



Asymptotic Analysis of the Lubrication Problem with Nonstandard Boundary Conditions for Microrotation

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Abstract. The purpose of this paper is to propose a new effective model describing lubrication process with incompressible micropolar fluid. Instead of usual zero Dirichlet boundary condition for the microrotation, we consider more general (and physically justified) type of boundary condition at the fluid-solid interface, linking the velocity and microrotation through a so-called boundary viscosity. Starting from the linearized micropolar equations, we derive the second-order effective model by means of the asymptotic analysis with respect to the film thickness. The resulting equations, in the form of the Brinkman-type system, clearly show the influence of new boundary conditions on the effective flow. We also discuss the rigorous justification of the obtained asymptotic model.

1. Introduction

Lubrication is mostly concerned with the behavior of a lubricant flowing through a narrow gap. If the gap between the moving surfaces becomes very small, the experimental results appearing in the lubrication literature (see e.g. [1, 10, 12, 13]) indicate that the fluid's internal structure and the intrinsic motion of its particles must be taken into account. A possible way to introduce the obtained experimental facts is to employ micropolar fluid model. Proposed by Eringen [9] in 60's, the model of micropolar fluid has gained much attention since it captures the effects of local structure and micro-motions of the fluid elements that cannot be described by the classical models. Physically, it represents fluids consisting of rigid, spherical particles suspended in a viscous medium, where the deformation of fluid particles is ignored. The related mathematical model is based on the introduction of new vector field, the angular velocity field of rotation of particles (microrotation), to the classical pressure and velocity fields. Correspondingly, one new (vector) equation is added, expressing the conservation of the angular momentum. As a result, a nonlinear coupled system of PDEs is obtained, representing a significant generalization of the Navier-Stokes equations with four new viscosities introduced. We refer the reader to the monograph [11] which provides a unified picture of the mathematical theory underlying this particular model.

To close up the governing problem, one should specify the reasonable boundary conditions for the velocity and microrotation. While the classical no-slip condition for the velocity is widely used (and physically

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clear), the situation is much more complicated for the microrotation as it reflects the fluid-solid interaction. Throughout the literature, using simple zero Dirichlet boundary condition for the microrotation¹⁾ has been a common practice. The reason lies in the fact that not much has been done in proving the well-posedness of the corresponding boundary-value problem for different types of boundary conditions. Recently, in [2], the micropolar flow associated with another type of boundary condition for microrotation has been treated from the mathematical point of view. This new type of boundary condition is based on the concept of the so-called *boundary viscosity* and was originally proposed in [6, 7]. As such, it is much more physically justified than the simple zero boundary condition. It links the value of microrotation with the rotation of the velocity in the following manner:

$$\mathbf{w} \times \mathbf{n} = \frac{\alpha}{2} (\text{rot } \mathbf{u}) \times \mathbf{n} \quad (1)$$

where \mathbf{n} denotes normal unit vector to the boundary. The coefficient α characterizes the microrotation on the solid surfaces and is computed from a boundary viscosity value and other viscosity coefficients (see (14)). The authors in [2] managed to prove the existence and uniqueness of the corresponding weak solution providing the well-posedness of the governing problem. It is shown that, in such setting, classical no-slip condition for the velocity should be replaced by the condition allowing the slippage at the wall:

$$(\mathbf{u} - \mathbf{s}) \times \mathbf{n} = \delta (\text{rot } \mathbf{w}) \times \mathbf{n}, \quad \mathbf{u} \cdot \mathbf{n} = 0. \quad (2)$$

Here \mathbf{s} is the given velocity of the wall, while δ is a real parameter allowing the control of the slippage. Such fundamental result enables us to perform an asymptotic study of the micropolar flow associated with new boundary conditions (1)-(2).

The goal of this paper is to derive the new asymptotic model for lubrication process explicitly acknowledging the effects of boundary conditions (1)-(2). We study the situation appearing naturally in numerous engineering applications consisting of moving machine parts: two rigid surfaces being in relative motion are separated by a thin layer of fluid, lower surface is assumed to be perfectly smooth, while the upper is rough with roughness described by some function h . We start from linearized micropolar equations posed in a thin three-dimensional domain describing real physical situation. Our method relies on the technique of two-scale asymptotic expansion of the solution in powers of the small parameter ε , where εh represents the film thickness. Noticing the analogy between porous medium flow and thin film flow²⁾, we follow the approach recently proposed in [15]. Instead of computing only the first term, we compute the successive terms in the asymptotic expansion of the solution in a way that they have *zero mean value*. That requirement forces us to correct the macroscopic equation and, as a result, we obtain new second-order model governing the flow. It has the form of the Brinkman-type system³⁾ clearly showing the influence of fluid microstructure and new boundary conditions on the effective flow. As far as we know, such contribution cannot be found in the context of tribology.

There are not many papers dealing with mathematical modeling of lubrication with micropolar fluid. In [5] (see also [4]), the authors consider two-dimensional linearized problem in which the microrotation is a scalar function. Assuming that viscosity coefficients depend on small gap parameter, they rigorously derive the corrected version of the standard Reynolds equation. In [14], the corresponding 3D problem has been addressed with no assumptions made on the viscosity coefficients. The higher-order effective model

¹⁾It means that the fluid microelements cannot rotate on the solid surface.

²⁾In both cases we have low permeable domain so the domain permeability can be used as the small parameter.

³⁾H. Brinkmann [8] in 1947 modified the well-known Darcy law to be able to impose the no-slip boundary condition on an obstacle submerged in a porous medium. Brinkman law has the form:

$$\mu \mathbf{B} \mathbf{u} - \mu_B \Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0, \quad \mathbf{B} = \mathbf{K}^{-1}, \quad (\mathbf{K} - \text{permeability}).$$

is obtained and compared with the one obtained in [16] for classical Newtonian fluid film lubrication. In all those references it is assumed that

$$\mathbf{u} = \mathbf{s} = (s_1, s_2), \quad \mathbf{w} = 0 \tag{3}$$

at the boundary. In [2], the generalized micropolar Reynolds equation is derived in the case of new boundary conditions (1)-(2) using weak convergence method. The authors again start from the simplified 2D problem with ε -dependent viscosity coefficients and derive the effective model in the specific critical case. In this paper, we manage to obtain completely new effective model in the form of the Brinkman-type system (see Sec. 3.3) starting from the original 3D problem with no assumptions made on the viscosity coefficients. Therefore, we believe that our result could be instrumental for creating more efficient numerical algorithms explicitly acknowledging the microstructure effects on the lubrication process. Last but not least, in Section 4 we discuss rigorous justification of the formally obtained asymptotic model. We especially focus on the mathematical difficulties associated with non-periodic physical boundary conditions. We first prove an important auxiliary result by introducing the appropriate geometric tools (see Lemma 4.1). The result presented there is rather general and can be applied in different situations, especially those involving boundary layer phenomena. Then, we prove a priori estimates for the solution of the obtained Brinkman approximation (see Theorem 4.2) and clearly indicate the main technical difficulties which prevent us to derive satisfactory L^2 and H^1 estimates. We also offer possible ways to avoid those difficulties.

