q(x_0,x_1)).$$

Then it follows that,

$$q(x_n, x_m) \preceq \frac{h^n}{1-h}(q(x_1,x_0)+q(x_0,x_1)) = u_n,$$

for $m > n$, where $\{u_n\}$ is a c-sequence. Thus Lemma 1(3) implies that $\{x_n\}$ is Cauchy sequence in X and since X is complete, $x_n \to x^* \in X$ $(n \to \infty)$. Since f is continuous, then $x_{n+1} = fx_n \to fx^*$, and since the limit of a sequence in tvs-cone metric space is unique, we get that $fx^* = x^*$, i.e. x^* is a unique fixed point of f. Suppose that $fv = v$. Then by [(c)(i)] we have,

$$\begin{aligned} q(v,v) &= q(fv,fv) \\ &\preceq k(x_0)q(v,v) + r(x_0)q(fv,v) + l(x_0)q(fv,v) + t(x_0)[q(fv,v)+q(fv,v)] \\ &= (k+r+l+2t)(x_0)q(v,v), \end{aligned}$$

since $(k+r+l+2t)(x_0) < 1$ and by property (P_5) implies that $q(v,v) = \theta$.

Proof of the uniqueness of the fixed point is similar as shown in the previous theorems. □

Example 3. *Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0,1]$ and define a mapping $d : X \times X \to E$ by $d(x,y) = |x-y|$ for all $x, y \in X$. Then (X,d) is a complete cone metric space. Define a mapping $q : X \times X \to E$ by $q(x,y) = 2d(x,y)$ for all $x, y \in X$. Then q is c-distance on X (cone metric version).*
Let $f : X \to X$ defined by $f(x) = \frac{x^2}{16}$ for all $x \in X$. Take mappings $k, r, l, t : X \to [0,1)$ by

$$k(x) = \frac{x+1}{16}, r(x) = \frac{2x+3}{16}, l(x) = \frac{3x+2}{16}, t(x) = \frac{x}{16}$$

for all $x \in X$. Observe that

(a) $k(fx) = \frac{(\frac{x^2}{16}+1)}{16} = \frac{1}{16}(\frac{x^2}{16}+1) \leq \frac{1}{16}(x+1) = k(x)$ *for all $x \in X$.*

(b) $r(fx) = \frac{(2(\frac{x^2}{16})+3)}{16} = \frac{1}{16}(\frac{2x^2}{16}+3) \leq \frac{1}{16}(2x+3) = r(x)$ *for all $x \in X$.*

(c) $l(fx) = \frac{(3(\frac{x^2}{16})+2)}{16} = \frac{1}{16}(\frac{3x^2}{16}+2) \leq \frac{1}{16}(3x+2) = l(x)$ *for all $x \in X$.*

(d) $t(fx) = \frac{(\frac{x^2}{16})}{16} = \frac{1}{16}(\frac{x^2}{16}) \leq \frac{1}{16}(x) = t(x)$ *for all $x \in X$.*

(e) $(k+r+l+2t)(x) = (\frac{x+1}{16}) + (\frac{2x+3}{16}) + (\frac{3x+2}{16}) + 2(\frac{x}{16}) = (\frac{8x+6}{16}) < 1$ *for all $x \in X$.*

(f) *for all $x, y \in X$, we have*

$$\begin{aligned} q(fx,fy) &= 2\Big|\frac{x^2}{16} - \frac{y^2}{16}\Big| \\ &= \frac{2|x+y||x-y|}{16} \\ &= \Big(\frac{x+y}{16}\Big)2|x-y| \\ &\leq \Big(\frac{x+1}{16}\Big)2|x-y| \\ &= k(x)q(x,y) \\ &\leq k(x)q(x,y) + r(x)q(fx,x) + l(x)q(fy,y) + t(x)[q(fx,y)+q(fy,x)]. \end{aligned}$$

In the similar manner

$$\begin{aligned} q(fy, fx) &= 2\Big|\frac{y^2}{16} - \frac{x^2}{16}\Big| \\ &= \frac{2|y+x||y-x|}{16} \\ &= \Big(\frac{y+x}{16}\Big)2|y-x| \\ &\leq \Big(\frac{y+1}{16}\Big)2|y-x| \\ &= k(y)q(y,x) \\ &\leq k(y)q(y,x) + r(y)q(x,fx) + l(y)q(y,fy) + t(y)[q(y,fx) + q(x,fy)]. \end{aligned}$$

Therefore, all the conditions of Theorem 6 are satisfied, which is real metric version with c-distance.

In above example, we take $X = [0, +\infty)$*, defined* $d : X \times X \to (E, \tau)$ *[*τ *is locally convex tvs] and* $d(x, y)(t) = 2|x-y|\phi(t)$ *for* $\phi \in P$*. Then,* (X, d) *is a tvs-cone metric space. Now we take two c-distance in this space;* $q_1(x, y)(t) = 2x\psi(t)$ *and* $q_2(x, y)(t) = 2y\psi(t)$*. Then* q_1 *and* q_2 *are c-distances on tvs-cone metric spaces.*

4. Conclusion

In this attempt, we prove some fixed point results in tvs-cone metric spaces. These results generalizes and improves the recent results of Fadail et al. [10] and Dordevic et al. [8] in the sense that in our results we employing c-distances and in contractive conditions, replacing the constants with functions, which extends the further scope of our results. For the usability of our results we present one interesting example in the last section of this paper.

5. Acknowledgements

The authors are thankful to the learned referee for his/her deep observations and their suggestions, which greatly helped us to improve the paper significantly. The second author gratefully acknowledges the support from the Chhattisgarh Council of Science and Technology, Raipur, India, Grant No. 1208/CCOST/MRP/2014.

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