2. Description of the Problem

2.1. The Domain

We consider a micropolar fluid flow in a three-dimensional domain Ω^ε defined by (see Fig. 1):

$$\Omega^\varepsilon = \{x = (x', x_3) \in \mathbf{R}^3 : x' \in \mathcal{O}, 0 < x_3 < \varepsilon h(x')\} . \tag{4}$$

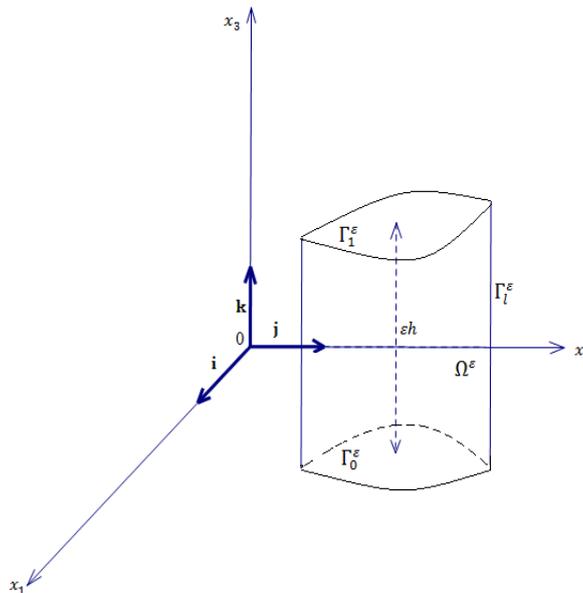


Figure 1: The domain considered.

Here $\mathcal{O} \subset \mathbf{R}^2$ is a bounded domain, $h : \overline{\mathcal{O}} \rightarrow \langle 0, +\infty \rangle$ is a smooth positive function and $\varepsilon > 0$ is a small parameter. We denote by Γ_0^ε , Γ_1^ε and Γ_ℓ^ε the lower, the upper and the lateral boundary of Ω^ε :

$$\begin{aligned}\Gamma_0^\varepsilon &= \{(x', x_3) \in \mathbf{R}^3 : x' \in \mathcal{O}, x_3 = 0\} , \\ \Gamma_1^\varepsilon &= \{(x', x_3) \in \mathbf{R}^3 : x' \in \mathcal{O}, x_3 = \varepsilon h(x')\} , \\ \Gamma_\ell^\varepsilon &= \{(x', x_3) \in \mathbf{R}^3 : x' \in \partial\mathcal{O}, 0 < x_3 < \varepsilon h(x')\} .\end{aligned}\tag{5}$$

2.2. The Equations and Boundary Conditions

In view of the application we want to model, we can assume a small Reynolds number and neglect the inertial terms in the governing equations. Thus, we assume that the flow in Ω^ε is governed by the following equations:

$$-(\nu + \nu_r) \Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = 2\nu_r \operatorname{rot} \mathbf{w}^\varepsilon ,\tag{6}$$

$$\operatorname{div} \mathbf{u}^\varepsilon = 0 ,\tag{7}$$

$$-(c_a + c_d) \Delta \mathbf{w}^\varepsilon - (c_0 + c_d - c_a) \nabla \operatorname{div} \mathbf{w}^\varepsilon + 4\nu_r \mathbf{w}^\varepsilon = 2\nu_r \operatorname{rot} \mathbf{u}^\varepsilon .\tag{8}$$

The unknown functions are \mathbf{u}^ε , \mathbf{w}^ε and p^ε representing the velocity, the microrotation and the pressure of the fluid respectively. Positive constants ν , ν_r , c_0 , c_a , c_d are the viscosity coefficients: ν is the usual kinematic Newtonian viscosity, while ν_r , c_0 , c_a , c_d are new viscosities connected with the asymmetry of the stress tensor and, consequently, with the appearance of the microrotation field \mathbf{w}^ε . For the sake of notational simplicity, external forces and moments are neglected and fluid density is assumed to be one.

As discussed in the Introduction, the following boundary conditions are imposed:

$$\mathbf{u}^\varepsilon = 0, \quad \mathbf{w}^\varepsilon = 0 \quad \text{on} \quad \Gamma_1^\varepsilon ,\tag{9}$$

$$\mathbf{u}^\varepsilon = \mathbf{g}, \quad \mathbf{w}^\varepsilon = 0 \quad \text{on} \quad \Gamma_\ell^\varepsilon ,\tag{10}$$

$$\mathbf{u}^\varepsilon \cdot \mathbf{k} = 0, \quad \mathbf{w}^\varepsilon \cdot \mathbf{k} = 0 \quad \text{on} \quad \Gamma_0^\varepsilon ,\tag{11}$$

$$\mathbf{w}^\varepsilon \times \mathbf{k} = \frac{\alpha}{2} (\operatorname{rot} \mathbf{u}^\varepsilon) \times \mathbf{k} \quad \text{on} \quad \Gamma_0^\varepsilon ,\tag{12}$$

$$(\operatorname{rot} \mathbf{w}^\varepsilon) \times \mathbf{k} = \frac{2\nu_r}{c_a + c_d} \beta (\mathbf{u}^\varepsilon - \mathbf{s}) \times \mathbf{k} \quad \text{on} \quad \Gamma_0^\varepsilon .\tag{13}$$

As we can see, the usual boundary conditions (9)-(10) for the velocity and microrotation are prescribed on $\Gamma_1^\varepsilon \cup \Gamma_\ell^\varepsilon$. However, on the lower part Γ_0^ε (corresponding to moving boundary), the impermeability of the wall leads to (11) associated with new type of boundary conditions (12)-(13). The coefficient $\alpha > 0$ appearing in (12) takes a microrotation retardation at the boundary into account and can be defined, according to [6, 7], by means of the boundary viscosity ν_b :

$$\alpha = \frac{\nu + \nu_r - \nu_b}{\nu_r} .\tag{14}$$

Finally, the coefficient β in (13) allows the control of the slippage at the wall when the value $\mathbf{u} - \mathbf{s}$ is not zero.

Remark 2.1. *The concept of boundary viscosity is motivated by the experimental results suggesting that chemical interactions between solid surface and the lubricant cannot be neglected especially in the case of thin film flow of a non-Newtonian fluid. This phenomenon can be taken into account by introducing a viscosity ν_b in the vicinity of the surface which is different from ν and ν_r . The slippage condition, expressed in (13), is related to the chemical properties of the surface and its effects are also enhanced by the non-Newtonian characteristics of the fluid.*

The well-posedness of the above setting is established in [2, Theorem 2.2] by proving that boundary-value problem (6)–(13) has only one weak solution $(\mathbf{u}^\varepsilon, p^\varepsilon, \mathbf{w}^\varepsilon)$ in appropriate spaces. Our goal is to find an effective law describing the asymptotic behavior of the flow in a thin domain Ω^ε . For the convenience of our analysis, let us first homogenize the boundary condition (10)₁. Assuming $\mathbf{g} \in H^{\frac{1}{2}}(\Gamma_\ell^\varepsilon)$, $\int_{\Gamma_\ell^\varepsilon} \mathbf{g} \cdot \mathbf{n} = 0$, using standard procedure we can construct a lift function $\mathbf{J}^\varepsilon \in H^1(\Omega^\varepsilon)$ such that

$$\operatorname{div} \mathbf{J}^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \quad \mathbf{J}^\varepsilon = 0 \text{ on } \Gamma_1^\varepsilon, \quad \mathbf{J}^\varepsilon = \mathbf{g} \text{ on } \Gamma_\ell^\varepsilon, \quad J_3^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon. \tag{15}$$

Introducing

$$\mathbf{v}^\varepsilon = \mathbf{u}^\varepsilon - \mathbf{J}^\varepsilon$$

the system (6)–(13) can be rewritten as

$$-(\nu + \nu_r) \Delta (\mathbf{v}^\varepsilon + \mathbf{J}^\varepsilon) + \nabla p^\varepsilon = 2\nu_r \operatorname{rot} \mathbf{w}^\varepsilon \quad \text{in } \Omega^\varepsilon, \tag{16}$$

$$\operatorname{div} \mathbf{v}^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon, \tag{17}$$

$$-(c_a + c_d) \Delta \mathbf{w}^\varepsilon - (c_0 + c_d - c_a) \nabla \operatorname{div} \mathbf{w}^\varepsilon + 4\nu_r \mathbf{w}^\varepsilon = 2\nu_r (\operatorname{rot} \mathbf{v}^\varepsilon + \operatorname{rot} \mathbf{J}^\varepsilon) \quad \text{in } \Omega^\varepsilon, \tag{18}$$

$$\mathbf{v}^\varepsilon = \mathbf{w}^\varepsilon = 0 \quad \text{on } \Gamma_1^\varepsilon \cup \Gamma_\ell^\varepsilon, \tag{19}$$

$$\mathbf{v}^\varepsilon \cdot \mathbf{k} = 0, \quad \mathbf{w}^\varepsilon \cdot \mathbf{k} = 0 \quad \text{on } \Gamma_0^\varepsilon, \tag{20}$$

$$\mathbf{w}^\varepsilon \times \mathbf{k} = \frac{\alpha}{2} (\operatorname{rot} (\mathbf{v}^\varepsilon + \mathbf{J}^\varepsilon)) \times \mathbf{k} \quad \text{on } \Gamma_0^\varepsilon, \tag{21}$$

$$(\operatorname{rot} \mathbf{w}^\varepsilon) \times \mathbf{k} = \frac{2\nu_r}{c_a + c_d} \beta (\mathbf{v}^\varepsilon + \mathbf{f} - \mathbf{s}) \times \mathbf{k} \quad \text{on } \Gamma_0^\varepsilon. \tag{22}$$

and this is the problem we are going to treat.

Remark 2.2. Note that only normal component of the velocity is known on Γ_0^ε , while the tangential component is not given, see (11)₁. Nevertheless, we can choose an artificial value $\mathbf{f} = (f_1, f_2)$ of the velocity on Γ_0^ε appearing in (22). We choose it in a way such that function $\mathbf{d} \in H^{\frac{1}{2}}(\partial\Omega^\varepsilon)$ defined by

$$\mathbf{d} = \begin{cases} 0 & \text{on } \Gamma_1^\varepsilon, \\ \mathbf{g} & \text{on } \Gamma_\ell^\varepsilon, \\ (\mathbf{f}, 0) & \text{on } \Gamma_0^\varepsilon \end{cases}$$

satisfies $\int_{\partial\Omega^\varepsilon} \mathbf{d} \cdot \mathbf{n} = 0$. Consequently, we are in position to construct the lift function \mathbf{J}^ε as in (15). It is important to observe that $\mathbf{J}^\varepsilon = (J_1^\varepsilon, J_2^\varepsilon, J_3^\varepsilon)$ depends on the small parameter ε and is, in fact, given by

$$J_i^\varepsilon(x', x_3) = J_i\left(x', \frac{x_3}{\varepsilon}\right), \quad i = 1, 2, \quad J_3^\varepsilon(x', x_3) = \varepsilon J_3\left(x', \frac{x_3}{\varepsilon}\right). \tag{23}$$

Here the function $\mathbf{J} = (\hat{\mathbf{J}}, J_3)$, $\hat{\mathbf{J}} = J_1 \mathbf{i} + J_2 \mathbf{j}$, is defined on ε -independent domain

$$\Omega = \{(x', y) \in \mathbf{R}^3 : x' \in \mathcal{O}, 0 < y < h(x')\}$$

satisfying the divergence equation $\operatorname{div}_{x'} \hat{\mathbf{J}} + \frac{\partial J_3}{\partial y} = 0$ in Ω and the corresponding boundary conditions.

3. Asymptotic Analysis

First, let us fix the notation employed in the sequel. We denote the fast variable by $y = \frac{x_3}{\varepsilon}$ and use the following partial differential operators:

$$\nabla_{x'}\phi = \frac{\partial\phi}{\partial x_1} \mathbf{i} + \frac{\partial\phi}{\partial x_2} \mathbf{j}, \quad \text{rot}_{x'}\phi = \frac{\partial\phi}{\partial x_2} \mathbf{i} - \frac{\partial\phi}{\partial x_1} \mathbf{j}, \quad \Delta_{x'}\mathbf{v} = \frac{\partial^2\mathbf{v}}{\partial x_1^2} + \frac{\partial^2\mathbf{v}}{\partial x_2^2}, \quad \text{div}_{x'}\mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$$

for a scalar function ϕ and $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$. Following the multiscale expansion technique, we construct an expansion of the unknowns $\mathbf{v}^\varepsilon, p^\varepsilon$ and \mathbf{w}^ε in the following form:

$$\mathbf{v}^\varepsilon = \mathbf{v}^0(x', y) + \varepsilon \mathbf{v}^1(x', y) + \varepsilon^2 \mathbf{v}^2(x', y) + \dots, \tag{24}$$

$$p^\varepsilon = \frac{1}{\varepsilon^2} p^0(x') + \frac{1}{\varepsilon} p^1(x', y) + p^2(x', y) + \dots, \tag{25}$$

$$\mathbf{w}^\varepsilon = \mathbf{w}^0(x', y) + \varepsilon \mathbf{w}^1(x', y) + \varepsilon^2 \mathbf{w}^2(x', y) + \dots. \tag{26}$$

We employ the standard approach: we replace formally $(\mathbf{v}^\varepsilon, p^\varepsilon, \mathbf{w}^\varepsilon)$ in (16)–(18) by its asymptotic expansions and determine the profiles $(\mathbf{v}^i, p^i, \mathbf{w}^i)$ by identifying all terms of the same order (with respect to ε) and taking into account the boundary conditions (19)–(22). However, an additional requirement is imposed in the process: the correctors in (24)–(26) are to be computed such that they have zero-mean value. As a consequence, we will obtain new higher-order model describing the macroscopic flow.

We begin by plugging the expansions into momentum equation (16). It leads to

$$\begin{aligned} & \frac{1}{\varepsilon^2} \left[-(v + \nu_r) \frac{\partial^2 \mathbf{v}^0}{\partial y^2} - (v + \nu_r) \frac{\partial^2 \hat{\mathbf{j}}}{\partial y^2} + \nabla_{x'} p^0 + \frac{\partial p^1}{\partial y} \mathbf{k} \right] \\ & + \frac{1}{\varepsilon} \left[-(v + \nu_r) \frac{\partial^2 \mathbf{v}^1}{\partial y^2} - (v + \nu_r) \frac{\partial^2 J_3}{\partial y^2} \mathbf{k} + \nabla_{x'} p^1 + \frac{\partial p^2}{\partial y} \mathbf{k} + 2\nu_r \left(\frac{\partial w_2^0}{\partial y} \mathbf{i} - \frac{\partial w_1^0}{\partial y} \mathbf{j} \right) \right] \\ & + \left[-(v + \nu_r) \frac{\partial^2 \mathbf{v}^2}{\partial y^2} - (v + \nu_r) \Delta_{x'} \mathbf{v}^0 - (v + \nu_r) \Delta_{x'} \hat{\mathbf{j}} + \nabla_{x'} p^2 + \frac{\partial p^3}{\partial y} \mathbf{k} + \right. \\ & \left. + 2\nu_r \left(\frac{\partial w_2^1}{\partial y} \mathbf{i} - \frac{\partial w_1^1}{\partial y} \mathbf{j} \right) - 2\nu_r \text{rot}_{x'} w_3^0 - 2\nu_r \left(\frac{\partial w_2^0}{\partial x_1} - \frac{\partial w_1^0}{\partial x_2} \right) \mathbf{k} \right] + \dots = 0. \end{aligned} \tag{27}$$

3.1. Main-order Approximation

For the moment, let us keep only the main order term in (27):

$$-(v + \nu_r) \frac{\partial^2 \mathbf{v}^0}{\partial y^2} - (v + \nu_r) \frac{\partial^2 \hat{\mathbf{j}}}{\partial y^2} + \nabla_{x'} p^0 + \frac{\partial p^1}{\partial y} \mathbf{k} = 0, \quad \mathbf{v}^0 = 0 \text{ for } y = h, \quad v_3^0 = 0 \text{ for } y = 0.$$

The above system can be solved by taking $p^1 = p^1(x')$ and

$$\mathbf{v}^0 = \frac{1}{2(v + \nu_r)} y (h(x') - y) \mathbf{r}(x') - \hat{\mathbf{j}} + \left(1 - \frac{y}{h(x')} \right) \hat{\mathbf{A}}(x'), \quad \mathbf{r} = -\nabla_{x'} p^0. \tag{28}$$

The unknown function $\hat{\mathbf{A}} = A_1 \mathbf{i} + A_2 \mathbf{j}$ is to be determined by taking into account the boundary condition (21). The main order term gives

$$\frac{1}{\varepsilon} : - \left(\frac{\partial v_2^0}{\partial y} + \frac{\partial J_2}{\partial y} \right) (\mathbf{i} \times \mathbf{k}) + \left(\frac{\partial v_1^0}{\partial y} + \frac{\partial J_1}{\partial y} \right) (\mathbf{j} \times \mathbf{k}) = 0 \quad \text{for } y = 0 \tag{29}$$

implying $\hat{\mathbf{A}} = \frac{h^2}{2(v+v_r)} \mathbf{r}$. On the other hand, the divergence equation (17) yields

$$1 : \quad \text{div}_{x'} \mathbf{v}^0 + \frac{\partial v_3^1}{\partial y} = 0 \quad \text{in } \Omega. \tag{30}$$

Integrating from 0 to $h(x')$ with respect to y and using a simple formula

$$\frac{\partial}{\partial x_i} \int_0^{h(x')} \phi(x', y) dy = \int_0^{h(x')} \frac{\partial \phi}{\partial x_i}(x', y) dy - \phi(x', h(x')) \frac{\partial h}{\partial x_i}$$

we get

$$\text{div}_{x'} \left(\int_0^h \mathbf{v}^0 dy \right) = \text{div}_{x'} \left(\frac{h^3}{12(v+v_r)} \mathbf{r} + \frac{h}{2} \hat{\mathbf{A}} \right) = 0. \tag{31}$$

Consequently,

$$\text{div}_{x'} (h^3 \mathbf{r}) = 0 \quad \text{in } \mathcal{O}. \tag{32}$$

Taking into account that $\hat{\mathbf{A}} = \frac{h^2}{2(v+v_r)} \mathbf{r}$, from (28) we deduce that we should impose $\mathbf{r} = 0$ on $\partial\mathcal{O}$. Indeed, by imposing $\mathbf{r} = 0$ on $\partial\mathcal{O}$ we would ensure that $\mathbf{v}^0 = 0$ for $x' \in \partial\mathcal{O}$, as requested by (19). However, such condition cannot be imposed on $\partial\mathcal{O}$ since then the equation (32) would lead to a trivial solution $\mathbf{r} = 0$ in \mathcal{O} . The best we can do is $\mathbf{r} \cdot \mathbf{n} = 0$ but that does not meet our purposes. All of that motivates us to continue the computation and to seek for the higher-order approximation. We compute the correctors in a way that they have zero mean value $\int_0^{h(x')} \cdot dy$. By doing so, the correctors do not contribute to the net flow rate. Furthermore, such requirement will force us to change the leading order term \mathbf{v}^0 which now has to carry the whole flow rate. However, those changes will be of the lower order. By computing the correctors in such way, the macroscopic equation for the mean flow rate \mathbf{r} is going to be changed. It means that \mathbf{r} will not obey the simple equation $\mathbf{r} + \nabla_{x'} p^0 = 0$ anymore. The new (small) terms will appear changing the order and, thus, the nature of the obtained effective equations (see (45)).

Before constructing correctors, let us just identify the leading term \mathbf{w}^0 in the expansion (26). From (18)-(20) and (22) we deduce

$$\left\{ \begin{array}{l} \frac{1}{\varepsilon^2} : \quad -(c_a + c_d) \frac{\partial^2 \mathbf{w}^0}{\partial y^2} - (c_0 + c_d - c_a) \frac{\partial^2 w_3^0}{\partial y^2} \mathbf{k} = 0 \quad \text{in } \Omega, \\ \mathbf{w}^0 = 0 \quad \text{for } y = h, \quad w_3^0 = 0 \quad \text{for } y = 0, \\ \frac{1}{\varepsilon} : \quad \frac{\partial w_1^0}{\partial y} = \frac{\partial w_2^0}{\partial y} = 0 \quad \text{for } y = 0 \end{array} \right. \tag{33}$$

implying $\mathbf{w}^0 = 0$.

3.2. Correctors

In this section we construct the correctors in the asymptotic expansions. The next term in the expansion for the velocity is given by the following problem:

$$\left\{ \begin{array}{l} \frac{1}{\varepsilon} : \quad -(\nu + \nu_r) \frac{\partial^2 v^1}{\partial y^2} - (\nu + \nu_r) \frac{\partial^2 J_3}{\partial y^2} \mathbf{k} + \nabla_{x'} p^1 + \frac{\partial p^2}{\partial y} \mathbf{k} = 0, \quad \text{in } \Omega, \\ 1 : \quad \operatorname{div}_{x'} \mathbf{v}^0 + \frac{\partial v_3^1}{\partial y} = 0 \quad \text{in } \Omega, \\ \quad \mathbf{v}^1 = 0 \text{ for } y = h, \quad v_3^1 = 0 \text{ for } y = 0, \\ 1 : \quad \frac{\partial v_1^1}{\partial y} = \frac{\partial v_2^1}{\partial y} = 0 \text{ for } y = 0. \end{array} \right. \quad (34)$$

From (34)₂ we obtain

$$\begin{aligned} v_3^1 &= - \int_0^y \operatorname{div}_{x'} \mathbf{v}^0(x', \xi) d\xi \\ &= \frac{1}{\nu + \nu_r} \left(\frac{y^3}{6} - \frac{h^2 y}{2} \right) \operatorname{div}_{x'} \mathbf{r} - \frac{h y}{\nu + \nu_r} \nabla_{x'} h \cdot \mathbf{r} - J_3. \end{aligned} \quad (35)$$

Note that $v_3^1|_{y=0,h} = 0$, in view of (32). We demand $\int_0^h v_1^1 dy = \int_0^h v_2^1 dy = 0$, so we deduce $v_1^1 = v_2^1 = 0, p^1 = 0$ and

$$p^2 = (\nu + \nu_r) \left(\frac{\partial v_3^1}{\partial y} + \frac{\partial J_3}{\partial y} \right) + Q^2(x') = -\frac{1}{2} \operatorname{div}_{x'}(h^2 \mathbf{r}) + \frac{y^2}{2} \operatorname{div}_{x'} \mathbf{r} + Q^2(x').$$

We determine $Q^2(x')$ such that $\int_0^h p^2 dy = 0$ implying

$$p^2 = -\frac{1}{2} \operatorname{div}_{x'}(h^2 \mathbf{r}) + \frac{y^2}{2} \operatorname{div}_{x'} \mathbf{r}. \quad (36)$$

Now we compute \mathbf{w}^1 from the expansion for microrotation. Taking into account the angular momentum equation (18) together with boundary conditions (20) and (22), we arrive at:

$$\left\{ \begin{array}{l} \frac{1}{\varepsilon} : \quad -(c_a + c_d) \frac{\partial^2 w^1}{\partial y^2} - (c_0 + c_d - c_a) \frac{\partial^2 w_3^1}{\partial y^2} \mathbf{k} \\ \quad = 2\nu_r \left[-\left(\frac{\partial v_2^0}{\partial y} + \frac{\partial J_2}{\partial y} \right) \mathbf{i} + \left(\frac{\partial v_1^0}{\partial y} + \frac{\partial J_1}{\partial y} \right) \mathbf{j} \right] + \mathbf{B}(x') \quad \text{in } \Omega, \\ \quad \mathbf{w}^1 = 0 \text{ for } y = h, \quad w_3^1 = 0 \text{ for } y = 0, \\ 1 : \quad \frac{\partial w_1^1}{\partial y} = \frac{2\nu_r}{c_a + c_d} \beta (v_2^0 + f_2 - s_2), \quad \frac{\partial w_2^1}{\partial y} = -\frac{2\nu_r}{c_a + c_d} \beta (v_1^0 + f_1 - s_1) \text{ for } y = 0. \end{array} \right. \quad (37)$$

We additionally impose $\int_0^h \mathbf{w}^1 dy = 0$. To satisfy that, we need to introduce an additional term, denoted by $\mathbf{B}(x')$, to angular momentum equation (see (37)₁). Since $w_3^1|_{y=0,h} = 0$, we immediately conclude $w_3^1 = 0$. Taking into account (28) and zero mean value condition, the remaining two components can be explicitly computed from the system (37):

$$\begin{aligned} w_1^1 &= \frac{1}{c_a + c_d} \left[\frac{\nu_r}{\nu + \nu_r} \left(-\frac{y^3}{3} + \frac{3}{8} h y^2 - \frac{h^3}{24} \right) r_2 \right. \\ &\quad \left. - 2\beta \nu_r \left(\frac{h^2}{2(\nu + \nu_r)} r_2 + f_2 - s_2 \right) \left(\frac{3}{4h} y^2 - y + \frac{h}{4} \right) \right], \end{aligned} \quad (38)$$

$$w_2^1 = -\frac{1}{c_a + c_d} \left[\frac{v_r}{v + v_r} \left(-\frac{y^3}{3} + \frac{3}{8}hy^2 - \frac{h^3}{24} \right) r_1 - 2\beta v_r \left(\frac{h^2}{2(v + v_r)} r_1 + f_1 - s_1 \right) \left(\frac{3}{4h}y^2 - y + \frac{h}{4} \right) \right]. \tag{39}$$

It remains to construct the corrector \mathbf{v}^2 . In view of (27), boundary conditions (19)-(21) and preceding calculation, it is given by

$$\left\{ \begin{array}{l} 1 : \quad -(\nu + \nu_r) \frac{\partial^2 \mathbf{v}^2}{\partial y^2} = (\nu + \nu_r) \Delta_{x'} \mathbf{v}^0 + (\nu + \nu_r) \Delta_{x'} \hat{\mathbf{J}} - \nabla_{x'} p^2 - \frac{\partial p^3}{\partial y} \mathbf{k} \\ \qquad \qquad \qquad + 2\nu_r \left(-\frac{\partial w_2^1}{\partial y} \mathbf{i} + \frac{\partial w_1^1}{\partial y} \mathbf{j} \right) + \mathbf{C}(x') \quad \text{in } \Omega, \\ \mathbf{v}^2 = 0 \text{ for } y = h, \quad v_3^2 = 0 \text{ for } y = 0, \\ \varepsilon : \quad \frac{\partial \mathbf{v}^2}{\partial y} = \frac{1}{\alpha} \cdot \frac{1}{c_a + c_d} \left[\frac{v_r}{v + v_r} \frac{h^3}{12} \mathbf{r} + \beta v_r h \left(\frac{h^2}{2(v + v_r)} \mathbf{r} + \mathbf{f} - \mathbf{s} \right) \right] \quad \text{for } y = 0, \\ \int_0^h \mathbf{v}^2 dy = 0. \end{array} \right. \tag{40}$$

Due to requirement (40)₄, we had to add an additional term to momentum equation, denoted by $\mathbf{C}(x')$. Notice that, by intervening in the second order corrector $\varepsilon^2 \mathbf{v}^2$, we, in fact, change the leading order term \mathbf{v}^0 which now has to carry the whole flow rate. However, those changes will be of the lower order. Now we solve (40). First, we set $p^3 = 0$, leading to $v_3^2 = 0$, in order to keep the divergence equation satisfied. Then, we need to compute $\Delta_{x'} \mathbf{v}^0$ and $\nabla_{x'} p^2$ appearing on the right-hand side in (40)₁. Using the decomposition

$$\Delta_{x'}(h^2 \mathbf{r}) = 2|\nabla_{x'} h|^2 \mathbf{r} + 2h(\Delta_{x'} h) \mathbf{r} + 4h\nabla_{x'} h \cdot (\nabla_{x'} \mathbf{r})^\tau + h^2 \Delta_{x'} \mathbf{r} \tag{41}$$

from (28) we deduce

$$\begin{aligned} (\nu + \nu_r) \Delta_{x'} \mathbf{v}^0 = & |\nabla_{x'} h|^2 \mathbf{r} + h(\Delta_{x'} h) \mathbf{r} + 2h\nabla_{x'} h \cdot (\nabla_{x'} \mathbf{r})^\tau \\ & + \frac{1}{2} (h^2 - y^2) \Delta_{x'} \mathbf{r} - (\nu + \nu_r) \Delta_{x'} \hat{\mathbf{J}}. \end{aligned} \tag{42}$$

Using (32) we get from (36):

$$\nabla_{x'} p^2 = \left(-\frac{3y^2}{2h} + \frac{h}{2} \right) \nabla_{x'} (\nabla_{x'} h \cdot \mathbf{r}) - \left(\frac{3y^2}{2} + \frac{h^2}{2} \right) \nabla_{x'} \left(\frac{1}{h} \right) (\nabla_{x'} h \cdot \mathbf{r}). \tag{43}$$

Straightforward calculation now yields the following expression for \mathbf{v}^2 :

$$\begin{aligned} \mathbf{v}^2 = & \frac{1}{2(\nu + \nu_r)} \left(\frac{y^4}{12} - \frac{h^2 y^2}{2} \right) \Delta_{x'} \mathbf{r} - \frac{y^2}{2(\nu + \nu_r)} \left(h\Delta_{x'} h + |\nabla_{x'} h|^2 \right) \mathbf{r} \\ & - \frac{hy^2}{(\nu + \nu_r)} \nabla_{x'} h \cdot (\nabla_{x'} \mathbf{r})^\tau + \frac{1}{\nu + \nu_r} \left(\frac{hy^2}{4} - \frac{y^4}{8h} \right) \nabla_{x'} (\nabla_{x'} h \cdot \mathbf{r}) \\ & - \frac{1}{\nu + \nu_r} \left(\frac{y^4}{8} + \frac{h^2 y^2}{4} \right) \nabla_{x'} \left(\frac{1}{h} \right) (\nabla_{x'} h \cdot \mathbf{r}) + \frac{1}{c_a + c_d} \left(\frac{\nu_r}{\nu + \nu_r} \right)^2 \left(\frac{y^4}{6} - \frac{hy^3}{4} \right) \mathbf{r} \\ & + 4\beta \frac{\nu + \nu_r}{c_a + c_d} \left(\frac{\nu_r}{\nu + \nu_r} \right)^2 \left(\frac{h^2}{2(\nu + \nu_r)} \mathbf{r} + \mathbf{f} - \mathbf{s} \right) \left(\frac{y^3}{4h} - \frac{y^2}{2} \right) \\ & - \frac{y^2}{2(\nu + \nu_r)} \mathbf{C}(x') - \frac{y}{\nu + \nu_r} \mathbf{D}(x') + \mathbf{E}(x'), \end{aligned} \tag{44}$$

with

$$\begin{aligned}
 \mathbf{C}(x') &= -|\nabla_{x'} h|^2 \mathbf{r} - h(\Delta_{x'} h) \mathbf{r} - 2h \nabla_{x'} h \cdot (\nabla_{x'} \mathbf{r})^\tau - \frac{2h^2}{5} \Delta_{x'} \mathbf{r} + \frac{h}{5} \nabla_{x'} (\nabla_{x'} h \cdot \mathbf{r}) \\
 &\quad - \frac{4h^2}{5} \nabla_{x'} \left(\frac{1}{h} \right) (\nabla_{x'} h \cdot \mathbf{r}) - \frac{\nu + \nu_r}{c_a + c_d} \left(\frac{\nu_r}{\nu + \nu_r} \right)^2 \frac{13h^2}{80} \mathbf{r} \\
 &\quad - \frac{7}{4} \beta \frac{\nu_r^2}{c_a + c_d} \left(\frac{h^2}{2(\nu + \nu_r)} \mathbf{r} + \mathbf{f} - \mathbf{s} \right) \\
 &\quad + \frac{1}{\alpha} \frac{\nu + \nu_r}{c_a + c_d} \left(\frac{\nu_r}{\nu + \nu_r} \frac{h^2}{8} \mathbf{r} + \frac{3}{2} \beta \nu_r \left(\frac{h^2}{2(\nu + \nu_r)} \mathbf{r} + \mathbf{f} - \mathbf{s} \right) \right) \\
 \mathbf{D}(x') &= -\frac{1}{\alpha} \frac{\nu + \nu_r}{c_a + c_d} \left(\frac{\nu_r}{\nu + \nu_r} \frac{h^3}{12} \mathbf{r} + \beta \nu_r h \left(\frac{h^2}{2(\nu + \nu_r)} \mathbf{r} + \mathbf{f} - \mathbf{s} \right) \right), \\
 \mathbf{E}(x') &= \frac{h^4}{120(\nu + \nu_r)} \Delta_{x'} \mathbf{r} - \frac{h^3}{40(\nu + \nu_r)} \nabla_{x'} (\nabla_{x'} h \cdot \mathbf{r}) - \frac{h^4}{40(\nu + \nu_r)} \nabla_{x'} \left(\frac{1}{h} \right) (\nabla_{x'} h \cdot \mathbf{r}) \\
 &\quad + \frac{1}{c_a + c_d} \left(\frac{\nu_r}{\nu + \nu_r} \right)^2 \frac{h^4}{480} \mathbf{r} + \beta \frac{\nu + \nu_r}{c_a + c_d} \left(\frac{\nu_r}{\nu + \nu_r} \right)^2 \left(\frac{h^2}{2(\nu + \nu_r)} \mathbf{r} + \mathbf{f} - \mathbf{s} \right) \frac{h^2}{8} \\
 &\quad + \frac{h}{4(\nu + \nu_r)} \mathbf{D}(x').
 \end{aligned}$$

3.3. Brinkman Approximation

After we have computed the correctors, we return to the referent equation (27) once again and integrate it with respect to y from 0 to $h(x')$. It follows

$$\begin{aligned}
 &\mathbf{r} + \nabla_{x'} p^0 - \varepsilon^2 \left\{ \frac{2h^2}{5} \Delta_{x'} \mathbf{r} + 2h \nabla_{x'} h \cdot (\nabla_{x'} \mathbf{r})^\tau - \frac{h}{5} \nabla_{x'} (\nabla_{x'} h \cdot \mathbf{r}) + (h \Delta_{x'} h + |\nabla_{x'} h|^2) \mathbf{r} \right. \\
 &\quad \left. + \frac{4h^2}{5} \nabla_{x'} \left(\frac{1}{h} \right) (\nabla_{x'} h \cdot \mathbf{r}) + \frac{\nu + \nu_r}{c_a + c_d} \left[\left(\frac{\nu_r}{\nu + \nu_r} \right)^2 \left(\frac{13}{80} + \frac{7}{8} \beta \right) - \frac{1}{\alpha} \left(\frac{\nu_r}{\nu + \nu_r} \right) \left(\frac{1}{8} + \frac{3}{4} \beta \right) \right] h^2 \mathbf{r} \right\} \\
 &= \beta \nu_r \varepsilon^2 \left(\frac{7}{4} \frac{\nu_r}{c_a + c_d} - \frac{3}{2\alpha} \frac{\nu + \nu_r}{c_a + c_d} \right) (\mathbf{f} - \mathbf{s}) \quad \text{in } \mathcal{O}, \tag{45}
 \end{aligned}$$

$$\operatorname{div}_{x'} (h^3 \mathbf{r}) = 0 \quad \text{in } \mathcal{O},$$

$$\mathbf{r} = 0 \quad \text{on } \partial \mathcal{O}.$$

This second-order system, in the form of the Brinkman-type law, satisfied by (\mathbf{r}, p^0) , describes two-dimensional macroscopic flow⁴⁾. Its solvability can be established in a standard manner since this is, in fact, a linear system. It is essential to emphasize that \mathbf{r} does not obey the equation $\mathbf{r} + \nabla_{x'} p^0 = 0$ but the new asymptotic model (45). As you can see, the new lower-order terms appear which changes the order and the nature of the obtained effective equations. That was exactly what we wanted as indicated in the discussion in Sec. 3.1. Since new effective equations contain the second order derivatives of \mathbf{r} , we can now impose zero boundary condition on $\partial \mathcal{O}$. Let us recall that we were not in position to do so for zero-order approximation satisfying (32). Furthermore, if we compare it to the effective system derived for the standard setting (3) and proposed in [14], we clearly detect the effects arising due to the new boundary conditions prescribed for the microrotation.

⁴⁾The third component of the velocity is then determined from (35).

4. Rigorous Justification

In the previous section, the higher-order asymptotic model for micropolar fluid film lubrication is derived. Though the derivation was just formal, it provides a very good platform for understanding the direct influence of the fluid microstructure on the lubrication process. However, from the strictly mathematical point of view, formally derived model should be rigorously justified by proving the corresponding error estimate. If we impose periodic boundary conditions on $\partial\mathcal{O}$, we can easily prove the satisfactory error estimates using classical techniques (see [15] for details). In the case of original, physically relevant, Dirichlet boundary conditions, the situation is much more complicated. The aim of this section is to present the main mathematical difficulties which prevent us to derive satisfactory L^2 and H^1 error estimates in case of non-periodic physical boundary conditions.

First we prove the following technical result being essential for further discussion:

Lemma 4.1. *Let \mathbf{v} be a given (smooth enough) function on $\bar{\mathcal{O}}$ such that $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\mathcal{O}$. Then for every $\varepsilon > 0$, there exists a function \mathbf{c} such that*

$$\operatorname{div} \mathbf{c} = 0 \text{ in } \mathcal{O}, \quad \mathbf{c} = \mathbf{v} \text{ on } \partial\mathcal{O}.$$

Furthermore, \mathbf{c} can be chosen such that

$$|\mathbf{c}|_{L^2(\mathcal{O})} \leq C \sqrt{\varepsilon}, \quad |\nabla \mathbf{c}|_{L^2(\mathcal{O})} \leq \frac{C}{\sqrt{\varepsilon}},$$

with constant $C > 0$ being independent of ε .

Proof. The main idea is to introduce the local curvilinear coordinates in the vicinity the domain boundary $\partial\mathcal{O}$ and to write the divergence operator in such coordinates. In view of that, let us suppose that the boundary is a smooth curve in \mathbf{R}^2 , denoted by γ and parameterized by its arc length $s \in [0, \ell]$. Let $\pi : [0, \ell] \rightarrow \mathbf{R}^2$ be its natural parametrization such that $\frac{d\pi}{ds} \neq 0$. At each point $\pi(s)$ of the curve γ we define the curvature as $\kappa(s) = \left| \frac{d^2\pi}{ds^2} \right|$ and introduce the local basis:

$$\mathbf{t} = \frac{d\pi}{ds} \quad (\text{the tangent}), \quad \mathbf{n} = \frac{1}{\kappa} \frac{d\mathbf{t}}{ds} \quad (\text{the normal}).$$

We assume that \mathbf{n} is extended by continuity in points where curvature is zero. It holds $\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}$, $\frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t}$. Now we introduce the mapping

$$\Phi(s, \eta) = \pi(s) + \eta\mathbf{n}(s)$$

and compute the corresponding *covariant* and *contravariant* basis. The covariant basis is defined as the gradient of the mapping Φ :

$$\mathbf{a}_1 = \frac{\partial\Phi}{\partial s} = (1 - \kappa\eta)\mathbf{t}, \quad \mathbf{a}_2 = \frac{\partial\Phi}{\partial \eta} = \mathbf{n}.$$

The contravariant basis is defined by the relation $\mathbf{a}^i \cdot \mathbf{a}_j = \delta_{ij}$ implying

$$\mathbf{a}^1 = \frac{1}{1 - \kappa\eta} \mathbf{t}, \quad \mathbf{a}^2 = \mathbf{n}.$$

Since the vectors of the contravariant basis represent the rows of $(\nabla\Phi)^{-1}$ we conclude

$$(\nabla\Phi_\varepsilon)^{-1} = \begin{bmatrix} \frac{1}{1-\kappa\eta} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{B}^\tau, \quad \mathbf{B} = \begin{bmatrix} \mathbf{t} & \mathbf{n} \end{bmatrix}.$$

Christoffel's symbols are given by

$$\Gamma_{1j}^i = \mathbf{a}^i \cdot \frac{\partial \mathbf{a}_j}{\partial s}, \quad \Gamma_{2j}^i = \mathbf{a}^i \cdot \frac{\partial \mathbf{a}_j}{\partial \eta} \quad i, j = 1, 2$$

and they are symmetric in lower indices. We put them in the matrices $\Gamma^i = [\Gamma_{jk}^i]_{j,k}$:

$$\Gamma^1 = \begin{bmatrix} -\frac{\kappa'\eta}{1-\kappa\eta} & -\frac{\kappa}{1-\kappa\eta} \\ -\frac{\kappa}{1-\kappa\eta} & 0 \end{bmatrix}, \quad \Gamma^2 = \begin{bmatrix} (1-\kappa\eta)\kappa & 0 \\ 0 & 0 \end{bmatrix}.$$

We employ the following formula for the nabla operator in the curvilinear coordinates:

$$(\nabla \mathbf{u}) \circ \Phi = (\nabla \Phi)^{-T} \left(\begin{bmatrix} \frac{\partial U_1}{\partial s} & \frac{\partial U_1}{\partial \eta} \\ \frac{\partial U_2}{\partial s} & \frac{\partial U_2}{\partial \eta} \end{bmatrix} - U_1 \Gamma^1 - U_2 \Gamma^2 \right) (\nabla \Phi)^{-1}, \quad \mathbf{U} = \mathbf{u} \circ \Phi = U_t \mathbf{t} + U_n \mathbf{n},$$

with $U_1 = \mathbf{U} \cdot \mathbf{a}_1 = (1 - \kappa\eta)U_t$, $U_2 = \mathbf{U} \cdot \mathbf{a}_2 = U_n$ being the contravariant components of \mathbf{U} . By direct calculation we obtain

$$(\nabla \mathbf{u}) \circ \Phi = \mathbf{B} \begin{bmatrix} \frac{1}{1-\kappa\eta} \left(\frac{\partial U_t}{\partial s} - \kappa U_n \right) & 0 \\ \frac{1}{1-\kappa\eta} \left(\frac{\partial U_n}{\partial s} + \kappa U_t \right) & \frac{\partial U_n}{\partial \eta} \end{bmatrix} \mathbf{B}^T. \tag{46}$$

Taking the trace in (46) leads to

$$\operatorname{div} \mathbf{u} \circ \Phi = \frac{1}{1 - \kappa\eta} \left(\frac{\partial U_t}{\partial s} - \kappa U_n \right) + \frac{\partial U_n}{\partial \eta}. \tag{47}$$

Now we are in position to construct the function \mathbf{c} with desired properties. We define

$$\mathbf{C} = \mathbf{c} \circ \Phi = C_t \mathbf{t} + C_n \mathbf{n}. \tag{48}$$

For arbitrary $\varepsilon > 0$ we put

$$C_t(s, \eta) = e^{-\frac{\eta}{\varepsilon}} V_t(s, 0), \tag{49}$$

where V_t stands for the tangential component of the given function $\mathbf{V} = \mathbf{v} \circ \Phi$. We demand

$$\operatorname{div} \mathbf{c} \circ \Phi = \frac{1}{1 - \kappa\eta} \left(\frac{\partial C_t}{\partial s} - \kappa C_n \right) + \frac{\partial C_n}{\partial \eta} = 0, \quad C_n(s, 0) = 0$$

implying the second component to be given by

$$C_n(s, \eta) = \varepsilon \frac{e^{-\frac{\eta}{\varepsilon}} - 1}{1 - \kappa\eta} \frac{\partial V_t(s, 0)}{\partial s}. \tag{50}$$

By a simple change of variables, one can easily verify that

$$\|\mathbf{c}\|_{L^2(\mathcal{O})} \leq C \sqrt{\varepsilon}, \quad \|\nabla \mathbf{c}\|_{L^2(\mathcal{O})} \leq \frac{C}{\sqrt{\varepsilon}}.$$

□

Now we prove sharp *a priori* estimates for the solution of the Brinkman system (45). To avoid notational

complexities, let us take $h = 1$. The general case can be treated following exactly the same arguments as presented in the sequel. For $h = 1$ the effective problem (45) reduces to

$$\begin{aligned} \mathbf{r} - \frac{2\varepsilon^2}{5} \Delta \mathbf{r} - C_1 \varepsilon^2 \mathbf{r} + \nabla p^0 &= C_2 \varepsilon^2 (\mathbf{f} - \mathbf{s}) \quad \text{in } \mathcal{O}, \\ \operatorname{div} \mathbf{r} &= 0 \quad \text{in } \mathcal{O}, \\ \mathbf{r} &= 0 \quad \text{on } \partial \mathcal{O}, \end{aligned} \tag{51}$$

with $C_1 = \frac{\nu + \nu_r}{c_a + c_d} \left[\left(\frac{\nu_r}{\nu + \nu_r} \right)^2 \left(\frac{13}{80} + \frac{7}{8} \beta \right) - \frac{1}{\alpha} \left(\frac{\nu_r}{\nu + \nu_r} \right) \left(\frac{1}{8} + \frac{3}{4} \beta \right) \right]$, $C_2 = \beta \nu_r \left(\frac{7}{4} \frac{\nu_r}{c_a + c_d} - \frac{3}{2\alpha} \frac{\nu + \nu_r}{c_a + c_d} \right)$ being constants independent of ε .

Theorem 4.2. *Suppose that $\mathbf{f} \in H^l(\mathcal{O})^2$ for some $l \geq 0$. Then there exists a constant $C > 0$, independent of ε , such that*

$$\|\mathbf{r}\|_{L^2(\mathcal{O})} \leq C, \tag{52}$$

$$\|\nabla \mathbf{r}\|_{L^2(\mathcal{O})} \leq \frac{C}{\sqrt{\varepsilon}}, \tag{53}$$

$$\|\mathbf{r}\|_{H^m(\mathcal{O})} \leq \frac{C}{\varepsilon^{m-\frac{1}{2}}}, \quad m \in \{0, \dots, l + 2\}, \tag{54}$$

$$\|p^0\|_{H^1(\mathcal{O})} \leq C. \tag{55}$$

Proof. We introduce (\mathbf{r}_0, q) as the solution of the following system

$$\begin{aligned} \mathbf{r}_0 + \nabla q &= C_2 \varepsilon^2 (\mathbf{f} - \mathbf{s}), \quad \operatorname{div} \mathbf{r}_0 = 0 \quad \text{in } \mathcal{O}, \\ \mathbf{r}_0 \cdot \mathbf{n} &= 0 \quad \text{on } \partial \mathcal{O} \end{aligned}$$

and compare it with (\mathbf{r}, p^0) from (51). Denoting their differences by $\mathbf{E} = \mathbf{r} - \mathbf{r}_0$ and $e = p^0 - q$, we deduce

$$\left(1 - C_1 \varepsilon^2\right) \mathbf{E} - \frac{2\varepsilon^2}{5} \Delta \mathbf{E} + \nabla e = \frac{2\varepsilon^2}{5} \Delta \mathbf{r}_0 + C_1 \varepsilon^2 \mathbf{r}_0, \quad \operatorname{div} \mathbf{E} = 0 \quad \text{in } \mathcal{O}, \tag{56}$$

$$\mathbf{E} = -\mathbf{r}_0 \quad \text{on } \partial \mathcal{O}. \tag{57}$$

According to Lemma 4.1., there exists $\mathbf{c} \in H^1(\mathcal{O})^2$ such that

$$\operatorname{div} \mathbf{c} = 0 \quad \text{in } \mathcal{O}, \quad \mathbf{c} = \mathbf{r}_0 \quad \text{on } \partial \mathcal{O}$$

and

$$\|\mathbf{c}\|_{L^2(\mathcal{O})} \leq C \sqrt{\varepsilon}, \quad \|\nabla \mathbf{c}\|_{L^2(\mathcal{O})} \leq \frac{C}{\sqrt{\varepsilon}}.$$

Using $\mathbf{E} + \mathbf{c}$ as a test-function in (56)₁ we obtain

$$\begin{aligned} &\left(1 - C_1 \varepsilon^2\right) \int_{\mathcal{O}} |\mathbf{E}|^2 + \frac{2\varepsilon^2}{5} \int_{\mathcal{O}} |\nabla \mathbf{E}|^2 = \\ &= \int_{\mathcal{O}} \left(-\left(1 - C_1 \varepsilon^2\right) \mathbf{E} \cdot \mathbf{c} - \frac{2\varepsilon^2}{5} \nabla \mathbf{E} \cdot \nabla \mathbf{c} + \frac{2\varepsilon^2}{5} \Delta \mathbf{r}_0 (\mathbf{E} + \mathbf{c}) + C_1 \varepsilon^2 \mathbf{r}_0 (\mathbf{E} + \mathbf{c}) \right) \leq \\ &\leq C \left(\sqrt{\varepsilon} \|\mathbf{E}\|_{L^2(\mathcal{O})} + \varepsilon^{3/2} \|\nabla \mathbf{E}\|_{L^2(\mathcal{O})} + \varepsilon^{3/2} \right) \end{aligned}$$

implying

$$|\mathbf{E}|_{L^2(O)} \leq C\sqrt{\varepsilon}, \quad |\nabla \mathbf{E}|_{L^2(O)} \leq \frac{C}{\sqrt{\varepsilon}}. \quad (58)$$

That proves (52) and (53). Introducing $\varrho = \frac{5}{2\varepsilon^2}e$ we can rewrite (56)-(57) in the following form

$$\begin{aligned} -\Delta \mathbf{E} + \nabla \varrho &= -\left(\frac{5}{2\varepsilon^2} - \frac{5}{2}C_1\right)\mathbf{E} + \Delta \mathbf{r}_0 + \frac{5}{2}C_1\mathbf{r}_0, \quad \operatorname{div} \mathbf{E} = 0 \quad \text{in } O, \\ \mathbf{E} &= -\mathbf{r}_0 \quad \text{on } \partial O. \end{aligned}$$

Applying the standard a priori estimate for the Stokes system (see e.g. [17]) with a right-hand side in L^2 and taking into account (58), we obtain

$$|\mathbf{E}|_{H^2(O)} \leq C\varepsilon^{-2}|\mathbf{E}|_{L^2(O)} \leq C\varepsilon^{-3/2}, \quad (59)$$

$$|\varrho|_{H^1(O)} \leq C\varepsilon^{-3/2} \Rightarrow |p^0|_{H^1(O)} \leq C\sqrt{\varepsilon}. \quad (60)$$

Supposing further regularity on \mathbf{f} , treating the right hand side as an H^1 function and using the estimate (53), we get

$$|\mathbf{E}|_{H^3(O)} + |\varrho|_{H^2(O)} \leq C\varepsilon^{-2}(|\mathbf{r}|_{H^1(O)} + \varepsilon^2|\mathbf{f}|_{L^2(O)}) \leq C\varepsilon^{-5/2}.$$

That process can be continued for higher Sobolev norms to obtain

$$|\mathbf{E}|_{H^4(O)} + |\varrho|_{H^3(O)} \leq C\varepsilon^{-7/2}.$$

□

As we can see, the negative powers of ε appear in the a priori estimates (53)-(54). Such outcome is essentially caused by the boundary layer effects and it represents the main obstacle for proving satisfactory L^2 and H^1 error estimates. Nevertheless, we can try to obtain error estimates in the weaker Sobolev norms, namely H^{-2} norm. That is the subject of our current investigation.

5. Conclusion

In the previous section, new second-order asymptotic model for micropolar fluid film lubrication has been proposed. We start from linearized micropolar equations and perform an asymptotic analysis with respect to the film thickness. No assumptions are made in order to simplify the original 3D problem. Instead of using simple zero boundary condition for microrotation (commonly used in the literature), we impose more complex type of boundary condition for microrotation linking the value of the microrotation with rotation of the velocity. It is based on the concept of boundary viscosity and, as such, turns out to be more physically justified. Observing the analogy between thin film flow and flow through a porous medium, we apply the idea recently proposed in [15]. As a result, we obtain second-order effective model in the form of the Brinkman system describing the macroscopic flow. We clearly detect the effects of new boundary conditions and that represent our main contribution. Of course, from the strictly mathematical point of view, formally derived Brinkman approximation should be rigorously justified by proving some kind of error estimate. As indicated in Section 4, proving satisfactory L^2 or H^1 error estimates is not possible. That is essentially due to boundary layer effects polluting those estimates. A possible way to avoid technical difficulties caused by the boundary layer is to prescribe periodicity as a boundary condition, or to try to obtain satisfactory error estimates in the weaker Sobolev norms. The latter is the subject of our current work. Nevertheless, we believe that the result presented in this paper provides a good platform for understanding the direct influence of the fluid microstructure on the lubrication process and, thus, could be important for developing more efficient numerical algorithms.

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